Local Gan-Gross-Prasad Conjecture: Real Symplectic-metaplectic Cases

Zhe Li

January 9, 2024

Setup

- F: a local field;
- σ: an automorphism on F with $\sigma^2 = 1$;
- F_0: the fixed field of σ;
- (V, q_V): a vector space over F with a non-degenerate σ-sesquilinear form q_V;
- $(W, q_V|_W)$: a non-degenerated subspace of V;
- $G(V)$ (resp. $G(W)$): the identity component of the $\mathrm{Aut}(V, q_V)$ (resp. $\mathrm{Aut}(W, q_V|_W)$).

Remark

There are four distinct cases: orthogonal, symplectic, hermitian or skew-hermitian.

Local Restriction Problem

- H: a subgroup of the locally compact group $G = G(V) \times G(W)$ containing the diagonally embedded subgroup $G(W)$;
- ν: a unitary representation of H;
- The local restriction problem is to determine

$$m(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \widehat{\otimes} \overline{\nu}, \mathbb{C})$$

where π is an irreducible complex representation of G.

Local GGP conjecture

- $W_{\mathbb{R}}$: be the Weil group of $\mathbb{R}$;
- ${}^LG$: the Langlands dual of G;
- Langlands parameter φ: a homomorphism of $W_{\mathbb{R}}$ into ${}^LG$;
- $\Pi(\varphi)$: a finite set of irreducible representations of G associates to φ, which is called L-packet associated to φ;
- $\Pi(\varphi)$ is called a generic L-packet if it contains a generic representation of G.

Conjecture

For any generic L-packet $\Pi(\varphi)$ of G, there exists a unique representation $\pi \in \Pi(\varphi)$, such that $m(\pi) = 1$.

Bessel models

1. Moeglin and Waldspurger(2010,2012): p-adic special orthogonal groups;
2. Beuzart-Plessis(2016): extended their work to the case of Bessel models for p-adic unitary groups.
3. Beuzart-Plessis(2020): developed a new local trace formula to establish integral formula for the multiplicities, which includes both p-adic and real unitary groups;
4. H. He(2017): established the GGP-conjecture for discrete series packets for the unitary groups via theta correspondence;
5. H. Xue(2023): inspired by He's method, completely settled the conjecture for real unitary groups;
6. C. Chen and Z. Luo(2022): claimed a proof of the conjecture for real special orthogonal groups in a series of papers.

Fourier-Jacobi models

1. Gan and Ichino(2014): proved the GGP-conjecture in the case of Fourier-Jacobi models for p-adic unitary groups;
2. Xue(recently): proved the same result for real unitary groups;
3. Atobe(2018): proved the GGP-conjecture for p-adic symplectic-metaplectic groups.

Slogan

All of the above works rely on the theta correspondence, which connects the Bessel models and the Fourier-Jacobi models.

Generic L-packet

- Let $W \subset V$ be symplectic spaces over $\mathbb{R}$ of rank m and n respectively, $m \leq n$;
- Let
$$\varphi_1 : W_{\mathbb{R}} \to \mathrm{SO}(N), \varphi_2 : W_{\mathbb{R}} \to \mathrm{Sp}(M)$$
be generic Langlands parameters of $G(V)$ and the double cover $\widehat{G}(W)$ of $G(W)$ respectively, where N is a $(2n+1)$-dimensional orthogonal space and M is a $2m$-dimensional symplectic space;
- Suppose $\pi_1 \in \Pi(\varphi_1)$ and $\pi_2 \in \Pi(\varphi_2)$.

Langlands-Vogan parameter

- C_{φ_1}: the centralizer of the image of φ_1;
- $C_{\varphi_1}^+ = \{a \in C_{\varphi_1} : \det(a) = +1\}$;
- Component group $A_{\varphi_1} := C_{\varphi_1}/C_{\varphi_1,0}$, where $C_{\varphi_1,0}$ is the identity component of C_{φ_1}; similarly $A_{\varphi_1}^+ := C_{\varphi_1}^+/C_{\varphi_1,0}^+$, then $A_{\varphi_1}^+$ is the subgroup of A_{φ_1};
- Let (φ_1, η_1) be the Langlands-Vogan parameter of π_1, where η_1 is a character of the component group $A_{\varphi_1}^+$.
- For $\varphi_2 \in \Pi(\varphi_2)$, we have the same defintions;

- Put $N_1 = N \oplus \mathbb{C}$ and $\varphi_1' : W_{\mathbb{R}} \to \mathrm{SO}(N_1)$. View $A^+_{\varphi_1}$ as a subgroup of $A^+_{\varphi_1'}$;
- For $a \in A_{\varphi_2}$ and $b \in A^+_{\varphi_1'}$, define

$$\chi_{N_1}(a) = \epsilon(M^a \otimes N_1)\det(M^a)(-1)^{\dim_{\mathbb{C}}(N_1)/2}\det(N_1)(-1)^{\dim_{\mathbb{C}}(M^a)/2},$$

$$\chi_M(b) = \epsilon(M \otimes N_1^b)\det(M)(-1)^{\dim_{\mathbb{C}}(N_1^b)/2}\det(N_1^b)(-1)^{\dim_{\mathbb{C}}(M)/2},$$

where $M^a = \{m \in M : am = -m\}$, $N_1^b = \{n \in N_1 : bn = -n\}$ and ϵ stands for the local root numbers.

Theorem

Let the notation be as above. Then $m(\pi) \neq 0$ if and only if

$$\eta_1 \times \eta_2 = \chi_M \times \chi_{N_1}|_{A^+_{\varphi_1} \times A_{\varphi_2}}. \tag{1}$$

Reduction to basic case

Definition

An irreducible unitary representation is tempered if its matrix coefficients are almost square integrable.

- By making use the Schwartz homology theory, we can reduce to the basic case: $t = \mathrm{rk}(V) - \mathrm{rk}(W) = 0$ and π_1 and π_2 both being tempered.

Basic case: setting

- Let $W = V$. Let π_1 (resp. π_2) be an irreducible tempered (resp. tempered genuine) representation of $\mathrm{Sp}(V)$ (resp. $\widehat{\mathrm{Sp}}(V)$).
- Let $G = \mathrm{Sp}(V) \times \widehat{\mathrm{Sp}}(V)$, $J(V) = H(V) \rtimes \mathrm{Sp}(V)$, $G^J = \mathrm{Sp}(V) \times J(V)$, and $H = \mathrm{Sp}(V)$. The group H embeds diagonally into G^J.
- Let **S** be the mixed model of the Weil representation ω of the double cover of Jacobi group $\widehat{J}(V)$, realized on the space of Schwartz functions **S**.
- We put π for a representation $\pi_1 \widehat{\otimes} \pi_2$ of G, which is a finite length tempered representation of G, and $\pi^J = \pi \widehat{\otimes} \bar{\omega}$, which is an irreducible representation of G^J.

$\mathcal{L}$-integrals

- Let $\mathrm{End}(\pi^J)$ be the algebra of (continuous) endomorphisms of π^J, which has an action of $G^J \times G^J$ by left and right multiplication.
- Let $\mathrm{End}(\pi^J)^\infty$ be the smooth vectors in $\mathrm{End}(\pi^J)$, which is identified with $\pi^J \widehat{\otimes} \overline{\pi^J}$.
- We define

$$\mathcal{L}_{\pi^J} : \mathrm{End}(\pi^J)^\infty = \pi^J \widehat{\otimes} \overline{\pi^J} \to \mathbb{C};\ T \mapsto \mathcal{L}_{\pi^J}(T) = \int_H \mathrm{Trace}(\pi(h)T)dh.$$

Remark

We have the similar $\mathcal{L}$-integral for the GGP-pair consisting of special orthogonal groups.

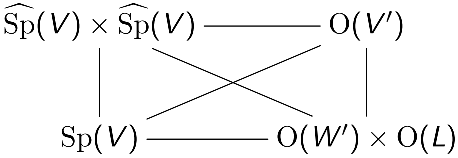
Theorem

Assume π is irreducible and tempered. Then $m(\pi) \neq 0$ if and only if $\mathcal{L}_{\pi^J} \neq 0$.

Remark

- We follow the strategy of Beuzart-Plessis for GGP pair of unitary groups to prove this theorem.
- This theorem is also true for GGP pair of special orthogonal groups by the work of Luo following the same strategy.

Consider the see-saw diagram:



where V' is a $(2n+2)$-dimensional orthogonal space and W' is a $(2n+1)$-dimensional orthogonal space with $V' = W' \oplus L$.

- Let $\pi_1' = \theta_{V,V'}(\pi_1)$ and $\pi_2' = \theta_{V,W'}(\pi_2)$ be the irreducible tempered representations of $\mathrm{SO}(V')$ and $\mathrm{SO}(W')$ via the theta correspondence for the dual pairs $(\mathrm{Sp}(V), \mathrm{O}(V'))$ and $(\mathrm{Sp}(V), \mathrm{O}(W'))$ respectively.
- Denote by $\pi' = \pi_1' \boxtimes \pi_2'$ the representation of $G' = \mathrm{SO}(V') \times \mathrm{SO}(W')$.
- For $S \in \mathrm{End}(\pi')$, define the integral

$$\mathcal{L}_{\pi'}(S) = \int_{\mathrm{SO}(W')} \mathrm{Trace}(\pi'(h)S)dh,$$

which is absolutely convergent.

Proposition

$\mathcal{L}_{\pi'} \neq 0$ if and only if $\mathcal{L}_{\pi^J} \neq 0$.

$$\begin{array}{ccccc}
\pi = \pi(\varphi, \eta) & \overset{?}{\Longleftrightarrow} & m(\pi) \neq 0 & \overset{(1)}{\Longleftrightarrow} & \mathcal{L}_{\pi^J} \neq 0 \\
\updownarrow (5)? & & & & \Updownarrow (2) \\
\pi' = \pi'(\varphi', \eta') & \overset{(4)}{\Longleftrightarrow} & m(\pi') \neq 0 & \overset{(3)}{\Longleftrightarrow} & \mathcal{L}_{\pi'} \neq 0
\end{array}$$

- ? is the local GGP conjecture we want to prove;
- the equivalences (1)(2) have been explained previously;
- (3) is proved by Luo;
- (4) is proved by Chen and Luo;
- We only need to study "(5)?": construct the explicit formula of Langlands-Vogan parameters between π and π' via theta correspondence.

- Let $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{R}) \otimes \mathbb{C}$;
- Let σ be the antiholomorphic involutive automorphism such that $\mathbf{G}^\sigma = \mathrm{Sp}_{2n}(\mathbb{R})$.

Definition

A Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$ is called fundamental if the following conditions are satisfied:
(i) $T_* = \mathbf{T}_*^\sigma$ is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$;
(ii) The set of roots of $\mathbf{T}_*$ in $\mathbf{B}_*$ is stable under $-\sigma$.
Moreover, a fundamental Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$ is called of Whittaker type if all the imaginary simple roots of $\mathbf{T}_*$ in $\mathbf{B}_*$ are non-compact.

Remark

$\mathbf{G}$ always has a fundental Borel pair of Whittaker type.

- The Levi factor of a parabolic subgroup P of $\mathrm{Sp}_{2n}(\mathbb{R})$ is isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^a \times \mathrm{GL}_2(\mathbb{R})^b \times \mathrm{Sp}_{2(n-a-2b)}(\mathbb{R});$$

- Let φ_0 be a discrete series Langlands parameter of $\mathrm{Sp}_{2(n-a-2b)}(\mathbb{R})$;
- Let $\Pi(\varphi_0)$ be the finite set of irreducible limit of discrete series representations of $\mathrm{Sp}_{2(n-a-2b)}(\mathbb{R})$;
- Taking Langlands quotients of parabolic inductions gives a bijection between $\Pi(\varphi)$ and $\Pi(\varphi_0)$.

Remark

It is enough to treat the limit of discrete series representations.

Theta correspondence using Harish-Chandra parameters

- Fix a fundamental Borel pair of Whittaker type $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$;
- Let $(X, \Phi, \Delta_*, X^\vee, \Phi^\vee, \Delta_*^\vee)$ be based root system of $\mathbf{G}$ associated to $(\mathbf{B}_*, \mathbf{T}_*)$;
- Let Ψ_* be the set of positive roots generated by Δ_* and $\Psi_{c,*}$ its subset of compact positive roots;
- Let $\mathfrak{t}_0$ be the Lie algebra of T_* and $\mathfrak{t}$ be its complexification;
- Let $\mathfrak{t}_0^*$ and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}_0$ and $\mathfrak{t}$ respectively;
- $\lambda \in \mathfrak{t}^*$ is called regular if $\langle \lambda, \alpha \rangle$ is a non-zero real number for all the imaginary root α of $\mathfrak{t}$ in $\mathfrak{sp}_{2n}(\mathbb{C})$.

Definition

1. A limit of discrete series Harish-Chandar parameter of $\mathrm{Sp}_{2n}(\mathbb{R})$ is an integral element $\lambda_d \in i\mathfrak{t}_0^* \subset \mathfrak{t}^*$. In particular, if λ_d is regular, then we say λ_d is a discrete series Harish-Chandra parameter.
2. A limit of discrete series Harish-Chandra parameter is a pair (λ_d, Ψ), where λ_d is a limit of discrete series Harish-Chandra paramter of $\mathrm{Sp}_{2n}(\mathbb{R})$, and $\Psi \subset \Phi$ is the corresponding set of positive roots satisfying:
    1. $\Psi_{c,*} \subset \Psi$;
    2. λ_d is dominant with respect to Ψ;
    3. if a simple root $\alpha \in \Psi$ satisfies $\langle \lambda_d, \alpha \rangle = 0$, then α is non-compact.

Theorem (Moeglin, Paul)

Let π_1 be a limit of discrete series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ and (λ, Ψ) be the Harish-Chandra parameter of π, where

$$\lambda = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_k}, \underbrace{0, \cdots, 0}_{z}, \underbrace{-\lambda_k, \cdots, -\lambda_k}_{q_k}, \cdots, \underbrace{-\lambda_1, \cdots, -\lambda_1}_{q_1}).$$

Let $w = [\frac{z}{2}]$, $p_0 = \sum_{i=1}^k p_i + w$ and $q_0 = \sum_{i=1}^k q_i + w$. There are exactly four pairs of integers (p, q) with $p + q = 2n$ or $2n + 2$ such that $\theta_{p,q}(\pi)$ is a non-zero limit of discrete series representation of $O(p, q)$.

(1) $z = 2w$: $\theta_{2p_0,2q_0}(\pi) \neq 0$ with the Harish-Chandra parameter $(\lambda_{0,0}, 1, \Psi_{0,0})$, where

$$\begin{aligned}\lambda_{0,0} = (&\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_k}, \underbrace{0, \cdots, 0}_{w}, \\ &\underbrace{\lambda_1, \cdots, \lambda_1}_{q_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{q_k}, \underbrace{0, \cdots, 0}_{w}),\end{aligned} \tag{2}$$

and $\Psi_{0,0}$ is obtained from Ψ as follows: for $1 \leq i \leq p_0$ and $1 \leq j \leq q_0$, the root $e_i - f_j \in \Psi_{0,0}$ if and only if $e_i + e_{n-j+1} \in \Psi$. (This determines $\Psi_{0,0}$ completely.)

(2) $z = 2w > 0$:

- If $e_{k+1} + e_{k+z} \in \Psi$, $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ with the parameter $(\lambda_{2,0}, 1, \Psi_{2,0})$, where $\lambda_{2,0}$ is obtained from $\lambda_{0,0}$ by adding a zero on the left and $\Psi_{0,0} \subset \Psi_{2,0}$.
- If $-e_{k+1} - e_{k+z} \in \Psi$, $\theta_{2p_0,2q_0+2}(\pi) \neq 0$ with the parameter $(\lambda_{0,2}, 1, \Psi_{0,2})$, where $\lambda_{0,2}$ is obtained from $\lambda_{0,0}$ by adding a zero on the right and $\Psi_{0,0} \subset \Psi_{0,2}$.

(3) $z = w = 0$: $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ with parameter $(\lambda_{2,0}, 1, \Psi_{2,0})$ and $\theta_{2p_0,2q_0+2}(\pi) \neq 0$ with parameter $(\lambda_{0,2}, 1, \Psi_{0,2})$, where $\lambda_{2,0}$ and $\lambda_{0,2}$ are obtained from $\lambda_{0,0}$ by adding a zero on the left and right respectively, and $\Psi_{0,0} \subset \Psi_{2,0}, \Psi_{0,2}$.

(4) $z = 2w + 1$:

- If $e_{k+1} + e_{k+z} \in \Psi$, then $\theta_{2p_0+2,2q_0+2}(\pi) \neq 0$ with the parameter $(\lambda_{1,1}, 1, \Psi_{1,1})$, where $\lambda_{1,1}$ is obtained from $\lambda_{0,0}$ by adding a zero on each side of the semicolon, and $\Psi_{0,0} \cup \{e_{p_0+1} - f_{q_0+1}\} \subset \Psi_{1,1}$. Moreover, $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ with parameter $(\lambda_{1,0}, 1, \Psi_{1,0})$, where $\lambda_{1,0}$ is obtained from $\lambda_{0,0}$ by adding a zero on the left, and $\Psi_{0,0} \subset \Psi_{1,0}$.
- If $-e_{k+1} - e_{k+z} \in \Psi$, then $\theta_{2p_0+2,2q_0+2}(\pi) \neq 0$ with the parameter $(\lambda_{1,1}, 1, \Psi_{1,1})$, where $\lambda_{1,1}$ is obtained from $\lambda_{0,0}$ by adding a zero on each side of the semicolon, and $\Psi_{0,0} \cup \{-e_{p_0+1} + f_{q_0+1}\} \subset \Psi_{1,1}$. $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ with parameter $(\lambda_{0,1}, 1, \Psi_{0,1})$, where $\lambda_{0,1}$ is obtained from $\lambda_{0,0}$ by adding a zero on the right, and $\Psi_{0,0} \subset \Psi_{0,1}$.

Remark

This theorem is just part of the theorem of Moeglin and Paul. In fact, there are exactly four representations of orthogonal groups corresponding to π_1. We choose one of them preserving the limit of discrete series, and such that the corresponding dual pair is of equal rank.

To classify the limit of discrete series representations of $\mathrm{Sp}_{2n}(\mathbb{R})$, we use the minimal K-type defined by Vogan.

Definition

A minimal K-type of a $(\mathfrak{g}, K)$-module π is a K-type that has minimal norm among all K-types occurring in $\pi_{|K}$.

Remark

If an infinitesimal character is specified, there are finitely many irreducible representations of K such that every irreducible representation π of G with that infinitesimal character contains one of these K-types as its minimal K-type .

Example

Consider the real group $\mathrm{Sp}_4(\mathbb{R})$. Let $\lambda_d = (2,0)$ be an infinitesimal character.

- There are two Harish-Chandra parameters $(2,0), (0,-2)$ as liftings of λ_d, each of which determines two limit of discrete series of $\mathrm{Sp}_{2n}(\mathbb{R})$;
- For the Harish-Chandra parameters $(2,0)$, we can associate two sets of simple roots $\{e_1 - e_2, 2e_2\}$ with positive non-compact roots $\{2e_1, 2e_2, e_1+e_2\}$ and $\{e_1+e_2, -2e_2\}$ with positive non-compact roots $\{2e_1, -2e_2, e_1+e_2\}$. The corresponding minimal K-types $\lambda_d + \rho_n - \rho_c$ are $(3,2)$ and $(3,0)$ respectively.

Strategy of the translation

- Regroup the L-packet $\Pi(\varphi_1)$ by the elements in the center;
- Construct the bijection between the generic L-packet containing π_1 and the set of strong involutions of **G**,
- Construct the bijection between the set of strong involutions and the characters of the component groups using the base points, which correspond to the fundamental Borel pair of Whittaker type.

Definition

A strong involution of $\mathbf{G}$ for the equal rank inner class is an elliptic element $\tilde{x} \in \mathbf{G}$ such that $\tilde{x}^2 \in Z$.

For a strong involution $\tilde{x}$, we set $\theta_{\tilde{x}} = \mathrm{int}(\tilde{x})$ and $\mathbf{K}_{\tilde{x}} = \mathbf{G}^{\theta_{\tilde{x}}} = \mathrm{Cent}_{\mathbf{G}}(\tilde{x})$.

Definition

A representation of a strong involution $\tilde{x}$ is a pair $(\tilde{x}, \pi)$ where π is a $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-module.

Definition

Let $\tilde{x}$ be a strong involution with $\tilde{x}^2 = z \in Z$. We denote by $\Pi(\tilde{x}, \varphi)$ the L-packet associate to the L-parameter φ and the strong involution $\tilde{x}$, i.e. the set of discrete series $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-modules with infinitesimal character λ. In other words, $\Pi(\tilde{x}, \varphi)$ is exactly the L-packet of φ and the real form of $\mathbf{G}$ determined by $\tilde{x}$.

Definition

For a Langlands parameter φ of a real form $G = \mathbf{G}^\sigma$ of equal rank, we define the L-packet associated to φ as

$$\Pi(\varphi) = \coprod_{G' \in \mathrm{Inn}(G)} \Pi(\varphi, G'),$$

where $\mathrm{Inn}(G)$ is the set of representatives of conjugacy classes of real forms in the inner class of G.

Regroup the L-packet

- Let $\tilde{\Pi}(\varphi_1)$ be set of all the representations of strong real forms with infinitesimal character λ;
- The L-packet $\Pi(\tilde{x}, \varphi_1)$ can be embedded into $\tilde{\Pi}(\varphi_1)$ as
$$\Pi(\tilde{x}, \varphi_1) = \{[\tilde{x}, \pi_{\tilde{x}}(w^{-1}\lambda)] : w \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\},$$
which induces a bijection $\Pi(\tilde{x}, \varphi_1) \leftrightarrow \{wx | w \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\}$;
- We regroup the L-packets of φ for inner forms using the central invariant: for $z \in Z$, we set
$$\Pi_z(\varphi_1) := \cup_{\tilde{x}^2 = z} \Pi(\tilde{x}, \varphi_1);$$
- For any $z \in Z$ and any discrete series Langlands parameter φ_1 of $\mathbf{G}$, we have a W-equivariant bijection between $\tilde{S}(z)$ and $\Pi_z(\varphi)$, given by
$$\tilde{x} \mapsto [\tilde{x}, \pi_{\tilde{x}}(\lambda)]$$
.

Base points

- The element $\rho \in X \otimes \mathbb{R}$ produces a basepoint of the strong involutions

$$x_b = \exp(i\pi\rho^\vee),$$

  which is independent of choice of the set of positive roots;
- For any simple root $\alpha \in \Delta$, we have $\langle \alpha, \rho^\vee \rangle = 1$;
- We can deduce that $\alpha(x_b) = \exp(i\pi\langle \alpha, \rho^\vee \rangle) = -1$, for any simple root α;
- For $z(\rho^\vee) = x_b^2$ and a discrete Langlands parameter φ of $\mathbf{G}$, we have a W-equivariant bijection

$$\Pi_{z(\rho^\vee)}(\varphi) \cong \tilde{S}(z(\rho^\vee)),$$

  where $\tilde{S}(z(\rho^\vee))$ is the set of strong involutions which satisfy $x^2 = z(\rho^\vee)$.

Example

For the symplectic group $\mathrm{Sp}(V)$ of rank n, suppose π_1 is a generic discrete series of $\mathrm{Sp}(V)$ with Langlands parameter

$$\varphi_1 = \oplus_{i=1}^{n} \rho_{2\lambda_i}.$$

- The corresponding base point of strong involution is $x_{b,\mathrm{Sp}} = i((-1)^{n-1}, \cdots, 1)$;
- The infiniestimal character of π_1 is $(\lambda_1, \cdots, \lambda_n)$;
- Then the Harish-Chandra parameter of π_1 has the form

$$(\lambda_1, \lambda_3, \cdots, \lambda_{n-1}, -\lambda_n, \cdots, -\lambda_2),$$

and the corresponding root system is generated by the simple roots

$$\{e_1 + e_2, -e_2 - e_3, \cdots, e_{n-1} + e_n, -2e_n\};$$

- Hence the Harish-Chandra-Langlands parameter of $\theta(\pi)$ is the pair $(\lambda_{2,0}, 1, \Psi_{2,0})$, where
$$\lambda_{2,0} = (\lambda_1, \lambda_3, \cdots, \lambda_{n-1}, 0; \lambda_2, \cdots, \lambda_n)$$
and the corresponding root system
$$\Psi_{2,0} = \langle e_1 - f_1, f_1 - e_2, \cdots, e_{\frac{n}{2}-1} - f_{\frac{n}{2}}, f_{\frac{n}{2}} - e_{\frac{n}{2}}, f_{\frac{n}{2}} + e_{\frac{n}{2}} \rangle;$$
- Since $\lambda_1 > \cdots > \lambda_n > 0$ and the root system is generated by the non-compact simple roots, the parameter $(\lambda_{2,0}, 1, \Psi_{2,0})$ is exactly the Harish-Chandra-Langlands parameter of the generic discrete series representation of $\mathrm{O}(n+2, n)$. As a result, $\theta(\pi)$ is a generic discrete series of $\mathrm{O}(n+2, n)$.