

# On the A-packets for genuine representations of $Mp(2n)$

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*Sveučilišta u Zagrebu*

**Wen-Wei Li**

Sveučilište u Pekingu

wwli@pku.edu.cn

<https://wwli.asia>

# Outline

Genuine representations of metaplectic groups

LLC of Gan–Savin

Automorphic set-up

Desiderata: local and global

Results of Gan–Ichino

Strategy à la Arthur

To-do list

Hecke algebra correspondences

References



## Local metaplectic covering

- $\mu_m := \mu_m(\mathbb{C})$ , all rep are over  $\mathbb{C}$ .
- $F$ : local field,  $\text{char}(F) = 0$ ; let  $\mathcal{L}_F$  denote its Weil–Deligne group.
- $W$ : symplectic  $F$ -vector space,  $\dim W = 2n$ .
- $\text{Sp}(W) = \mathbf{Sp}(W, F)$  (or  $\mathbf{Sp}(2n, F)$ ): the symplectic group.

### Definition

If  $F \neq \mathbb{C}$ , the metaplectic group is THE non-trivial central extension of locally compact groups

$$1 \rightarrow \mu_2 \rightarrow \text{Mp}(W) \rightarrow \text{Sp}(W) \rightarrow 1.$$

If  $F = \mathbb{C}$  we put  $\text{Mp}(W) = \mu_2 \times \text{Sp}(W)$ .

It is customary to write  $\text{Mp}(2n)$  or  $\text{Mp}(2n, F)$ .



# Genuine representations

Fixing an additive character  $\psi$  of  $F$  and the symplectic form  $\langle \cdot | \cdot \rangle$  on  $W$ , one can describe  $\mathrm{Mp}(W)$  by explicit 2-cocycles (Rao, Lion–Perrin).

*Representations of  $\mathrm{Mp}(W)$* : HC-modules or Casselman–Wallach representations if  $F \supset \mathbb{R}$ ; smooth if  $F \supset \mathbb{Q}_p$ .

## Definition

A representation  $(\pi, V_\pi)$  of  $\mathrm{Mp}(W)$  is *genuine* if  $\pi(z) = z \cdot \mathrm{id}_{V_\pi}$  for all  $z \in \mu_2$ .

For instance, the Weil/oscillator representation  $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$  of  $\mathrm{Mp}(W)$  is genuine. They depend on  $\psi \circ \langle \cdot | \cdot \rangle$ .

## Goal

Understand the genuine representation theory of  $\mathrm{Mp}(W)$ .

# The L-group

*Question:* Langlands program for genuine representations of  $\mathrm{Mp}(W)$ ?

Fix  $\psi \circ \langle \cdot | \cdot \rangle$ . There are strong evidences (from  $\Theta$ -correspondence, geometric Satake, etc.) for the

## **Definition**

The L-group of  $\mathrm{Mp}(W)$  is  $\mathrm{Sp}(2n, \mathbb{C}) \times \mathrm{Weil}_F$ , i.e. same as the L-group of the split  $\mathrm{SO}(2n + 1)$ .

This is also compatible with Weissman's definition of L-groups for coverings.

- L-parameters for  $\mathrm{Mp}(W)$  = symplectic representations  $\phi = \bigoplus_{i \in I} m_i \phi_i$  of  $\mathcal{L}_F$ , where  $m_i \geq 1$ , the  $\phi_i$ 's are distinct simple representations of  $\mathcal{L}_F$ , and  $\sum_i m_i \dim \phi_i = 2n$ .
- A-parameters for  $\mathrm{Mp}(W)$  = symplectic representations  $\psi = \bigoplus_{i \in I} m_i \psi_i$  of dimension  $2n$  of  $\mathcal{L}_F \times \mathrm{SL}(2, \mathbb{C})$  as above; the  $\mathcal{L}_F$  factor of each  $\psi_i$  is bounded.
- $S_\phi$  = centralizer of  $\phi$  in  $\mathrm{Sp}(2n, \mathbb{C})$  (same for  $S_\psi$ ).
- $\mathcal{S}_\phi = \pi_0(S_\phi)$  (same for  $\mathcal{S}_\psi$ ).

One can describe  $S_\phi$  and  $\mathcal{S}_\phi$  explicitly;  $\mathcal{S}_\phi$  is finite abelian, and:

$$\mathcal{S}_\phi^\vee = \mu_2^{I^+}, \quad I^+ := \{i \in I : \phi_i \text{ symplectic.}\}$$

Same for  $S_\psi$ ,  $\mathcal{S}_\psi$  and  $\mathcal{S}_\psi^\vee$ .

# Local Langlands correspondences

Let  $\Phi_{\text{symp}}(2n)$  be the set of equivalence classes of L-parameters for  $\text{Mp}(W)$  (= those for  $\text{SO}(2n + 1)$ ). The following is due to Adams–Barbasch ( $F \supset \mathbb{R}$ ) and Gan–Savin ( $F \supset \mathbb{Q}_p$ ).

## Theorem (LLC)

There is a decomposition

$$\text{Irr}_{\text{gen}}(\text{Mp}(W)) = \bigsqcup_{\phi \in \Phi_{\text{symp}}(2n)} \Pi_{\phi},$$

together with bijections  $\Pi_{\phi} \leftrightarrow \mathcal{S}_{\phi}^{\vee}$  + various properties, eg.

- $\pi \in \Pi_{\phi}$  is tempered (resp. discrete series)  $\iff \phi$  is bounded (resp. does not factor through proper Levi);
- LLC reduces to tempered/bounded case via Langlands quotients;
- if  $\phi$  is bounded, then  $\mathbf{1} \in \mathcal{S}_{\phi}^{\vee}$  corresponds to generic representation.

- The LLC depends on  $\psi$  and the symplectic form  $\langle \cdot | \cdot \rangle$  on  $W$ .
- It is proved by reduction to the LLC of  $\mathrm{SO}(V^\pm)$  (Arthur, Ishimoto) where  $V^\pm$  is the quadratic vector space with
  - dimension  $2n + 1$ ,
  - discriminant  $1$ ,
  - Hasse invariant  $\pm 1$ ,
 via  $\Theta$ -correspondence for the reductive dual pair  $(\mathrm{Sp}(W), \mathrm{O}(V^\pm))$ .
- Note:  $\mathcal{S}_\phi^\vee$  is also in bijection with the packet  $\Pi_\phi^{\mathrm{Vogan}}$  for  $\mathrm{SO}(V^\pm)$ .
- There is a more direct proof for  $F = \mathbb{C}$ .



## The endoscopic viewpoint (Adams, Renard, L.)

The set of elliptic endoscopic data of  $\mathrm{Mp}(W)$  is defined as

$$\begin{aligned} \mathcal{E}_{\mathrm{ell}}(\mathrm{Mp}(W)) &:= \{s \in \mathrm{Sp}(2n, \mathbb{C}) : s^2 = 1\} / \mathrm{conj}. \\ &= \{(\underbrace{n'}_+, \underbrace{n''}_-) \in \mathbb{Z}_{\geq 0}^2 : n' + n'' = n\}. \end{aligned}$$

The endoscopic group is  $\mathrm{SO}(2n' + 1) \times \mathrm{SO}(2n'' + 1)$ .

Similar to elliptic endoscopic data for  $\mathrm{SO}(2n + 1)$ , but without symmetry  $(n', n'') \leftrightarrow (n'', n')$ .

### Known results

- Transfer of orbital integrals (Renard for  $F = \mathbb{R}$ , L. '11 for  $F \supset \mathbb{Q}_p$ )
- Fundamental lemma for units, including the weighted case (L. '11)
- Fundamental lemma for spherical Hecke algebra (C. Luo '18).

Let  $\mathbf{G}^! \in \mathcal{E}_{\text{ell}}(\text{Mp}(W))$  with endoscopic group  $G^!$ . The transfer of orbital integrals dualizes to a map

$$\check{\mathcal{J}}_{\mathbf{G}^!, \text{Mp}(W)} : \{\text{st. dist. on } G^!(F)\} \rightarrow \{\text{genuine dist. on } \text{Mp}(W)\}$$

sending stable virtual characters to genuine virtual characters.

- For every BOUNDED L-parameter  $\phi^!$  for  $G^!$ , we have a stable tempered distribution  $S\Theta_{\phi^!}^{G^!}$ .
- $(G^!)^\vee = \text{Sp}(2n', \mathbb{C}) \times \text{Sp}(2n'', \mathbb{C}) \hookrightarrow \text{Sp}(2n, \mathbb{C})$  up to conjugacy, hence  $\phi^!$  maps to a bounded  $\phi \in \Phi_{\text{symp}}(2n)$ .

### Endoscopic character relations (ECR) — C. Luo

Let  $\phi \in \Phi_{\text{symp}}(2n)$  be bounded. The L-packet  $\Pi_\phi$  for  $\text{Mp}(W)$  can be characterized in terms of  $\check{\mathcal{J}}_{\mathbf{G}^!, \text{Mp}(W)} \left( S\Theta_{\phi^!}^{G^!} \right)$  for various  $(\mathbf{G}^!, \phi^!)$  such that  $\phi^! \mapsto \phi$ , and **some  $\epsilon$ -factors**.

The precise formulation will be given later on.

## The adélic covering

Let  $F$  be a number field, and  $W$  a symplectic  $F$ -vector space of dimension  $2n$ . The metaplectic group is the non-trivial central extension

$$1 \rightarrow \mu_2 \rightarrow \mathrm{Mp}(W, \mathbb{A}_F) \rightarrow \mathbf{Sp}(W, \mathbb{A}_F) \rightarrow 1.$$

- It splits uniquely over  $\mathbf{Sp}(W, F)$ , hence it makes sense to study
  - *genuine automorphic forms* on  $\mathrm{Mp}(W, \mathbb{A}_F)$ , eg. Siegel modular forms of  $\frac{1}{2} + \mathbb{Z}$ -weights,
  - genuine  $L^2$ -automorphic spectrum.
- It is the quotient of  $\prod'_v \mathrm{Mp}(W_v)$  by  $\{(z_v)_v \in \bigoplus_v \mu_2 : \prod_v z_v = 1\}$ , hence irreducible admissibles of  $\mathrm{Mp}(W, \mathbb{A}_F)$  decompose into  $\bigotimes'_v \pi_v$ .

Here we use the fact that  $\mathrm{Mp}(W_v)$  splits over  $\mathbf{Sp}(W_v, \mathcal{O}_v)$  with commutative Hecke algebras, for almost all  $v$ .

## Local desiderata for Arthur packets

Let  $F$  be local,  $\dim_F W = 2n$  and  $\psi$  is fixed. For the study of

1. unitary duals,
2. Gelfand–Kirillov dimensions, or
3. global  $L^2$ -automorphic spectrum,

the LLC is not enough: one has to go beyond tempered L-packets and study Arthur packets. My main motivation is 3.

Recall:  $A$ -parameters for  $\mathrm{Mp}(W)$  are symplectic representations

$$\psi = \bigoplus_{i \in I} m_i \phi_i \boxtimes r(b_i) : \mathcal{L}_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(2n, \mathbb{C}),$$

where  $r(b_i) := \mathrm{Sym}^{b_i-1}(\mathrm{std})$  and  $\phi_i$  is bounded. Let

$\Psi_{\mathrm{symp}}(2n) = \{\text{such } \psi\}$ . Define  $\Psi_{\mathrm{symp}}^+(2n)$  by dropping boundedness.

## Characterization via ECR

Given a pair  $(\mathbf{G}^!, \psi^!)$  where  $\mathbf{G}^! \in \mathcal{E}_{\text{ell}}(\text{Mp}(W))$  and  $\psi^! \in \Psi^+(G^!)$ , we obtain  $(\psi, s)$  where  $\psi^! \mapsto \psi$  and  $s \in \mathcal{S}_{\psi, 2\text{-tors}}/\text{conj}$  corresponds to  $\mathbf{G}^!$ . This is actually a bijection.

Given  $(\psi, s)$ , Arthur's theory for  $G^!$  provides stable virtual characters  $S\Theta_{\psi^!}^{G^!}$  on  $G^!(F)$ . Consider the  $(-1)$ -eigenspace of  $\text{std} \circ \psi$  under  $s$  and set

$$\epsilon(\psi^{s=-1}) := \epsilon\left(\frac{1}{2}, \psi^{s=-1} \Big|_{\mathcal{L}_F}, \psi\right).$$

### Definition-Lemma

The following depends only on  $\psi$  and the image  $x$  of  $s$  in  $\mathcal{S}_{\psi}$ .

$$T_{\psi, s} := \epsilon(\psi^{s=-1}) \cdot \check{\mathcal{J}}_{\mathbf{G}^!, \text{Mp}(W)} \left( S\Theta_{\psi^!}^{G^!} \right).$$

Every  $x \in \mathcal{S}_\psi$  arises from some  $s \in S_{\psi,2\text{-tors}}$ , hence we may write  $T_{\psi,x} = T_{\psi,s}$ , and consider the Fourier expansion of  $x \mapsto T_{\psi,x}$  or its translates.

### Main local Theorem (L.)

Given  $\psi \in \Psi_{\text{symp}}^+(2n)$ , set  $s_\psi := \psi \left( 1, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right) \in S_\psi$  and let  $x_\psi$  be its image in  $\mathcal{S}_\psi$ . Then

$$\pi_{\psi,\chi} := |\mathcal{S}_\psi|^{-1} \sum_{x \in \mathcal{S}_\psi} \chi(x_\psi x) T_{\psi,x}$$

is a  $\mathbb{Z}_{\geq 0}$ -linear combination (possibly zero) of genuine irreducible characters of  $\text{Mp}(W)$ , for all  $\chi \in \mathcal{S}_\psi^\vee$ . If  $\psi \in \Psi_{\text{symp}}(2n)$ , then these irreducible characters arise from unitary representations.

We also call it the *endoscopic character relation* (ECR) associated with  $\psi$ .

## Definition of A-packets

Collecting the constituents of  $\pi_{\psi, \chi}$  for various  $\chi$ ), we obtain the A-packet  $\Pi_{\psi}$  as a multi-set of genuine irreducibles. Rigorously,  $\Pi_{\psi}$  is a finite set equipped with two maps

$$\text{Irr}_{\text{gen}}(\text{Mp}(W)) \leftarrow \Pi_{\psi} \rightarrow \mathcal{S}_{\psi}^{\vee}.$$

- The structure above is characterized completely by ECR.
- The  **$\epsilon$ -factor** in  $T_{\psi, s}$  is a **metaplectic feature**: it does not appear in Arthur's original version.
- If  $\psi$  is a bounded L-parameter (i.e. all  $b_i = 1$ ), then we recover the L-packet and Luo's ECR for the tempered LLC for  $\text{Mp}(W)$ .

Furthermore,  $\pi_{\psi, \chi}$  has the following properties established in [L.].

1. Reduction to good parity case (i.e. each simple summand in  $\psi$  is symplectic) via full parabolic induction.
2. Infinitesimal characters expressed in terms of  $\psi$  when  $F \supset \mathbb{R}$ .
3. Assume the covering is unramified. If  $\psi$  is trivial on  $I_F \times \mathrm{SL}(2, \mathbb{C}) \subset \mathcal{L}_F$ , then  $\Pi_\psi$  has a unique spherical member, which is multiplicity-free and parametrized by  $\chi = \mathbf{1}$ . If  $\psi$  is not unramified then  $\Pi_\psi$  has no spherical members.
4. Central characters expressed in terms of  **$\epsilon$ -factors** and  $(\psi, \chi)$ .
5.  $\Pi_{\phi_\psi}$  embeds canonically into  $\Pi_\psi^{\mathrm{mult}=1}$ , where for all  $w \in \mathcal{L}_F$ ,

$$\phi_\psi(w) = \psi \left( w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix} \right) \quad (\text{L-parameter}).$$

6. The effect of variation of  $\psi$  can be explicitly described, generalizing the recipe for L-packets due to Gan–Savin.
7. Normalization of int. op. via A-parameters.



## Global desiderata

Let  $F$  be a number field. As before,  $\dim_F W = 2n$  and

$\psi = \bigotimes_v \psi_v : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$  is fixed. Put

$$L_{\text{gen,disc}}^2 := L_{\text{genuine,discrete}}^2(\mathbf{Sp}(W, F) \backslash \text{Mp}(W, \mathbb{A}_F)).$$

- A-parameters  $\psi$  are defined as formal sums of  $\phi_i \boxtimes r(b_i)$  where  $\phi_i$ : cuspidal automorphic representations of  $\text{GL}(n_i, \mathbb{A}_F)$ , and  $b_i \in \mathbb{Z}_{\geq 1}$  with parity conditions (Arthur).

They are defined without resort to the hypothetical **automorphic Langlands group**  $\mathcal{L}_F$ .

- Also defined:  $S_\psi$  and  $\mathcal{S}_\psi$  equipped with localization maps,  $\forall v$ .
- Given  $\psi$ , can define the summand  $L_\psi^2$  of  $L_{\text{gen,disc}}^2$  via “near equivalence classes”, using the Satake parameters attached to  $\psi_v$  for almost all  $v$ .
- We say  $\psi$  is *elliptic* if  $\psi = \bigoplus_{i \in I} \phi_i \boxtimes r(b_i)$  where all the  $\phi_i \boxtimes r(b_i)$  are distinct and symplectic.

**Theorem 1 (Gan–Ichino)**

$$L_{\text{gen, disc}}^2 = \widehat{\bigoplus}_{\psi:\text{elliptic}} L_{\psi}^2.$$

The above is proved using  $\Theta$ -correspondence in stable range. Define

$$\Pi_{\psi} := \left\{ \pi = (\pi_v)_v \mid \begin{array}{l} \pi_v \in \Pi_{\psi_v} \text{ (multi-set!),} \\ \text{spherical for almost all } v \end{array} \right\},$$

and let  $\Pi_{\psi}(\epsilon_{\psi})$  be the subset given by

$$\prod_v \langle s_v, \pi_v \rangle = \underbrace{\epsilon_{\psi}^{\text{Art}}(s)}_{\text{same as } \text{SO}(2n+1)} \in \left( \frac{1}{2}, \psi^{s=-1} |_{\mathcal{L}_F}, \psi \right), \quad \forall s \in S_{\psi} = \mathcal{S}_{\psi}.$$

**Main global Theorem (L., conjectured by Gan in ICM 2014)**

Grosso modo,  $L_{\psi}^2 \simeq \bigoplus_{\pi \in \Pi_{\psi}(\epsilon_{\psi})} \bigotimes'_v \pi_v$  for all elliptic  $\psi$ .

**Remark.** Levi subgroups of  $\mathbf{Sp}(W)$  are of the form

$$\mathbf{M} = \mathbf{Sp}(W^b) \times \prod_{k=1}^r \mathbf{GL}(n_k), \quad \begin{array}{l} W^b \subset W : \text{symp. subspace,} \\ \dim W^b + 2 \sum_k n_k = \dim W. \end{array}$$

One has to formulate and prove these assertions for preimages of  $\mathbf{M}(F)$  (resp.  $\mathbf{M}(\mathbb{A}_F)$ ) in  $\mathrm{Mp}(W)$  (resp.  $\mathrm{Mp}(W, \mathbb{A}_F)$ ).

1. The twofold covering does not split over  $\mathbf{GL}$  factors, but this can be handled as in Hanzer–Muić (10), using some genuine characters made from Weil constants.
2. Alternatively,  $\mathrm{Mp}(W)$  can be enlarged to  $\widetilde{\mathrm{Sp}}(W)$  by pushing out via  $\mu_2 \hookrightarrow \mu_8$ . Genuine representation theory is unaffected, but the preimage of  $\mathbf{M}(F)$  becomes  $\widetilde{\mathrm{Sp}}(W^b) \times \prod_{k=1}^r \mathrm{GL}(n_k, F)$ .

Both approaches rely on the choice of  $\psi$  and symplectic forms.

# Waldspurger and Gan–Ichino

Consider both the local and global settings.

- When  $n = 1$ , these are known to Waldspurger.
- When  $\psi$  is generic (i.e. all  $b_i = 1$ ), the main global theorem is due to Gan–Ichino.
- When  $n = 2$ , Gan–Ichino (‘21) obtained both main theorems “by hand” with the help of Hanzer–Matić (‘10).

These are all based on  $\Theta$ -correspondence, not endoscopy.  
Compatibilities are shown in [L.]

## Example: the most degenerate case

Take  $\psi = \mathbf{1} \boxtimes r(2n)$ .

- Locally,  $\Pi_{\psi_v} = \left\{ \omega_{\psi_v}^+, \omega_{\psi_v}^- \right\}$ ,  $\forall v$  (known to Adams).
- Globally,  $L_{\psi}^2$  are generated by *elementary*  $\vartheta$ -series.

## Proofs: Strategy à la Arthur

Try to imitate Arthur's *endoscopic classification* (Chapters 4 and 7) to prove the local and global theorems altogether. Ingredients:

- **Stabilization of trace formula.** Done for  $\mathrm{Mp}(W)$  (L. '21).
- **Spectral decomposition of the stable side.** Done by Arthur since the endoscopic groups are  $\mathbf{SO}(2n' + 1) \times \mathbf{SO}(2n'' + 1)$ .
- **Local intertwining relation (LIR).** DIFFICULT, only known for generic  $\psi$  (due to Ishimoto).

Specifically, Arthur used LIR to prove that the  $L^2$ -automorphic spectrum involves only elliptic A-parameters (the “no embedded Hecke eigenvalues” property), for quasi-split classical groups.

It seems difficult to prove LIR directly for  $\mathrm{Mp}(W_v)$ .

## Shortcut (+ suggestions from Waldspurger)

Thanks to Gan–Ichino,  $L_{\text{gen, disc}}^2 = \widehat{\bigoplus}_{\psi: \text{ell.}} L_{\psi}^2$  is directly available to us.

1. Main global theorem (= decomposition of  $L_{\psi}^2$ ) follows easily from STF for  $\text{Mp}(W, \mathbb{A}_F)$ . Though A-packets are not yet available, the global theorem can be formulated as a character relation involving various  $\pi_{\psi_v, \chi_v}$ .
2. The main local theorem is proved by global means via the main global theorem (phrased as above). Data put at the auxiliary places:
  - either from the L-packet inside an A-packet, or
  - suitable co-tempered representations ( $v \nmid \infty$ ).

### Theorem (F. Chen, '24)

Transfer for  $\text{Mp}(W)$  commutes with Aubert dual for  $F \supset \mathbb{Q}_p$ .

To get the co-tempered ECR from Luo's ECR via Aubert dual, some sign equality is needed; we globalize carefully and reduce it to SO case (AGIKMS, Ishimoto, Liu–Lou–Shaihi... ) via  $\Theta$ .

# To-do list

- Explicit construction of A-packets when  $F \supset \mathbb{Q}_p$ , after Mœglin, Xu, Atobe..., and multiplicity-one (work in progress by J. Chen).
- Relation to  $\Theta$ -correspondence (Xu's student?).
- Relation to translation functors and cohomological induction when  $F = \mathbb{R}$ , à la Mœglin–Renard; Adams–Johnson packets.
- Explicit construction for  $F = \mathbb{C}$  as predicted by Mœglin–Renard.
- Prove LIR.
- Application to number theory, eg. Ikeda–Yamana lifting from  $\mathrm{PGL}(2)$  to  $\mathrm{Mp}(2n)$  for  $n$  odd and  $F$  totally real.
- Can we use these results to study **global root numbers**?

## Postscript: affine Hecke algebras

Suppose  $F \supset \mathbb{Q}_p$  and  $\psi$  has conductor  $4\mathcal{O}_F$ . Let  $\mathcal{G}_{\psi}^{\pm}$  be the Bernstein block  $\ni \omega_{\psi}^{\pm}$ . Let  $\mathcal{G}^{\pm}$  be the Bernstein block  $\ni \mathbf{1}_{\mathrm{SO}(V^{\pm})}$ .

- Gan–Savin ( $p > 2$ ) and Takeda–Wood ( $p = 2$ ) showed  $\mathcal{G}_{\psi}^{\pm} \simeq \mathcal{G}^{\pm}$  by constructing types for  $\mathcal{G}_{\psi}^{\pm}$  and giving an explicit isomorphism between Hecke algebras.
- It is stronger (being a categorical equivalence) and looks more natural than LLC in many aspects. It also preserves unitarity, temperedness and discrete series.
- $\mathrm{Mp}(W)$  and  $\mathrm{SO}(V^{\pm})$  share the same L-group;  $\mathcal{G}_{\psi}^{\pm} \simeq \mathcal{G}^{\pm}$  preserves L-parameters  $\phi$  but not the  $\chi \in \mathcal{S}_{\phi}^{\vee}$ .

**Natural question 1:** How do the  $\chi$ 's differ under  $\mathcal{G}_{\psi}^{\pm} \simeq \mathcal{G}^{\pm}$ ?





Suppose that  $\pi \in \mathcal{G}_\psi^\pm$  is irreducible and corresponds to  $\sigma \in \mathcal{G}^\pm$ ;

$$\pi = \pi_{\phi, \chi}, \quad \sigma = \sigma_{\phi^\circ, \chi^\circ} \quad \text{under LLC.}$$

Write  $\phi = \bigoplus_{i \in I} m_i \phi_i$ . Identify  $\mathcal{S}_\phi^\vee$  with  $\mu_2^{I^+}$  where  $I^+ \subset I$  indexes the symplectic summands in  $\phi$ .

### Theorem (F. Chen–L. '25)

We have  $\phi^\circ = \phi$  (known to GS+TW) and  $\chi^\circ = \chi \nu_\phi$ , where







$$\nu_\phi = (\nu_{\phi, i})_{i \in I^+} \in \mathcal{S}_\phi^\vee, \quad \nu_{\phi, i} = \epsilon \left( \frac{1}{2}, \phi_i, \psi \right).$$

Modulo [Chen–L. '23], the argument is largely “endoscopic”.







**Naive question 2:** How about A-packets?

**Naive question 3:** Other blocks?

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# Thanks for your attention

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