On the distinction of Harish-Chandra modules and its Ext-analogues

Wen-Wei Li

Based on a talk given in the RTAG Seminar, 29.07.2020

Abstract

These are notes for my talk[1] in the Seminar of Representation Theory and Algebraic Geometry, Weizmann Institute on July 29th, 2020. The original abstract is reproduced below.

One core problem in relative harmonic analysis is to study the space of H-invariant linear functionals on an admissible representation, where H is a spherical subgroup of a reductive group G over a local field. In this talk, I will focus on the Archimedean case in the setting of Harish-Chandra modules. I will review the interpretation of these Hom spaces in terms of certain regular holonomic D-modules on G/H (arXiv:1905.08135), under mild conditions on H. Then I will try to sketch a possible extension of this strategy to the Ext-analogues and the Euler–Poincaré numbers. This is a work IN PROGRESS.

Warning. The discussions below are speculative. We will make an Assumption 5.1 and a Conjecture 6.1.

1 Introduction

Let $\mathbf{G}$ be a connected reductive group over a local field F, and consider a subgroup $\mathbf{H}$. Write $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$ and let $\mathcal{Z}(\mathfrak{g})$ be its center. All the representations under consideration will be realized on $\mathbb{C}$-vector spaces.

The central problem of *distinction* of representations of G by H can be formulated as follows.

- ⋆ Suppose that F is non-Archimedean. We want to study $\mathrm{Hom}_H(\pi, \mathbb{C})$ where π is an admissible smooth representation of G, and $\mathbb{C}$ stands for the trivial representation of H. More generally, we can consider $\mathrm{Hom}_H(\pi, \chi)$ for a character $\chi : H \to \mathbb{C}^\times$, but we will not pursue this level of generality here.
- ⋆ Suppose that F is Archimedean, say $F = \mathbb{R}$. There are at least two formulations of our problem.
  - Let π^∞ be a Casselman–Wallach representation of G, and consider $\mathrm{Hom}_H(\pi^\infty, \mathbb{C})$ where Hom_H is the continuous Hom of H-representations.
  - Let π be a Harish-Chandra module, and consider $\mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C})$; here $\mathrm{Hom}_{\mathfrak{h}}$ is the algebraic Hom-space of $\mathfrak{h}$-equivariant maps.

  To be precise, we fix a maximal compact subgroup $K \subset G$; a Harish-Chandra module here means a $(\mathfrak{g}, K)$-module which is finitely generated over $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and locally $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-finite (i.e. each vector lies in a $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-stable finite-dimensional subspace).

[1] On Zoom.

Starting from a Casselman–Wallach representation π^∞, let π be the $(\mathfrak{g}, K)$-module of its K-finite vectors. Then $\mathrm{Hom}_H(\pi^\infty, \mathbb{C}) \subset \mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C})$. The conjecture of *automatic continuity* asserts that equalities holds under further assumptions on $\mathbf{H}$. We do not need any results of this type in this talk.

For non-Archimedean F, Dipendra Prasad [11] proposed to study the Ext-analogues of distinction. In our framework, the problem is thus to study various $\mathrm{Ext}^i_H(\pi, \mathbb{C})$ and the Euler–Poincaré characteristic

$$\mathrm{EP}_H(\pi, \mathbb{C}) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_H(\pi, \mathbb{C}),$$

which is expected to be a more manageable quantity. provided that the sum is finite and Ext^i_H is finite-dimensional for all i.

What is the Archimedean version of this? In order to do homological algebra, it seems easier to work with the algebraic version and study

$$\mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C}), \quad \mathrm{EP}_{\mathfrak{h}}(\pi, \mathbb{C}) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C})$$

where π is a Harish-Chandra module and $\mathrm{Ext}^i_{\mathfrak{h}}$ is calculated in $\mathfrak{h}\text{-}\mathsf{Mod}$.

We will focus on the case of Harish-Chandra modules in what follows. The problem so formulated being *algebraic*, one can replace $(\mathfrak{g}, K)$ by its complexification $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$.

2 Geometric setting

Let us change the notations somehow. Let $\mathbf{G}$ be a connected reductive $\mathbb{C}$-group, and let $\mathbf{K}$ be the fixed locus of a Cartan involution. Let's identify $\mathbf{G} \supset \mathbf{K}$ with their groups of $\mathbb{C}$-points $G \supset K$ and so forth, for typographic reasons.

Consider an algebraic homogeneous G-space X, with G acting on the right. Assume that

- ⋆ X is affine,
- ⋆ X is spherical, i.e. there exists an open Borel orbit.

We fix $x \in X$ so that $\mathrm{Stab}_G(x) = H$ and $H\backslash G \xrightarrow{\sim} X$. Let

$$\mathcal{O}_x \subset \mathcal{O}_x^{\mathrm{an}} \subset \widehat{\mathcal{O}_x}$$

be the local ring at x, its analytic avatar, and its formal completion, respectively. In what follows, $(\cdots)^{\mathrm{an}}$ will always denote the evident analytification functor, and $(\cdots)_x$ denote the stalk at x of quasi-coherent sheaves on X.

Let D_X be the ring of algebraic differential operators on X. There are also sheaf-versions $\mathcal{D}_X$, $\mathcal{D}_{X^{\mathrm{an}}}$ (the complex-analytic differential operators), and the stalks $D_{X,x}$, $D^{\mathrm{an}}_{X,x}$, etc.

Note that $D_{X,x}$ acts on $\mathcal{O}_x$, $\mathcal{O}_x^{\mathrm{an}}$, $\widehat{\mathcal{O}_x}$, whilst $D^{\mathrm{an}}_{X,x}$ acts only on $\mathcal{O}_x^{\mathrm{an}}$ and $\widehat{\mathcal{O}_x}$.

Also, G acts on the left of D_X by transport of structure, namely $P \xmapsto{g \in G} gPg^{-1}$ for every algebraic differential operator P on X.

The algebraic G-action induces a homomorphism of algebras $\mathcal{U}(\mathfrak{g}) \to D_X$.

3 Localization functor: degree zero

Keep the same conventions on $\mathfrak{g}$, K, etc.

Proposition 3.1. *Let π be a $(\mathfrak{g}, K)$-module. There are canonical isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C}) &\simeq \mathrm{Hom}_{D_X}\left(D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi,\ \widehat{\mathcal{O}_x}\right) \\ &\simeq \mathrm{Hom}_{D_{X,x}}\left(D_{X,x} \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi,\ \widehat{\mathcal{O}_x}\right) \\ &\simeq \mathrm{Hom}_{D^{\mathrm{an}}_{X,x}}\left(D^{\mathrm{an}}_{X,x} \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi,\ \widehat{\mathcal{O}_x}\right)\end{aligned}$$

of $\mathbb{C}$-vector spaces.

Proof. This is based on $H\backslash G \xrightarrow{\sim} X$ and the well-known isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}\left(\pi, \widehat{\mathcal{O}_x}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C}),$$

see for instance the proof of [8, Lemma 10.1]. Perform a change-of-ring via

$$\mathcal{U}(\mathfrak{g}) \to D_X \to D_{X,x} \to D^{\mathrm{an}}_{X,x}$$

to get the remaining Hom spaces. □

Here we encountered the *localization functor*

$$\begin{aligned}\mathrm{Loc}_X : \mathcal{U}(\mathfrak{g})\text{-}\mathsf{Mod} &\to D_X\text{-}\mathsf{Mod} \\ \pi &\mapsto D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi.\end{aligned}$$

It preserves free (resp. projective, finitely generated) objects. Also, $D^{\mathrm{an}}_{X,x} \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi \simeq (\mathrm{Loc}_X(\pi))^{\mathrm{an}}_x$.

Theorem 3.2. *Let π be a Harish-Chandra module. Then $\mathrm{Loc}_X(\pi)$ is regular holonomic. Consequently $\mathrm{Loc}_X(\pi)^{\mathrm{an}}$ is regular holonomic as well (see [6, Theorem 6.1.12]).*

Notice that the holonomicity can be proved by the arguments from [1].

What does it mean for a D^{an}_X-module M to be regular holonomic? A necessary and sufficient condition (see [6, Remark 7.3.2] or [4, Definition 5.3.1]) is that

$$\mathrm{Hom}_{D^{\mathrm{an}}_{X,x}}(M, \mathcal{O}^{\mathrm{an}}_x) \to \mathrm{Hom}_{D^{\mathrm{an}}_{X,x}}(M, \widehat{\mathcal{O}_x}) \tag{3.1}$$

is an isomorphism for every x. In other words, every formal solution of the corresponding differential system converges locally, at every point. Since analytification preserves regularity. We obtain the following

Corollary 3.3. *Let π be a Harish-Chandra module. Then there is a canonical isomorphism $\mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C}) \simeq \mathrm{Hom}_{D^{\mathrm{an}}_{X,x}}(\mathrm{Loc}_X(\pi)^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}}_x)$. In particular, $\mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C})$ is finite-dimensional.*

Proof. The finite-dimensionality of $\mathrm{Hom}_{D^{\mathrm{an}}_{X,x}}(M, \mathcal{O}^{\mathrm{an}}_x)$ is known (Kashiwara's result, [6, §4.6]) for holonomic M. □

Concerning the finiteness of $\dim \mathrm{Hom}_{\mathfrak{h}}(\pi, \mathbb{C})$, a simpler and broader result is obtained in [2, Theorem 4.1.1]. However, it seems possible to access the higher Ext-analogues and Euler–Poincaré characteristics via D-modules. We will return to this point later on.

4 Regularity

The proof of Theorem 3.2 is based on the following machinery from [8]. Let G, X be as before.

Theorem 4.1 (Special case of [8, Corollary 5.7])**.** *Let M be a D_X-module. Suppose that*

- ⋆ *M is finitely generated,*
- ⋆ *M is locally $\mathcal{Z}(\mathfrak{g})$-finite,*
- ⋆ *M is equipped with a structure of K-equivariant D_X-module.*

Then M is regular holonomic.

The proof given in *loc. cit.* is not very original: it is somehow a "replay" of Ginsburg's work [5]. Note that the K-equivariance of D_X-modules is understood in the *strong* sense: roughly speaking, M is endowed with a K-action, so that

- ⋆ the multiplication map $D_X \otimes M \to M$ is K-equivariant,
- ⋆ the derivative of K-action coincides with that coming from $\mathcal{U}(\mathfrak{k}) \hookrightarrow \mathcal{U}(\mathfrak{g}) \to D_X$.

Lemma 4.2. *Let M be a D_X-module that is generated by a finite-dimensional $\mathbb{C}$-vector subspace M_0, such that M_0 is $\mathcal{Z}(\mathfrak{g})$-stable. Then M is locally $\mathcal{Z}(\mathfrak{g})$-finite.*

Proof. This is implicit in [8, Remark 3.3] without proof. It turns out to be tricky when I was preparing for this talk, so a proof is given here.

Let $M^\flat \subset M$ be the $\mathfrak{g}$-submodule generated by M_0, so $M^\flat$ is $\mathcal{Z}(\mathfrak{g})$-finite. Recall that G acts on D_X by $P \overset{k}{\mapsto} gPg^{-1}$; its derivative is $P \overset{\xi\in\mathfrak{g}}{\longmapsto} [\xi, P]$. The latter action makes D_X into a $\mathfrak{g}$-module. Now consider the surjection

$$D_X \underset{\mathbb{C}}{\otimes} M^\flat \to M, \quad P \otimes m \mapsto m.$$

It is a map of $\mathfrak{g}$-modules if we let $\xi \in \mathfrak{g}$ act on the left by $P \otimes m \mapsto [\xi, P] \otimes m + P \otimes (\xi \cdot m)$, since

$$\xi \cdot (Pm) = [\xi, P] \cdot m + P \cdot (\xi m) \tag{4.1}$$

for all $m \in M$, $\xi \in \mathfrak{g}$ and $P \in D_X$.

By the algebraic nature of the G-action on D_X, we see that D_X is a locally finite $\mathcal{U}(\mathfrak{g})$-module. Hence it remains to show that $F \underset{\mathbb{C}}{\otimes} M^\flat$ is $\mathcal{Z}(\mathfrak{g})$-finite for every finite-dimensional $\mathfrak{g}$-module F, where $\xi \in \mathfrak{g}$ acts by $f \otimes m \mapsto \xi f \otimes m + f \otimes \xi m$. This reduces to Kostant's Theorem, see [7, Theorem 7.133]. □

Proof of Theorem 3.2. We are going to check the conditions in Theorem 4.1. First, since π is finitely generated, so is the D_X-module $D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi$.

As for the local $\mathcal{Z}(\mathfrak{g})$-finiteness, note that π is generated over $\mathcal{U}(\mathfrak{g})$ by a finite-dimensional $\mathcal{Z}(\mathfrak{g})$-stable subspace, and apply Lemma 4.2.

Letting K act diagonally on $D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} \pi$ gives the K-equivariant structure; here we use some analogue of (4.1) to check the compatibility between $\mathfrak{k}$-actions. □

More generally, we will be interested in the subcategory $\mathsf{D}^b_{\mathrm{rh}}(D_X\text{-}\mathsf{Mod})$ (resp. $\mathsf{D}^b_{\mathrm{h}}(D_X\text{-}\mathsf{Mod})$) of $\mathsf{D}^b(D_X\text{-}\mathsf{Mod})$ consisting of complexes with regular holonomic (resp. holonomic) cohomologies.

Definition 4.3 (Solution complex)**.** For every $C \in \mathsf{D}^b_{\mathrm{h}}(D_X\text{-}\mathsf{Mod})$, let

$$\mathrm{Sol}_X(C) := \mathrm{R}\mathcal{H}om_{D_X^{\mathrm{an}}}(C^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}).$$

After Kashiwara [6, §4.6], we know that $\mathrm{Sol}_X(C)[\dim X]$ is a constructible sheaf on X; in fact it is perverse. Thus $\mathrm{Sol}_X(C)_x \simeq \mathrm{RHom}_{D_{X,x}^{\mathrm{an}}}\left(C^{\mathrm{an}}, \mathcal{O}_{X,x}^{\mathrm{an}}\right)$ has finite-dimensional cohomologies, for each $x \in X$.

5 Speculations on Ext-analogues

Let π be a $(\mathfrak{g}, K)$-module. Take any projective resolution of $\mathfrak{g}$-modules

$$\cdots \to P^{-1} \to P^0 \to \pi.$$

Note that each P^{-i} is also projective as a $\mathfrak{h}$-module, since projective modules are direct summands of free ones, and $\mathcal{U}(\mathfrak{g})$ is free as a left (resp. right) $\mathcal{U}(\mathfrak{h})$-module by the Poincaré–Birkhoff–Witt Theorem. Therefore, for each $i \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C}) &\simeq \mathrm{H}^i \operatorname{Hom}_{\mathfrak{h}}\left(P^\bullet, \mathbb{C}\right) \\ &\simeq \mathrm{H}^i \operatorname{Hom}_{D_{X,x}}\left(D_{X,x} \underset{\mathcal{U}(\mathfrak{g})}{\otimes} P^\bullet,\ \widehat{\mathcal{O}_x} \right) \end{aligned}$$

by Proposition 3.1. Note that $D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} P^\bullet$ is a complex of projective D_X-modules representing $D_X \underset{\mathcal{U}(\mathfrak{g})}{\overset{\mathsf{L}}{\otimes}} \pi$ in $\mathsf{D}^b(D_X\text{-}\mathsf{Mod})$. Ditto after replacing D_X by its stalk $D_{X,x}$.

Consider the derived localization functor

$$\begin{aligned} \mathbf{Loc}_X : \mathsf{D}^b(\mathfrak{g}\text{-}\mathsf{Mod}) &\to \mathsf{D}^b\left(D_X\text{-}\mathsf{Mod}\right) \\ C &\mapsto D_X \underset{\mathcal{U}(\mathfrak{g})}{\overset{\mathsf{L}}{\otimes}} C. \end{aligned}$$

It does preserve boundedness, by using the bar (or Koszul, standard) resolution of $\mathfrak{g}$-modules.

Now we make the *key assumption* of this talk.

Assumption 5.1. Let π be a given Harish-Chandra module. We have

$$\mathbf{Loc}_X(\pi) \in \mathsf{D}^b_{\mathrm{rh}}(D_X\text{-}\mathsf{Mod}).$$

The regularity at H^0 is just Theorem 3.2.

Proposition 5.2. *Let π be a Harish-Chandra module. Under our key assumption, we have $\mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C}) \simeq \mathrm{H}^i\left(\mathrm{Sol}_X\left(\mathbf{Loc}_X(\pi)\right)_x\right)$ for each i. In particular, $\mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C})$ is finite-dimensional.*

Proof. Use the extension of (3.1) in which M is taken in $\mathsf{D}^b_{\mathrm{rh}}(D_X\text{-}\mathsf{Mod})$, and Hom is replaced by RHom; see for example [4, Definition 5.3.1]. □

The finite-dimensionality of $\mathrm{Ext}^i_{\mathfrak{h}}(\pi, \mathbb{C})$ is also proved in [2, Proposition 4.2.2] when $\mathfrak{h}$ and $\mathfrak{k}$ are in good relative position. In contrast, there is no condition on $x \in X$ in the result above.

Corollary 5.3. *Let π be a Harish-Chandra module. Under our key assumption,*

$$\begin{aligned} \mathrm{EP}_{\mathfrak{h}}(\pi, \mathbb{C}) &= \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{H}^i\left(\mathrm{Sol}_X\left(\mathbf{Loc}_X(\pi)\right)_x\right) \\ &=: \mathrm{EP}\left(\mathrm{Sol}_X\left(\mathbf{Loc}_X(\pi)\right)\right)(x) \end{aligned}$$

and the right-hand side can be expressed in terms of Kashiwara's Local Index Theorem [6, Theorem 4.6.7].

Let $C \in \mathsf{D}^b_{\mathrm{rh}}(D_X\text{-}\mathsf{Mod})$. Briefly speaking, the Local Index Theorem expresses $\mathrm{EP}\left(\mathrm{Sol}_X(C)\right)$, as a function on X, in terms of the *Euler obstructions* of the *characteristic cycle* of C, the latter one being a Lagrangian conic cycle in T^*X. Thus far, I have absolutely no idea about explicit computations of these geometric/topological quantities.

6 On the Key Assumption (in progress...)

Let π be a Harish-Chandra module. We hope to verify the premises in Theorem 4.1 for the cohomologies of $\mathbf{Loc}_X(\pi)$, modulo a conjecture.

Finite generation Since π is finitely generated over the Noetherian ring $\mathcal{U}(\mathfrak{g})$, it has a resolution by free modules of finite rank. Hence $\mathrm{Loc}_X(\pi)$ is represented by a complex of free D_X-modules of finite rank. Now recall that D_X is Noetherian.

Local $\mathcal{Z}(\mathfrak{g})$-finiteness This is currently unclear to me. Let's be cautiously optimistic[2]:

Conjecture 6.1. The cohomologies of $\mathbf{Loc}_X(\pi)$ are locally $\mathcal{Z}(\mathfrak{g})$-finite D_X-modules.

***K*-equivariance** This is troublesome for the following reason.

- ⋆ We defined $\mathbf{Loc}_X(\pi)$ using the $\mathfrak{g}$-module structure only. Where is K?
- ⋆ If we take resolutions of π as a $(\mathfrak{g}, K)$-module instead (see [7] for the general theory), we run into a bigger trouble because projective $(\mathfrak{g}, K)$-modules are rarely projective over $\mathfrak{g}$.

As a workaround for the aforementioned obstacle, we consider the *h-complexes* in the sense of [3] or [9, 10]. Since the precise formulas can be found in *loc. cit.*, I will only give a sketch. The h-complexes for $(\mathfrak{g}, K)$ are complexes $C^\bullet$ of weak $(\mathfrak{g}, K)$-modules (i.e. no compatibility imposed between the $\mathfrak{k}$-actions coming from K and $\mathfrak{g}$), together with a $i : \mathfrak{k} \to \mathrm{Hom}^{-1}_{\mathfrak{g}}(C^\bullet, C^\bullet)$ giving homotopies between the two $\mathfrak{k}$-actions, plus some other compatibilities. Hence the cohomologies of h-complexes are $(\mathfrak{g}, K)$-modules.

One can do homological algebra with h-complexes, define the $\oplus$ and $\underset{\mathbb{C}}{\otimes}$ of h-complexes, and so on. For example, if $A^\bullet$ and $B^\bullet$ are h-complexes with homotopies i^A, i^B, one endow $(A \underset{\mathbb{C}}{\otimes} B)^\bullet$ (which is obviously a complex of weak $(\mathfrak{g}, K)$-modules) with the homotopies

$$i_\xi^{A\otimes B}(u \otimes v) = i_\xi^A u \otimes v + (-1)^{\deg u} u \otimes i_\xi^B v, \quad \xi \in \mathfrak{k}.$$

A key example is the bar complex $N\mathfrak{g} := \mathcal{U}(\mathfrak{g}) \underset{\mathbb{C}}{\otimes} \bigwedge^{-\bullet} \mathfrak{g}$, with

- ⋆ $\mathfrak{g}$ acting only on $\mathcal{U}(\mathfrak{g})$,
- ⋆ K acting diagonally.

For any $\xi \in \mathfrak{k}$, $u \in \mathcal{U}(\mathfrak{g})$ and $\lambda \in \bigwedge^{-i} \mathfrak{g}$, set

$$i_\xi(u \otimes \lambda) = -u \otimes (\xi \wedge \lambda).$$

This turns out to be an h-complex. For any $(\mathfrak{g}, K)$-module π, the h-complex $P^\bullet := N\mathfrak{g} \underset{\mathbb{C}}{\otimes} \pi$ is a projective resolution of π in $\mathfrak{g}$-Mod. By a K-equivariant change-of-rings via $\mathcal{U}(\mathfrak{g}) \to D_X$, we shall get a similar structure after taking $D_X \underset{\mathcal{U}(\mathfrak{g})}{\otimes} P^\bullet$. Granting this fact, the cohomologies of $\mathbf{Loc}_X(\pi)$ should carry canonical K-equivariant structures.

References

[1] Avraham Aizenbud, Dmitry Gourevitch, and Andrei Minchenko. "Holonomicity of relative characters and applications to multiplicity bounds for spherical pairs". In: *Selecta Math. (N.S.)* 22.4 (2016), pp. 2325–2345. ISSN: 1022-1824 (cit. on p. 3).

[2] The arguments sketched during the talk contain a gap.

[2] Avraham Aizenbud et al. "Hausdorffness for Lie algebra homology of Schwartz spaces and applications to the comparison conjecture". In: *Math. Z.* 283.3-4 (2016), pp. 979–992. ISSN: 0025-5874 (cit. on pp. 3, 5).

[3] Joseph Bernstein and Valery Lunts. "Localization for derived categories of $(\mathfrak{g}, K)$-modules". In: *J. Amer. Math. Soc.* 8.4 (1995), pp. 819–856. ISSN: 0894-0347 (cit. on p. 6).

[4] Jan-Erik Björk. *Analytic D-modules and applications*. Vol. 247. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1993, pp. xiv+581. ISBN: 0-7923-2114-6 (cit. on pp. 3, 5).

[5] Victor Ginsburg. "Admissible modules on a symmetric space". In: *Astérisque* 173-174 (1989). Orbites unipotentes et représentations, III, pp. 9–10, 199–255. ISSN: 0303-1179 (cit. on p. 4).

[6] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*. Vol. 236. Progress in Mathematics. Translated from the 1995 Japanese edition by Takeuchi. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. xii+407. ISBN: 978-0-8176-4363-8 (cit. on pp. 3–5).

[7] Anthony W. Knapp and David A. Vogan Jr. *Cohomological induction and unitary representations*. Vol. 45. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1995, pp. xx+948. ISBN: 0-691-03756-6 (cit. on pp. 4, 6).

[8] Wen-Wei Li. *On the regularity of D-modules generated by relative characters*. 2019. arXiv: 1905.08135 [math.RT] (cit. on pp. 3, 4).

[9] Pavle Pandžić. "A simple proof of Bernstein-Lunts equivalence". In: *Manuscripta Math.* 118.1 (2005), pp. 71–84. ISSN: 0025-2611 (cit. on p. 6).

[10] Pavle Pandžić. "Zuckerman functors between equivariant derived categories". In: *Trans. Amer. Math. Soc.* 359.5 (2007), pp. 2191–2220. ISSN: 0002-9947 (cit. on p. 6).

[11] Dipendra Prasad. *Ext-analogues of Branching laws*. 2013. arXiv: 1306.2729 [math.NT] (cit. on p. 2).

Wen-Wei Li
Beijing International Center for Mathematical Research / School of Mathematical Sciences, Peking University. No. 5 Yiheyuan Road, Beijing 100871, People's Republic of China.
E-mail address: wwli@bicmr.pku.edu.cn

Updated on August 1, 2020.