Modular Forms and Number Theory
2019, Peking University

Problem Sheet # 1

Deadline: January 1, 2020

NOTE: You may choose any 3 problems among the following ones.

1. Let $N_0 \in \mathbb{Z}_{\geq 1}$. Prove that there exists $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ and $N \in \mathbb{Z}_{\geq 1}$ such that

$$f \in M_k(\Gamma(N_0)) \implies f \big|_k \alpha \in M_k(\Gamma_1(N)).$$

☞ **Hint**. Take $\alpha = \begin{pmatrix} N_0 & \\ & 1 \end{pmatrix}$ and a sufficiently divisible $N \in \mathbb{Z}_{\geq 1}$ such that $\alpha \Gamma_1(N) \alpha^{-1} \subset \Gamma(N_0)$.

2. Let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, and $f \in M_k(\Gamma)$ where $k \in \{1, 2\}$. Show that $f(\eta) = 0$ when $\eta \in \mathscr{H}$ is an elliptic point for Γ.

3. There is a modular form $f(\tau) = q^2 + 192q^3 - 8280q^4 + 147200q^5 + \cdots$ in $S_{28}(\mathrm{SL}(2, \mathbb{Z}))$, where $q = e^{2\pi i \tau}$. Granting this fact, express f as a polynomial in E_4, E_6.

☞ **Hint**. We have $f/\Delta^2 \in M_4(\mathrm{SL}(2, \mathbb{Z})) = \mathbb{C}E_4$.

4. Sketch a proof that $\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ is surjective for any $N \in \mathbb{Z}_{\geq 1}$. Prove that

$$(\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p \mid N \\ p:\ \text{prime}}} \left(1 - \frac{1}{p^2}\right).$$

☞ **Hint**. The computation for $(\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N))$ reduces easily to the case $N = p^e$. We also have $\ker[\mathrm{GL}(2, \mathbb{Z}/p^e\mathbb{Z}) \to \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})] = 1 + p\mathrm{M}_2(\mathbb{Z}/p^e\mathbb{Z})$.

5. Let $\alpha_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Q})^+$.

(a) Show that $\alpha_N \Gamma_0(N) \alpha_N^{-1} = \Gamma_0(N)$, thus $\tau \mapsto \alpha_N(\tau)$ descends to an automorphism of $Y_0(N)$.

(b) Give a moduli interpretation of this automorphism, in terms of complex tori with $\Gamma_0(N)$-level structures.

☞ **Hint**. The moduli interpretation is $(E, B) \mapsto (E/B, E[N]/B)$, where E is a complex torus and $B \subset E[N]$ is a subgroup $\simeq \mathbb{Z}/N\mathbb{Z}$. Show that this is indeed an automorphism of $Y_0(N)$.

6. For $(z, \tau) \in \mathbb{C} \times \mathscr{H}$, define $q := e^{\pi i \tau}$, $\eta := e^{2\pi i z}$ and

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}\eta^n,$$
$$P(z;\tau) := \prod_{n\geq 1}\left(1 + q^{2n-1}\eta\right)\left(1 + q^{2n-1}\eta^{-1}\right).$$

Define the lattice $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$ in $\mathbb{C}$.

(a) Prove that

$$\vartheta(z+\tau;\tau) = (q\eta)^{-1}\vartheta(z;\tau),$$
$$P(z+\tau;\tau) = (q\eta)^{-1}P(z;\tau),$$

and show that $z \mapsto \vartheta(z;\tau)/P(z;\tau)$ is a Λ_τ-periodic meromorphic function on $\mathbb{C}$.

(b) Fix τ and show that the zeros of $z \mapsto P(z;\tau)$ are precisely $z = \frac{1}{2} + \frac{\tau}{2} + \Lambda_\tau$. Show that they are also the zeros of $\vartheta(z;\tau)$, and $\vartheta(z;\tau)/P(z;\tau)$ depends only on q. Put $\phi(q) := \vartheta(z;\tau)/P(z;\tau)$.

(c) Prove that

$$\vartheta\left(\frac{1}{2}; 4\tau\right) = \vartheta\left(\frac{1}{4};\tau\right),$$
$$P\left(\frac{1}{2};4\tau\right) = P\left(\frac{1}{4};\tau\right)\cdot\prod_{n\geq 1}\left(1-q^{4n-2}\right)\left(1-q^{8n-4}\right),$$
$$\lim_{q\to 0}\phi(q) = 1.$$

(d) Apply the previous result to show $\phi(q) = \prod_{n\geq 1}(1-q^{2n})$. Make the change of variables $(z;\tau) \rightsquigarrow \left(-\frac{\tau}{4}+\frac{1}{2}, \frac{3\tau}{2}\right)$ (accordingly, $(q,\eta) \rightsquigarrow (q^{3/2}, -q^{-1/2})$) to deduce *Jacobi's triple product identity*[1]

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}} = \prod_{n\geq 1}(1-q^n).$$

Note that it yields the Fourier expansion for Dedekind's η-function.

☞ **Hint**. Just some basic operations on infinite sums and products.

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[1]More precisely, Euler's Pentagonal Numbers Theorem

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Problem Sheet # 2

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Note. You may choose any 2 problems among the following ones.

Conventions. We write

$$\mathrm{GL}(2,\mathbb{Q})^+ := \left\{ g \in \mathrm{GL}(2,\mathbb{Q}) : \det g > 0 \right\}, \quad \mathscr{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

as usual. Let $N \in \mathbb{Z}_{\geq 1}$; Fourier expansions of modular forms of level $\Gamma_1(N)$ will be written as $f = \sum_{n\geq 0} a_n(f) q^n$, where $q := e^{2\pi i \tau}$, and the Hecke operators T_p act on $M_k(\Gamma_1(N))$. The stabilizer of an element x under a group Γ is denoted as $\mathrm{Stab}_\Gamma(x)$, and so forth. We write $\sigma_h(n) = \sum_{d|n} d^h$ for every $h \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 1}$. Define automorphy factor as $j(\gamma, \tau) = c\tau + d$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$.

1. Consider the congruence subgroups $\Gamma_0(4) = \{\pm 1\} \cdot \Gamma_1(4) \rhd \Gamma_1(4)$. Note that $\begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix} \infty = \frac{1}{2}$.

(a) Show that

$$\mathrm{Stab}_{\Gamma_0(4)}\left(\frac{1}{2}\right) = \pm \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix}^{-1}$$

and this group is generated by -1 together with the element

$$\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}.$$

(b) Show that $\mathrm{Stab}_{\Gamma_1(4)}\left(\frac{1}{2}\right)$ is generated by $\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$. Conclude that $\frac{1}{2}$ represents an *irregular cusp* for $\Gamma_1(4)$.

2. Let $N, k \in \mathbb{Z}_{\geq 1}$. Prove that a modular form $f = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_1(N))$ is uniquely determined by $(a_n)_{n\geq 1}$.

3. For every $\tau \in \mathscr{H}$, put $\Lambda_\tau := \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$. The endomorphism ring of the complex torus $\mathbb{C}/\Lambda_\tau$ is denoted as $\mathrm{End}(\mathbb{C}/\Lambda_\tau)$, which is a subring of $\mathbb{C}$. Show that $\mathrm{End}(\mathbb{C}/\Lambda_\tau) \supsetneq \mathbb{Z}$ if and only if τ is a quadratic irrational in $\mathscr{H}$, i.e. there exist $A, B, C \in \mathbb{Z}$ such that $A \neq 0$ and $A\tau^2 + B\tau + C = 0$.

☞ **Hint**. First, show that $\mathrm{End}(\mathbb{C}/\Lambda_\tau) \supsetneq \mathbb{Z}$ if and only if $\gamma\tau = \tau$ for some $\gamma \in \mathrm{GL}(2, \mathbb{Q})^+$ which is not a scalar. Show that the quadratic irrationals in $\mathscr{H}$ are precisely the fixed points of elements of $\mathrm{GL}(2, \mathbb{Q})^+$.

4. Let $k \geq 4$ be an even integer. Show that the Eisenstein series E_k is orthogonal to $S_k(\mathrm{SL}(2, \mathbb{Z}))$ with respect to the Petersson inner product.

☞ **Hint**. Write $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ and $\Gamma_\infty := \mathrm{Stab}_\Gamma(\infty)$. Argue that, for all $f \in S_k(\mathrm{SL}(2, \mathbb{Z}))$,

$$\int_{\Gamma\backslash\mathscr{H}} f(\tau) \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \overline{j(\gamma, \tau)^{-k}} \mathrm{Im}(\tau)^k \, \mathrm{d}\mu(\tau) = \int_{\Gamma_\infty\backslash\mathscr{H}} f(\tau)\mathrm{Im}(\tau)^{k-2} \, \mathrm{d}x \, \mathrm{d}y$$

where $\mathrm{d}\mu(\tau) = y^{-2}\mathrm{d}x\mathrm{d}y$ (with $\tau = x + iy$) is the hyperbolic measure on $\mathscr{H}$. Find a fundamental domain for $\mathscr{H}$ under Γ_∞-action, and observe that $\int_0^1 f(x + iy) \, \mathrm{d}x = 0$ for each $y \in \mathbb{R}$.

5. For each even integer $k \geq 2$, use the Eisenstein series G_k to define

$$\mathscr{G}_k := \frac{(k-1)!}{2(2\pi i)^k} \cdot G_k \in M_k(\mathrm{SL}(2, \mathbb{Z}))$$

so that $a_n\left(\mathscr{G}_k\right) = \sigma_{k-1}(n)$ for all $n \geq 1$. Show that $\mathscr{G}_k$ is a normalized Hecke eigenform satisfying $T_p\mathscr{G}_k = \left(1 + p^{k-1}\right)\mathscr{G}_k$ for every prime number p.

☞ **Hint**. Compare $a_n\left(T_p\mathscr{G}_k\right)$ and $a_n\left(\mathscr{G}_k\right)$. The case $n = 0$ is straightforward. As for the case $n \geq 1$, one has to determine $\left(1 + p^{k-1}\right)\sigma_{k-1}(n) = \sigma_{k-1}(p)\sigma_{k-1}(n)$ in terms of $\sigma_{k-1}(pn)$ and $\sigma_{k-1}(n/p)$ (when $p \mid n$).

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