代数学 II 期末试题参考解答

2014 年 1 月 16 日, 考试时间 13:30-15:10

- ⋆ 请用中文或英文答题.
- ⋆ 一切符号与定义以讲义为准.
- ⋆ 论证中可使用讲义业已证明或预设的结果.
- ⋆ 本卷总分为 100 分.

Note: Unless otherwise specified, all rings are assumed to be nonzero and have a unit element 1. All representations are assumed to be finite-dimensional.

The solutions below are neither unique nor the best ones.

1. **(10 points)** Let R be a simple ring. Show that its center $Z(R)$ is a field.

   Solution. Let $z \in Z(R)$, $z \neq 0$. The set $Rz = zR$ is a two-sided ideal of R containing $z \neq 0$. Hence $Rz = zR = R$ by the simplicity of R. Therefore there exist $v, w \in R$ with $vz = 1 = zw$, i.e. $z \in R^\times$, $v = w = z^{-1}$. It follows that $z^{-1} \in Z(R)$ as well. Hence $Z(R)$ is a field.

2. **(10 points)** Let A be a finite-dimensional algebra over a field F. Denote by $\mathrm{rad}(A)$ its Jacobson radical. For every left A-module M, define its socle as the submodule

$$\mathrm{soc}(M) := \sum_{\substack{N \subset M \\ \text{simple submodule}}} N.$$

   Show that $\mathrm{soc}(M) = \{m \in M : \mathrm{rad}(A) \cdot m = 0\}$.

   Solution. The inclusion $\subset$ follows from the fact that for every simple left A-module M, we have $\mathrm{rad}(A)M = \{0\}$. To prove $\supset$, note that $A/\mathrm{rad}(A)$ is a semisimple ring since it is finite-dimensional, hence left artinian under our assumption. Thus $\{m \in M : \mathrm{rad}(A) \cdot m = 0\}$ decomposes into a sum of simple left $A/\mathrm{rad}(A)$-modules and is contained in $\mathrm{soc}(M)$.

3. **(10 points)** Let F be a finite field. Use Wedderburn's little theorem on finite division rings to show the Brauer group $\mathrm{Br}(F)$ is trivial.

   Solution. Wedderburn's little theorem says that finite division rings are fields. Let A be a central simple F-algebra. We may write $A \simeq M_n(D)$ for some $n \in \mathbb{Z}_{\geq 1}$ and D a central division F-algebra. Since D is a field, it must equal F. Hence $\mathrm{Br}(F)$ is trivial by its definition.

4. **(15 points)** Let R and S be rings. Assume that R and S are Morita equivalent. Show that R is finite if and only if S is finite.

   Solution. It can be proved using Morita's theorems. Here we give a direct proof. Claim: a ring R is finite if and only if for every finitely generated left module P, the endomorphism ring $\mathrm{End}({}_RP)$ is finite. The latter is a property of the category R-**Mod** (see Lecture 4, Definition 1.3), hence is preserved under Morita equivalence. In the hint we suggested a version for finitely generated projective modules, which is slightly more complicated.

   Let us prove the claim. Right translation gives rise to a ring isomorphism $R \xrightarrow{\sim} \mathrm{End}({}_RR)$. Since ${}_RR$ is finitely generated, the ring R is finite. Conversely, assume R finite and let ${}_RP$ be finitely generated. Then P is a finite module, hence the finiteness of $\mathrm{End}({}_RP)$.

5. **(15 points)** For every ring R, let $[R,R]$ be the subgroup of the additive group $(R,+)$ generated by elements of the form $xy - yx$ ($x, y \in R$). Let $\overline{R} := R/[R,R]$ be the quotient group. Show that if two rings R and S are Morita equivalent, then $\overline{R} \simeq \overline{S}$ as abelian groups.

Solution. By Morita theory for left modules, there exists a Morita context $(R, {}_R P_S, {}_S Q_R, S; \alpha, \beta)$ with $\alpha : P \otimes_S Q \xrightarrow{\sim} R$ as (R,R)-bimodules, and $\beta : Q \otimes_R P \xrightarrow{\sim} S$ as (S,S)-bimodules. We shall write $\alpha(p \otimes q) = pq$ and $\beta(q \otimes p) = qp$ as usual. From the maps

$$\begin{array}{ccccc} R & \xleftarrow[\alpha]{\sim} & P \otimes_S Q & \longrightarrow & S/[S,S] \\ pq & \longleftarrow\!\!\shortmid & p \otimes q & \longmapsto & \overline{qp} = \overline{\beta(q \otimes p)} \end{array}$$

one obtains $\Phi : R \to \overline{S}$ characterized by $pq \mapsto \overline{qp}$; note that the right hand side is well-defined only after modulo $[S,S]$. For any $r \in R$ we have $\Phi((rp)q) = \overline{qrp} = \Phi(p(qr))$, hence Φ induces a group homomorphism $\bar{\Phi} : \overline{R} \to \overline{S}$ characterized by $\overline{pq} \mapsto \overline{qp}$. Likewise, the maps

$$\begin{array}{ccccc} S & \xleftarrow[\beta]{\sim} & Q \otimes_R P & \longrightarrow & R/[R,R] \\ qp & \longleftarrow\!\!\shortmid & q \otimes p & \longmapsto & \overline{qp} = \overline{\alpha(p \otimes q)} \end{array}$$

yield a group homomorphism $\bar{\Psi} : \overline{S} \to \overline{R}$ characterized by $\overline{qp} \mapsto \overline{pq}$. They are mutually inverse, hence $\overline{R} \simeq \overline{S}$.

6. **(15 points)** Describe the conjugacy classes $c_1, c_2, \ldots$ of the symmetric group $\mathfrak{S}_3$ by prescribing representatives. Construct all the irreducible representations $V_1, V_2, \ldots$ of $\mathfrak{S}_3$ over $\mathbb{C}$ and compile the character table in the following format.

	c_1	c_2	$\cdots$
V_1	$\chi_{V_1}(c_1)$	$\chi_{V_1}(c_2)$	$\cdots$
V_2	$\chi_{V_2}(c_1)$	$\chi_{V_2}(c_2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

Solution. There are three conjugacy classes, say with representatives $1, (12), (123)$ (as cycles). Since $|\mathfrak{S}_3| = 3! = 2^2 + 1 + 1$, there are exactly three irreducible representations over $\mathbb{C}$: $V_1 := \mathbb{1}$, $V_2 := \mathrm{sgn}$ and a 2-dimensional irreducible representation V_3. As $\chi_{V_3}(1) = \dim V_3 = 2$, the character table takes the form

	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
V_3	2		

We offer two constructions for V_3, both are overkill somehow.

(i) Since $\mathfrak{S}_3$ is clearly supersolvable, the only subgroup of index 2 being the alternating subgroup $\mathfrak{A}_3 = \langle (123) \rangle \simeq \mathbb{Z}/3\mathbb{Z}$, we must have $V_3 \simeq \mathrm{ind}_{\mathfrak{A}_3}^{\mathfrak{S}_3}(\xi)$ for some 1-dimensional representation $\xi : \mathfrak{A}_3 \to \mathbb{C}^\times$. If $\xi = \mathbb{1}$ then $\mathrm{ind}_{\mathfrak{A}_3}^{\mathfrak{S}_3}(\xi)$ contains $\mathbb{1}$ as a subrepresentation, hence reducible. Thus the remaining candidates are $\xi((123)) = e^{\pm 2\pi i/3}$. They are conjugate under $\mathfrak{S}_3$ thus give the same induced representation V_3. The character values can be calculated by the induced character formula (Lecture 6, Proposition 4.1): $\chi_{V_3}((12)) = 0$ since $(12) \notin \mathfrak{A}_3$, whilst $\chi_{V_3}((123)) = e^{2\pi i/3} + e^{-2\pi i/3} = -1$.

(ii) Another way is via Specht modules. The Young diagram corresponding to V_3 is $\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.

Define the following tableaux of shape λ:

$$t_1 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad t_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad t_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then the associated tabloids

$$\{t_1\} = \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}, \quad \{t_2\} = \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \quad \{t_3\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}.$$

are precisely all the tabloids of shape λ; they form a basis for the permutation module $M^\lambda \simeq \mathrm{ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}(\mathbb{1})$. A mental calculation of polytabloids leads to $e_{t_1} = \{t_1\} - \{t_2\}$, $e_{t_2} = \{t_2\} - \{t_1\}$, $e_{t_3} = \{t_3\} - \{t_1\}$. Hence the Specht module is

$$V_3 \simeq S^\lambda = \left\{ \sum_{i=1}^{3} c_i\{t_i\} : c_1 + c_2 + c_3 = 0 \right\}, \quad M^\lambda / S^\lambda \simeq \mathbb{1}.$$

One may calculate $\chi_{V_3}(\cdot) = \chi_{M^\lambda}(\cdot) - 1$ by the induced character formula applied to M^λ.

In fact there is no need to calculate χ_{V_3} by hand. Since the columns of the character table satisfy orthogonality relations (Lecture 5, Theorem 4.5), the missing entries $\chi_{V_3}((12))$ and $\chi_{V_3}((123))$ are immediately determined. All in all, we have

	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
V_3	2	0	-1

7. **(10 points)** Let (V, π) be an absolutely irreducible representation of a finite group G over a field F. Show that there exists a group homomorphism $\omega_\pi : Z(G) \to F^\times$ such that $\pi(z)v = \omega_\pi(z)v$ for each $v \in V$ and $z \in Z(G)$. Here $Z(G)$ denotes the center of G.

   Solution. For every $z \in Z(G)$, the F-linear isomorphism $\pi(z) : V \to V$ satisfies $\pi(z)\pi(g) = \pi(g)\pi(z)$ for all $g \in G$, therefore $\pi(z) \in \mathrm{Hom}_G(V, V) = F \cdot \mathrm{id}$, as (V, π) is absolutely irreducible. We obtain the required group homomorphism $\omega_\pi : Z(G) \to F^\times$.

8. **(15 points)** Let $(V, \sigma), (W, \pi)$ be irreducible representations of a finite group G over a field F. Assume there exists an F-bilinear mapping $B : V \times W \to F$ such that (i) B is G-invariant in the sense that $\forall g \in G,\ B(\sigma(g)\cdot, \pi(g)\cdot) = B(\cdot, \cdot)$, and (ii) B is not identically zero. Show that V is isomorphic to the contragredient $W^\vee$ of W as representations.

   Solution. Define an F-linear map $b : V \to W^\vee$ by sending $v \in V$ to $B(v, \cdot) \in W^\vee$. It is nonzero since B is not identically zero. It is a homomorphism between representations of G. Indeed,

$$b(\sigma(g)v) = B(\sigma(g)v, \cdot) = B(v, \pi(g)^{-1}\cdot) = \check{\pi}(g)\left(B(v, \cdot)\right)$$

   for all $g \in G$, where $\check{\pi}$ denotes the contragredient representation of π. Since V, W are irreducible, b is an isomorphism of representations. Here we used the easy property W irreducible $\Leftrightarrow$ $W^\vee$ irreducible: the representations are finite-dimensional by assumption.