On the elliptic part of the trace formula for $\widetilde{\mathrm{Sp}}(2n)$

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Arthur-Selberg trace formula

Start with the case of linear reductive groups.

- F: number field, $\mathbb{A}$ its ring of adèles,
- G: a connected reductive group over F,
- $G(\mathbb{A})^1 := \mathrm{Ker}(H_G)$ where $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ is the Harish-Chandra homomorphism, $G(\mathbb{A}) = G(\mathbb{A})^1$ whenever G is semisimple,
- R: right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A})^1)$,
- $f \in C_c^\infty(G(\mathbb{A}))$, define the operator $R(f)$;
- K_G: the kernel of $R(f)$, $k(x) := K_G(x, x)$ for $x \in G(\mathbb{A})$.

The Arthur-Selberg trace formula calculates a truncated integral of $k(x)$ over $G(F)\backslash G(\mathbb{A})^1$:

$$(\text{geometric expansion}) = J(f) = (\text{spectral expansion}).$$

Roughly speaking:

- the geometric side: distributions on $G(\mathbb{A})^1$ indexed by rational conjugacy classes (eg. orbital integrals),
- the spectral side: distributions on $G(\mathbb{A})^1$ indexed by automorphic representations (eg. characters).

Some of the applications

- Base change and Jacquet-Langlands correspondence for $\mathrm{GL}(n)$.
- Endoscopic classification of representations of classical groups.
- Formula for the trace of Hecke operators.
- Various results in local harmonic analysis (character identities, etc.)

Bottleneck 1 : invariant trace formula

Bottleneck 2 : stabilization

Elliptic terms on the geometric side

$$T_{\mathrm{ell}}(f) = \sum_{\delta:\text{ell.ss./conj}} \tau(G_\delta) J_{G_\delta}(\delta, f)$$

where

- $G_\delta := Z_G(\delta)^\circ$,
- $\tau(\cdot)$: the Tamagawa number,
- $J_{G_\delta}(\delta, f) := \int_{G_\delta(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\delta x)dx$ (the orbital integrals), using Tamagawa measures.

Note: Ellipticity guarantees absolute convergence. One can use other Haar measures as well.

Idea of stabilization: endoscopy

Express $J_{\mathrm{ell}}(f)$ in terms of stable orbital integrals $S_H(\gamma, \cdot) = \int_{(H_\gamma\backslash H)(\mathbb{A})} \cdots$ on the endoscopic groups H attached to elliptic endoscopic data.

Need (a) the transfer $f \mapsto f^H$ of test functions to the elliptic endoscopic groups, by matching reg. ss. orbital integrals; (b) the fundamental lemma.

Desideratum

$$T_{\mathrm{ell}}(f) = \sum_H \iota(G,H) ST^H_{\mathrm{ell,equi}}(f^H)$$

where

$$ST^H_{\mathrm{ell,equi}}(f^H) := \tau(H) \sum_\gamma [Z_H(\gamma) : Z_H(\gamma)^\circ]^{-1} S_H(\gamma, f^H),$$

$\iota(G,H)$ is an explicit constant and γ ranges over the ell. ss. classes in $H(F)$ that are equisingular with G.

Here: Stable conjugacy = conjugacy classes over $\bar{F}$.
Mixture of arithmetic and harmonic analysis!

History of stabilization

Standard case

- Langlands (1982): the elliptic regular terms.
- Kottwitz (1986): the elliptic terms, assuming the transfer of equisingular semisimple orbital integrals.
- Langlands+Shelstad (1990): Regular transfer implies equisingular transfer.
- Arthur (2002-): Full stabilization, assuming the weighted fundamental lemma.

Twisted case (cf. Mezo's talk)

- Kottwitz+Shelstad (1999): the elliptic regular terms.
- Labesse (2003): the elliptic terms, assuming the transfer of equisingular twisted semisimple orbital integrals.
- Mœglin+Waldspurger (ongoing): full stabilization.

Metaplectic groups of Weil

F: local field, $\mathrm{char}(F) = 0$. $\psi : F \to U(1)$ nontrivial character.

- The metaplectic covering is a topological central extension

$$1 \to \{1, \epsilon\} \to \widetilde{\mathsf{Sp}}(2n) \xrightarrow{\mathbf{p}} \mathsf{Sp}(2n) \to 1.$$

  It is nonalgebraic when $F \neq \mathbf{C}$.
- The Weil representation of $\widetilde{\mathsf{Sp}}(2n)$: $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where $\omega_\psi^\pm$ are distinct unitary irreps.
- Study the irreps π of $\widetilde{\mathsf{Sp}}(2n)$ which are *genuine*, i.e. $\pi(\epsilon) = -\mathrm{id}$. Eg. $\omega_\psi^\pm$.
- Test functions: *antigenuine* $f \in C_c^\infty(\widetilde{\mathsf{Sp}}(2n))$, i.e. $f(\epsilon \cdot) = -f(\cdot)$.

The adélic case

F a number field, $\mathbb{A} = \mathbb{A}_F$, $\psi : \mathbb{A}/F \to U(1)$ a nontrivial character.

- Construct the adélic $\widetilde{\mathrm{Sp}}(2n, \mathbb{A})$ and ω_ψ as before.
- The same notions of genuine/antigenuine objects.
- We can show that

$$\widetilde{\mathrm{Sp}}(2n, \mathbb{A}) = \left({\prod_v}' \widetilde{\mathrm{Sp}}(2n, F_v) \right) / \mathbf{N},$$

$$\mathbf{N} := \left\{ (\epsilon_v)_v \in \bigoplus_v \{\pm 1\} : \prod_v \epsilon_v = 1 \right\}.$$

Relevance of $\widetilde{\mathrm{Sp}}(2n)$: Howe correspondence, modular forms of half-integral weights, ϑ functions, etc.

Goal

Study the genuine representations of $\widetilde{\mathsf{Sp}}(2n)$ (local) / the genuine automorphic spectrum of $\widetilde{\mathsf{Sp}}(2n)$ (global).

Motto: $\widetilde{\mathsf{Sp}}(2n)^\vee$ should be $\mathsf{Sp}(2n, \mathbb{C})$, i.e. $\underbrace{\mathsf{SO}(2n+1)}_{\text{split}}{}^\vee$.

Road map

- Endoscopy of the nonalgebraic group $\widetilde{\mathsf{Sp}}(2n)$.
- Invariant trace formula for $\widetilde{\mathsf{Sp}}(2n, \mathbb{A})$.
- Stabilization for the elliptic terms.
- Relate with the Howe correspondence for the dual pair $(Sp(2n), O(V))$ with $\dim V = 2n+1$.
- Full stabilization (long-term goal)

Endoscopic data

Set $\tilde{G} := \widetilde{\mathrm{Sp}}(2n)$, $G := \mathrm{Sp}(2n)$.

Definition

The elliptic endoscopic data of $\tilde{G}$ are the pairs (n', n'') with $n', n'' \in \mathbb{Z}_{\geq 0}$ such that $n' + n'' = n$. To a pair (n', n''), the associated endoscopic group is

$$H := \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1)$$

(the split one).

Can define (following Adams and Renard):

- a correspondence of semisimple classes via eigenvalues (up to a twist by -1) between G and H;
- a notion of stable conjugacy for regular ss. elements;
- transfer factors Δ for regular ss. classes.

Old results

- Endoscopic character identities for F local: Adams (1998), Renarnd (1999) for F archimedean, and Schultz (1998) for $n = 1$, F non-archimedean.
- Transfer of antigenuine orbital integrals from G to H: Renard for $F = \mathbb{R}$ (1999), `wwli` for all local F (2011). Denote this by $f \mapsto f^H$.
- Fundamental lemma: `wwli` (2011).
- Weighted fundamental lemma: `wwli` (2012).
- Invariant trace formula à la Arthur for certain covering groups: `wwli` (2012-).

The next step: stabilize the elliptic part of the geometric side of the trace formula.

Stabilization

In the global setting, define the adélic $\tilde{G}$, and $T_{\text{ell}}(f)$ for antigenuine f on $\tilde{G}$.

Theorem (`wwli`, 2013)

$$T_{ell}(f) = \sum_{n', n''} \iota(\tilde{G}, H) ST^H_{ell,equi}(f^H)$$

where $H = SO(2n'+1) \times SO(2n''+1)$ and $\iota(\tilde{G}, H) = \tau(H)^{-1}$ (formulas for this are known), for an appropriate notion of equisingular matching of ss. conjugacy classes.

Idea of the proof

Two steps (after Labesse)

1. Pre-stabilization: orbital integrals $\to$ κ-orbital integrals. Based upon Galois cohomology. Input: stable conjugacy.
2. Transfer: κ-orbital integrals $\to$ stable orbital integrals on an appropriate H. This is a local problem. Input: transfer of orbital integrals $+$ properties of transfer factors.

The regular elliptic semisimple terms

The stabilization of regular terms are actually embodied in the required properties of the transfer factors (eg. the cocycle condition, product formula...) Just use the arguments of Langlands.

Extension to all elliptic semisimple terms

We follow Labesse's approach. Some extra efforts are needed.

1. Extend the notion of stable conjugacy on $\tilde{G}$ to all ss. elements.
2. Define the transfer factors Δ for such elements, and obtain the necessary properties.
3. Prove the equisingular transfer of orbital integrals.

Descent à la Langlands-Shelstad

In the local setting, we define the transfer factor for equisingular ss. elements on $H(F) \times \tilde{G}$ by reducing to the regular case. This is done by passage to the limit + descent of transfer factors.

The next step: obtain the equisingular transfer by a limiting process.

Equisingular matching of orbital integrals

Theorem (wwli, 2013)

Let F be local, f^H be a transfer of an antigenuine $f \in C_c^\infty(\tilde{G})$, then

$$\sum_{\delta} \Delta(\gamma, \tilde{\delta}) J_{\tilde{G}}(\tilde{\delta}, f) = |2|_F^{-t} S_H(\gamma, f^H)$$

for all equisingular $\delta \leftrightarrow \gamma$, where $\tilde{\delta} \mapsto \delta$ and $2t$ is the sum of the multiplicities of eigenvalue ± 1 of δ. The orbital integrals in sight are defined w.r.t. Gross' canonical Haar measures.

- The factor $|2|_F^{-t}$ does not appear in the classical case.
- Tamagawa measure (global) on a reductive group I can be factorized into Gross' measures (local), up to the Artin conductor of the Artin-Tate motive attached to I (Gross 1997).

Idea of proof (Langlands-Shelstad): use Harish-Chandra's limit formula (F archimedean) or Rogawski's formula (F non-archimedean) to pass from regular orbital integrals to singular ones. In the classical setting, the two sides are related by inner twists after a descent argument.
Hard core: In the metaplectic case, one must treat a comparison between $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ (eg. by taking H corresponding to $(n, 0)$ and $\tilde{\delta} = 1$, $\gamma = 1$). Comparison of limit formulas becomes somehow tricky. What makes the comparison work are the functional equations for Gross' measures and the

Fact

The groups $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ have the same Artin-Tate motive

$$\mathbb{Q}(-1) \oplus \cdots \oplus \mathbb{Q}(1-2n).$$

Some applications

1. Local endoscopic character relations. Perhaps some of Arthur's arguments in the endoscopic classification can be reused?
2. Relation with Howe correspondence (Gan).
3. Comparison of formal degrees and μ-functions between $\tilde{G}$ and $\mathrm{SO}(2n+1)$, cf. Gan-Ichino.
4. Towards the full stabilization: try the case $n = 1$. Cf. Hiraga-Ikeda for more general coverings for $\mathrm{Sp}(2) = \mathrm{SL}(2)$.

Thank you! [1]

[1] The picture on the title page is taken from the website of PRIMA 2013