Modular Forms and Number Theory
2021, Peking University

Problem Sheet # 1

Deadline: May 6, 2021

NOTE: Please choose any 3 problems among the following ones. In each problem, you may make free use of the results enunciated in the previous ones. You can also use the results covered in class.

1. Denote $\mathrm{SL}(2,\mathbb{Z})$ by $\Gamma(1)$. Let $f \in M_k(\Gamma(1))$ and $N \in \mathbb{Z}_{\geq 1}$, prove that $f(N\tau) \in M_k(\Gamma_0(N))$.

➾ **Hint.** It might be helpful to know that $f \mid_k \alpha$ is uniformly bounded when $\Im(\tau) \to +\infty$ and $\Re(\tau)$ remains in a given interval, for all $\alpha \in \mathrm{GL}(2,\mathbb{Q})^+$. To show this, take $\beta \in \mathrm{SL}(2,\mathbb{Z})$ such that $\alpha\infty = \beta\infty$. Describe the behavior of $f \mid_k \beta \mid_k \beta^{-1}\alpha$ as $\Im(\tau) \to +\infty$.

2. Let $\Delta = q\prod_{n\geq 1}(1-q^n)^{24} \in S_{12}(\Gamma(1))$, where $q = e^{2\pi i\tau}$ as usual. Show that

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

➾ **Hint.** Use the fact that $\dim S_{12}(\Gamma(1)) = 1$, and compute the coefficient of q in $E_4^3 - E_6^2$.

3. Let k be any positive even integer.

(i) Show that if $\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{ab}E_4^aE_6^b = 0$ where $c_{ab} \in \mathbb{C}$, then $c_{ab} = 0$ for all a, b.

➾ **Hint.** Show that $E_4(\rho) = E_6(i) = 0$, where $\rho = \frac{1+\sqrt{-3}}{2}$, but E_4 and E_6 have no common roots since Δ is non-vanishing on $\mathcal{H}$.

(ii) Show that $M_k(\Gamma(1))$ is generated by $\left\{E_4^aE_6^b : a, b \in \mathbb{Z}_{\geq 0},\ 4a + 6b = k\right\}$ as a vector space.

➾ **Hint.** Reduce first to the case $f \in M_k(\Gamma(1))$ with $k > 6$, then reduce to $f \in S_k(\Gamma(1))$, and divide by Δ if $f \neq 0$.

(iii) Show that the $\mathbb{C}$-algebra $M(\Gamma(1)) := \bigoplus_k M_k(\Gamma(1))$ is isomorphic to $\mathbb{C}[X,Y]$. How are the gradings related?

4. Denote by $M_k(\mathbb{Z})$ the $\mathbb{Z}$-submodule of $M_k(\Gamma(1))$ consisting of modular forms $f = \sum_{n\geq 0} a_n(f)q^n$ with $a_n(f) \in \mathbb{Z}$ for all $n \geq 0$. Show that it has the following $\mathbb{Z}$-basis

$$\begin{aligned} &E_4^a \Delta^b, && 4a + 12b = k && k \equiv 0 \pmod 4, \\ &E_4^a E_6 \Delta^b, && 6 + 4a + 12b = k && k \equiv 2 \pmod 4. \end{aligned}$$

Show that it is also a $\mathbb{C}$-basis of $M_k(\Gamma(1))$. Therefore, one obtains a reasonable integral structure on $M_k(\Gamma(1))$ by just looking at the Fourier coefficients.

Hint. One can use $B_4 = \frac{-1}{30}$ and $B_6 = \frac{1}{42}$ to check that $E_4 \in M_4(\mathbb{Z})$ and $E_6 \in M_6(\mathbb{Z})$. To get the $\mathbb{Z}$-basis, reuse the ideas from the previous problem.

5. Prove *Ramanujan's Congruence* as follows. Write $\Delta = \sum_{n\geq 1} \tau(n)q^n$ and let $\sigma_b(n) := \sum_{\substack{d|n \\ d\geq 1}} d^b$ for all $n \in \mathbb{Z}_{\geq 1}$.

(i) Show that

$$\frac{E_4^3}{720} + \frac{E_6^2}{1008} \in \frac{1}{420} + q^2\mathbb{Z}[\![q]\!].$$

(ii) Show that Δ and $\frac{E_4^3}{720} + \frac{E_6^2}{1008}$ form a basis of $M_{12}(\Gamma(1))$.

(iii) Consider the following rescaled Eisenstein series of weight 12

$$\mathscr{G}_{12} := \frac{-B_{12}}{24} + \sum_{n\geq 1} \sigma_{11}(n)q^n.$$

Show that $\mathscr{G}_{12} = \Delta + \frac{691}{156}\left(\frac{E_4^3}{720} + \frac{E_6^2}{1008}\right)$. You may use the fact that $B_{12} = \frac{-691}{2730}$.

(iv) Deduce that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \geq 1$.

6. For every $N \in \mathbb{Z}_{\geq 1}$, prove that the reduction modulo N map $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ is surjective. Show that

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p|N \\ \text{prime}}} \left(1 - \frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(N)) = N^2 \prod_{\substack{p|N \\ \text{prime}}} \left(1 - \frac{1}{p^2}\right).$$

Last update: June 23, 2021