Hida theory for GSpin Shimura varieties

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Content

Fix an odd prime p and compatible field embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \simeq \overline{\mathbb{Q}}_p$.
We are interested in the following interpolation problem: for a function $f\colon I \to \overline{\mathbb{Q}}$ ($I \subset \mathbb{Z}$), is there an analytic function $F(s)$ with $s \in \mathbb{Z}_p$ (or a finite extension of $\mathbb{Z}_p$), such that $F(k) = f(k)$ for all $k \in I$?
Here is an elementary example: fix a positive integer d prime to p and set $f(s) = d^s$ ($s \in \mathbb{N}$).
Recall a theorem of Euler/Fermat, for any $r > 0$,

$$d^{k+\phi(p^r)} \equiv d^k (\mathrm{mod}\, p^r).$$

Thus we can define a function

$$F\colon \mathbb{Z}_p^\times \simeq \varprojlim_r \mathbb{Z}/\phi(p^r) \to \mathbb{Z}_p, \quad (s_r)_r \mapsto (d^{s_r})_r.$$

Then F is analytic and moreover $F(k) = f(k)$ for all $k \in \mathbb{N}$.
We can try to generalize this construction to modular forms.

We try to generalize this construction to modular forms.
Fix an even integer $k > 2$ and a congruence subgroup $\Gamma = \Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$ (consisting of those $\gamma \equiv 1_2 (\mathrm{mod}\, N)$), A a $\mathbb{Z}[\frac{1}{N}]$-algebra.
Then we write $M_k(\Gamma, A)$ for the space of modular forms of weight k, of level Γ with coefficients (of the q-expansion) in A and its subspace $S_k(\Gamma, A)$ of cuspidal modular forms.
Typical examples of such modular forms are Eisenstein series

$$E(k, z) = \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(mz+n)^k}, \quad z = x + iy \,\mathrm{with} y > 0.$$

Then we have the q-expansion ($q = \exp(2i\pi z)$)

$$\widetilde{E}(k, z) = \mathrm{const} \times E(k, z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) = \sum_{1 \le d | n} d^{k-1}$.

The p-stabilization

$$\sigma_{k-1}^{(p)}(n) = \sum_{p\nmid d|n} d^{k-1}$$

when viewed as a function of k, admits a p-adic interpolation just as the case of power function $k \mapsto d^k$.

Thus we see that in the p-stabilization

$$\begin{aligned} E^{(p)}(k,z) &= \widetilde{E}(k,z) - p^{k-1}\widetilde{E}(k,pz) \\ &= \frac{\zeta(1-k)}{2}(1-p^{k-1}) + \sum_{n=1}^{\infty} \sigma_{k-1}^{(p)}(n) q^n, \end{aligned}$$

the coefficients of non-constant terms have p-adic interpolations.

A theorem of Serre deduces from this that the constant term $\frac{\zeta(1-k)}{2}(1-p^{k-1})$ also admits a p-adic interpolation:

we write the weight space

$$\begin{aligned} W &= \{s \in \mathbb{Z}_p^\times | s (\mathrm{mod}\, p-1) \text{ is even}\}, \\ \Lambda &= \mathcal{O}_{\mathrm{an}}(1+p\mathbb{Z}_p, \mathbb{Z}_p), \\ \widetilde{\Lambda} &= \mathcal{O}_{\mathrm{an}}(W, \mathbb{Z}_p) = \Lambda^{\frac{p-1}{2}}. \end{aligned}$$

Theorem (Serre.73')

There is a formal power series $E^{(p)}(s) \in \Lambda[[q]]$ such that for any $2 < k \in W \cap 2\mathbb{N}$, the evaluation at k gives

$$E^{(p)}(k) = E^{(p)}(k, z).$$

Remark

The existence of the p-adic interpolation of the constant term $\frac{\zeta(1-k)}{2}(1-p^{k-1})$ is part of the theorem, related to the p-adic zeta function.

More generally, we introduce the notion of Λ-adic modular forms

Definition

We write

$$M(\Gamma, \Lambda) = \{F \in \Lambda[[q]] | \text{ for a.a. } k \in W \cap 2\mathbb{N}_{>2}, F(k) = M_k(\Gamma, \mathbb{Z}_p)\},$$
$$S(\Gamma, \Lambda) = \{F \in \Lambda[[q]] | \text{ for a.a. } k \in W \cap 2\mathbb{N}_{>2}, F(k) = S_k(\Gamma, \mathbb{Z}_p)\},$$

$$S(\Gamma,\Lambda)=\{F\in\Lambda[[q]] \mid \text{for a.a. } k\in W\cap\mathbb{N}_{>2},\ F(k)=S_k(\Gamma,\mathbb{Z}_p)\},$$

Now for each cuspidal modular form $f\in S_k(\Gamma,\mathbb{Z}_p)$, the product $f\cdot E^{(p)}\in S(\Gamma,\Lambda)$, which gives an inclusion

$$\bigcup_k S_k(\Gamma,\mathbb{Z}_p)\hookrightarrow S(\Gamma,\Lambda).$$

We know that the rank of $S_k(\Gamma,\mathbb{Z}_p)$ grows (linearly) with k, thus the space $S(\Gamma,\Lambda)$ may be very large. We want however a subspace of $S(\Gamma,\Lambda)$ which contains $\cup_k S_k(\Gamma,\mathbb{Z}_p)$ as a dense subspace and is of finite rank over Λ. The idea of Hida is to consider ordinary modular forms: we have the U_p Hecke operator acting on $M_k(\Gamma,A)$ which is given by

$$(U_pf)(z)=\sum_{n=0}^{\infty}a_{pn}q^n,\quad (f=\sum_{n=0}^{\infty}a_nq^n).$$

We say an eigenform f for U_p is p-ordinary if $U_p f = uf$ for some p-unit u. In general f is p-ordinary if f is a $\mathbb{Z}_p$-linear combination of such eigenforms. Moreover, if we put

$$e = \lim_{n\to\infty} U_p^{n!} \in \mathrm{End}_{\mathbb{Z}_p}(M_k(\Gamma, \mathbb{Z}_p)).$$

Then for any $f \in M_k(\Gamma, \mathbb{Z}_p)$, $e(f)$ is p-ordinary.

Theorem (Hida. 89′)

1. *$eS(\Gamma, \Lambda)$ is a finite free Λ-module;*
2. *For any $2 < k \in W \cap 2\mathbb{N}$, we have the specialization isomorphism*

$$eS(\Gamma, \Lambda) \otimes_{\Lambda,k} \mathbb{Z}_p = eS_k(\Gamma, \mathbb{Z}_p).$$

Remark

Applications such as the weight two case of Mazur-Tate-Teitelbaum conjecture by Greenberg and Stevens, some cases of Artin conjecture by Buzzard, Dickinson, Shephard-Barron and Taylor, the modularity lifting theorem by Wiles and Taylor-Wiles.

There are various generalizations of Hida theory to other groups than $\mathrm{GL}_2/\mathbb{Q}$, for example compact unitary group (D.Geraghty), $\mathrm{GSp}_{2n}/\mathbb{Q}$ (H.Hida and V.Pilloni), PEL type unitary Shimura varieties (H.Hida, R.Brasca-G.Rosso and E.Ellen-E.Mantovan). In this talk, we want to generalize this to GSpin Shimura varieties.
The machinery of studying ordinary families of p-adic modular forms is roughly as follows:

1. the rank of the "ordinary" part of the space of weight κ modular forms is bounded independently of κ,
2. there is a Hasse invariant whose non-vanishing locus is the ordinary locus of the Shimura variety,
3. a space of p-adic modular forms containing as a dense subset all the classical modular forms. We can take the space of functions on a certain Igusa tower.

We fix a non-degenerate self-dual quadratic space (L, Q) over $\mathbb{Z}_{(p)}$ of rank $n+2$ $(n \geq 3)$ such that the quadratic form $Q_{\mathbb{R}}$ is of signature $(n, 2)$. Then the Clifford algebra $C_L = C_{(L,Q)}$ associated to (L, Q) is the quotient tensor algebra

$$C_L = L^{\otimes}/\langle x \otimes x - Q(x) \cdot 1 \rangle$$

which decomposes according to the parity of the degree(length) of the elements in C_L:

$$C_L = C_L^+ \oplus C_L^-,$$

then C_L^+ is a subalgebra of C_L of rank 2^{n+1}.
The general spin group $G = \mathrm{GSpin}(L, Q)$ is the reductive algebraic group over $\mathbb{Z}_{(p)}$ whose S-points are given by

$$G(S) = \left\{x \in (C_L^+ \otimes S)^\times | x(L \otimes S)x^{-1} = L \otimes S\right\}.$$

There is a natural embedding of G into a general symplectic group. More precisely, there is an anti-automorphism $*$ on C sending $x_1 \otimes x_2 \otimes \cdots \otimes x_r$ to $(-1)^r x_r \otimes x_{r-1} \otimes \cdots \otimes x_1$. Fix one element $\delta \in C_L^+$ such that $*(\delta) = -\delta$.
Then the map

$$C_L^+ \times C_L^+ \to \mathbb{Z}_{(p)}, \quad (x, y) \mapsto \mathrm{Trd}(x \cdot \delta \cdot *(y))$$

is a non-degenerate symplectic form on C_L^+.
Moreover, the left multiplication of G on C_L^+ preserves this symplectic form, so we get

$$G \to \mathrm{GSp}(C_L^+).$$

We can find a tensor $\mathfrak{t} \in (C_L^+)^{\otimes 2} \otimes ((C_L^+)^\vee)^{\otimes 2}$ such that $G = \mathrm{Stab}_{\mathrm{GSp}(C_L^+)}(\mathfrak{t})$.
In defining a Shimura datum from G, we need a $G(\mathbb{R})$-conjugacy class X of morphisms $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}$. Here we can take

$$X = \{\text{oriented negative-definite 2-dim subspace in } L_{\mathbb{R}}\}.$$

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Each $x = \mathbb{R}\langle e_x, f_x\rangle \in X$ gives rise to a Hodge structure on $L_{\mathbb{Q}}$ by (suppose $Q|_x = -1_2$)

$$L_x^{p,q} = \begin{cases} \mathbb{C}\langle e_x + \sqrt{-1} f_x\rangle \subset L_{\mathbb{C}}, & (p,q) = (-1,+1); \\ \mathbb{C}\langle e_x - \sqrt{-1} f_x\rangle, & (p,q) = (+1,-1); \\ (x_{\mathbb{C}})^{\perp}, & (p,q) = (0,0), \\ 0 & \text{oterwise.} \end{cases}$$

We have then a morphism $h_x \colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \to \mathbb{G}_{\mathbb{R}}$ whose $\mathbb{R}$-points are given by sending $z = r\exp(i\theta) \in \mathbb{C}^{\times}$ to $h_x(z)$ which acts by $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}$ on $x_{\mathbb{C}}$ and by 1 on $(x_{\mathbb{C}})^{\perp}$.
Then (G, X) is a Shimura datum. Moreover, all these h_x are defined over $\mathbb{Q}$, the reflex field of (G, X).

Remark

The assumption that (L, Q) is self-dual over $\mathbb{Z}_{(p)}$ shows that G is quasi-split at p and moreover the adjoint group is $G^{\mathrm{ad}} = \mathrm{SO}(L, Q)$.

Similarly, for $\mathrm{GSp}(C_L^+)$, we can construct a $\mathrm{GSp}(C_L^+)$-conjugacy class $X_{C_L^+}$ of morphisms $\mathbb{S} \to \mathrm{GSp}(C_L^+)_{\mathbb{R}}$ and we have moreover an embedding $X \hookrightarrow X_{C_L^+}$ via the embedding $G \to \mathrm{GSp}(C_L^+)$.
Thus (G, X) is a Shimura datum of Hodge type.
Now we fix a compact open subgroup $K = K_p K^p \subset G(\mathbb{A}_f)$ with $K_p = G(\mathbb{Z}_p)$ hyperspecial and K^p sufficiently small.
Then the Shimura variety

$$Sh_K(G, X) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

has a smooth integral model over $\mathbb{Z}_p$, which we denote by Sh.
Over the integral model Sh, there is an abelian scheme $\mathcal{A}$ (the pull-back of the universal abelian scheme $\mathcal{A}'$ over the Siegel Shimura variety $Sh_{C_L^+} = Sh(\mathrm{GSp}(C_L^+))$ to Sh). Then we write $\omega = e^*(\det\Omega_{\mathcal{A}/Sh})$, an ample line bundle (added after the talk: over the minimal compactification), where $e\colon Sh \to \mathcal{A}$ is the unit section of $\mathcal{A} \to Sh$.
We put

$$Sh_m = Sh \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m, \quad Sh_\infty = \varinjlim_m Sh_m$$

(the completion of Sh along Sh_1)

Theorem (Koskivirta-Wedhorn,15')

There is a positive integer $N_G > 1$ such that

$$\dim H^0(Sh_1, \omega^{\otimes N_G}) = 1.$$

(Added after the talk: the statement of the above theorem is not correct due to a misunderstanding of a result of Koskivirta-Wedhorn. The correct statement can be found in their article (Generalized Hasse invariant for Shimura varieties of Hodge type). More precisely, there is a line bundle $\omega_G^\flat$ (see after (4.11) of *loc.cit*) on $G - \mathrm{Zip}^\chi$, a certain stack associated to G (see (4.5) of *loc.cit*). The pull-back $\zeta_G^*(\omega_G^\flat)$ to Sh is the Hodge line bundle ω. Then Theorem 4.12, (4.12) and Proposition 1.18 of *loc.cit* shows that for any integer r,

$$\dim H^0(G - \mathrm{Zip}^\chi, (\omega_G^\flat)^{\otimes r}) \leq 1$$

and there is some positive integer $r = N_G$ such that the above dimension is 1. Then our Hasse invariant Ha is the pull-back of a generator of $H^0(G - \mathrm{Zip}^\chi, (\omega_G^\flat)^{\otimes N_G})$. Moreover, the non-vanishing locus of Ha is independent of multiples of N_G. Many thanks to the audience for their nice questions)

Then we have the following result

Theorem (Koskivirta-Wedhorn,15')

There is a positive integer $N_G > 1$ such that

$$\dim H^0(Sh_1, \omega^{\otimes N_G}) = 1.$$

Definition

We fix a generator $\mathrm{Ha} \in H^0(Sh_1, \omega^{\otimes N_G})$, the Hasse invariant and define the μ-ordinary locus

$$Sh_1^{\mu} := Sh_1 \backslash V(\mathrm{Ha}).$$

By work of Wortmann, we know that Sh^{μ} is open and dense in Sh. Since ω is ample, we can lift some positive power Ha^t from Sh_1 to Sh and then we put

$$Sh^{\mu} = Sh \backslash V(\mathrm{Ha}^t).$$

Using Koecher principal, Ha^t extends to a section $\overline{\mathrm{Ha}^t}$ over a toroidal compactification $\overline{Sh}$ of Sh and we set $\overline{Sh}^{\mu} = \overline{Sh} \backslash V(\overline{\mathrm{Ha}^t})$.

Now we construct modular forms and Hecke operators on Sh as in the classical case.
For each $x \in X$, we have a cocharacter $\nu\colon \mathbb{G}_m \xrightarrow{z \mapsto (z,1)} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} G_{\mathbb{C}}$. Then we write $P \subset G$ for the parabolic subgroup stabilizing (the Hodge filtration induced by) the cocharacter ν. We fix a Borel subgroup and maximal torus

$$T \subset B \subset P \subset G.$$

Now we have a P-torsor $\mathcal{P}$ over Sh:

$$\mathcal{P} := \underline{\mathrm{Isom}}_{Sh,\mathrm{fil},\mathrm{pol}}\Big((\mathcal{O}_{Sh} \otimes_{\mathbb{Z}_{(p)}} C_L^+, \mathfrak{t}), (e^* H^1_{\mathrm{dR}}(\mathcal{A}/Sh), \mathfrak{t}_{\mathrm{dR}})\Big)$$

preserving the Hodge filtrations and tensors $\mathfrak{t}, \mathfrak{t}_{\mathrm{dR}}$ as well as the polarizations.
Write $P = LU$ for the Levi decomposition for P, U the unipotent radical of P. Then $\mathcal{L} = \mathcal{P}/U$ is an L-torsor over Sh.

For any character $\lambda \in X^*(T)$, we write $\mathrm{Ind}_{B\cap L}^{L}(\lambda)$ for the induction representation from the character λ and then we put

$$\mathcal{V}_\lambda = \mathcal{L} \times^L \mathrm{Ind}_{B\cap L}^{L}(\lambda)$$

for the contracted product over L. This is a quasi-coherent sheaf over Sh.

Definition

For any $\mathbb{Z}_p$-algebra A, we write

$$M_\lambda(K, A) := H^0(Sh_A, (\mathcal{V}_\lambda)_A) = H^0(\overline{Sh}_A, (\overline{\mathcal{V}_\lambda})_A)$$

for the space of modular forms on Sh of weight λ, of level K with coefficients in A.
$S_\lambda(K, A)$ is the subspace of $M_\lambda(K, A)$ consisting of global sections vanishing at the cusp $\overline{Sh}_A \backslash Sh_A$.

Another important ingredient in Hida theory is Igusa towers.

Definition

For any $k, j > 0$, fix a point $p_0 \in Sh_k^\mu$, we set

$$\mathrm{Ig}_{k,j} := \underline{\mathrm{Isom}}_{\overline{Sh}_k^\mu, \mathrm{pol}} \left((\mathcal{A}[p^j], \mathfrak{t}_\mathcal{A}), (\mathcal{A}_{p_0}[p^j], \mathfrak{t}_{\mathcal{A}_{p_0}}) \right),$$

an $L'(\mathbb{Z}/p^j)$-torsor over $\overline{Sh}_k^\mu$. Here L' is a certain inner form of L. Then we put

$$\mathbb{V}_{k,j} := H^0(\mathrm{Ig}_{k,j}, \mathcal{O}_{\mathrm{Ig}_{k,j}}), \quad \mathbb{V}_k := \varprojlim_j \mathbb{V}_{k,j}$$

$$\mathbb{V} := \varinjlim_k \mathbb{V}_k, \quad \mathbb{V}^* = \mathrm{Hom}(\mathbb{V}, \mathbb{Q}_p/\mathbb{Z}_p).$$

These are the spaces of p-adic modular forms we will use to p-adically interpolate the spaces $S_\lambda(K, A)$.

Next we construct the analogue of U_p operators.
For each dominant coroot $\alpha \in X_*(T) \subset X_*(T_{C_L^+})$, we define a correspondence $\mathrm{Ig}_{C_L^+,\alpha}$ over the Siegel Shimura variety $Sh_{C_L^+}$ (not over Sh) which classifies the quintuples $(\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \psi_j, \widetilde{\psi}_j)$ where

1. $\mathcal{A}$ is a principally polarized abelian scheme over $\mathbb{Z}_p$,
2. $\psi_j \in \underline{\mathrm{Isom}}(\mathcal{A}_{p_0}[p^j], \mathcal{A}[p^j])/P(\mathbb{Z}/p^j)$ with a lift $\psi_\infty \in \underline{\mathrm{Isom}}(\mathcal{A}_{p_0}[p^\infty], \mathcal{A}[p^\infty])/P(\mathbb{Z}_p)$ (similarly for $\widetilde{\mathcal{A}}$ and $\widetilde{\psi}_j$),
3. $\pi : \mathcal{A} \to \widetilde{\mathcal{A}}$ a p-isogeny such that the induced morphism on their Dieudonné modules satisfies

$$\mathbb{D}(\widetilde{\psi}_\infty^{-1} \circ \pi \circ \psi_\infty) = \alpha(p) \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{D}(\mathcal{A}_{p_0}[p^\infty])) \simeq \mathrm{GSp}(C_L^+).$$

There is a universal quintuple $(\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \psi_j, \widetilde{\psi}_j)$ over $\mathrm{Ig}_{C_L^+,\alpha}$. Now we write Ig_α for the pull-back of $\mathrm{Ig}_{C_L^+,\alpha}$ along $Sh^\mu \to Sh_{C_L^+}$ and we get two natural projections

$$\mathrm{pr}_1 : \mathrm{Ig}_\alpha \to \mathrm{Ig}_{1,j} \quad (\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \psi_j, \widetilde{\psi}_j) \mapsto \mathcal{A},$$

$$\mathrm{pr}_2 : \mathrm{Ig}_\alpha \to \mathrm{Ig}_{1,j} \quad (\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \psi_j, \widetilde{\psi}_j) \mapsto \widetilde{\mathcal{A}}.$$

Proposition

Set $m_\alpha = [U(\mathbb{Z}_p) : U(\mathbb{Z}_p) \cap \alpha(p) U(\mathbb{Z}_p) \alpha(p)^{-1})]$*,*
then for any sheaf $\mathcal{F}$ *over* $\mathrm{Ig}_{1,j}$*, we have a well-defined map*

$$U_\alpha : H^0(\mathrm{Ig}_{1,j}, \mathcal{F}) \xrightarrow{\mathrm{pr}_2^*} H^0(\mathrm{Ig}_\alpha, \mathrm{pr}_2^* \mathcal{F}) \xrightarrow{\pi^*} H^0(\mathrm{Ig}_\alpha, \mathrm{pr}_1^* \mathcal{F}) \xrightarrow{\frac{1}{m_\alpha} \mathrm{Tr}(\mathrm{pr}_1^*)} H^0(\mathrm{Ig}_{1,j}, \mathcal{F}).$$

Now we put

$$e = \lim_{n \to \infty} \left(\prod_{\alpha \text{ dom}} U_\alpha \right)^{n!}.$$

Now we can state the control theorem: recall

$$\mathbb{V}_\infty = \varprojlim_j \mathbb{V}_{\infty,j}, \quad P = LU,$$

$$\mathcal{L} = \mathcal{P}/U, \quad \mathcal{V}_\lambda = \mathcal{L} \times^L \mathrm{Ind}_{B\cap L}^L(\lambda).$$

Theorem (Z.19')

(1) For any character $\lambda \in X^*(T)$, $e\mathbb{V}_{\infty,\mathrm{cusp}}^U[\lambda]$ is free of finite rank over $\mathbb{Z}_p$, bounded independently of λ.
(2) For any dominant character $\lambda \in X^*(T)$, one has

$$eH^0(\mathrm{Ig}_{\infty,1}/U, \mathcal{V}_\lambda) \simeq e\mathbb{V}_\infty^U[\lambda].$$

If moreover λ is sufficiently regular, one can descent this map to $\overline{Sh}^\mu$:

$$eH^0(\overline{Sh}^\mu, \mathcal{V}_\lambda) \simeq e\mathbb{V}_\infty^U[\lambda]$$

$$eS_\lambda(K, \mathbb{Z}_p) \simeq eH^0(\overline{Sh}^\mu, \mathcal{V}_\lambda)_{\mathrm{cusp}} \simeq e\mathbb{V}_{\infty,\mathrm{cusp}}^U[\lambda].$$

$$\mathbb{V}^* = \mathrm{Hom}(\varinjlim_k \mathbb{V}_k, \mathbb{Q}_p/\mathbb{Z}_p).$$

Theorem (Z.19')

(3) Set $\Lambda = \mathbb{Z}_p[[\mathrm{Ker}(T(\mathbb{Z}_p) \to T(\mathbb{Z}/p))]]$. *Then* $e\mathbb{V}^{*,U}_{\mathrm{cusp}}$ *is a finite free* Λ*-module.*
Moreover, for each character $\lambda \in X^*(T)$, *we have the specialization map*

$$e\mathbb{V}^{*,U}_{\mathrm{cusp}} \otimes_{\Lambda,\lambda} \mathbb{Z}_p \simeq \left(e\mathbb{V}^{U}_{\infty,\mathrm{cusp}}[\lambda]\right)^*,$$

which is $(eS_\lambda(K, \mathbb{Z}_p))^*$ *for* λ *sufficiently regular.*

Remark

(1) For the case $n = 3$, $G = \mathrm{GSpin}_{3,2} \simeq \mathrm{GSp}_4$,
then the μ*-ordinary locus* Sh_1^μ *is coincides with the ordinary locus, the set of points in* Sh_1 *classifying ordinary abelian schemes* $\mathcal{A}$ *of dimension 2 over* $\overline{\mathbb{F}}_p$ *(i.e.* $\mathcal{A}[p^\infty]$ *is an extension of* $(\mathbb{Q}_p/\mathbb{Z}_p)^2$ *by* $\mu^2_{p^\infty}$*)*

Remark

(2) H.Hida constructed such a theory for (G, X) of PEL type with G unitary group such that the ordinary locus $Sh_1^{\mathrm{ord}} \neq \emptyset$, R.Brasca-G.Rosso and E.Ellen-E.Mantovan for the case (G, X) of PEL type with G unitary such that $Sh_1^{\mathrm{ord}} = \emptyset$.
(3)The same strategy works for Shimura varieties of Hodge type (G, X) where G^{ad} has no factor isomorphic to $\mathrm{PGL}_2/\mathbb{Q}$ and G is quasi-split at p.

Sketch of proof: one can construct a map of Hodge-Tate to relate the classical modular forms $H^0(\overline{Sh}^{\mu}, \mathcal{V}_\lambda) = H^0(Sh^{\mu}, \mathcal{V}_\lambda)$ to the space of p-adic modular forms $\mathbb{V}_{k,j} = H^0(\mathrm{Ig}_{k,j}, \mathcal{O}_{\mathrm{Ig}_{k,j}})$.
More precisely, any point $\phi\colon \mathcal{A}[p^\infty] \xrightarrow{\sim} \mathcal{A}_{p_0}[p^\infty]$ in Ig_m induces an isomorphism

$$\mathrm{HT}_m(\phi)\colon e^* H^1_{\mathrm{dR}}(\mathcal{A}/\overline{Sh}^{\mu}_m) \xrightarrow{\sim} e^* H^1_{\mathrm{dR}}(\mathcal{A}_{p_0}/\overline{Sh}^{\mu}_m)$$

which preserves the Hodge tensors $\mathfrak{t}_{\mathrm{dR}}$, Hodge filtrations and polarizations on both sides. This gives a point in the L-torsor $\mathcal{L} = \mathcal{P}/U$ over $\overline{Sh}^{\mu}_m$.

Therefore we get the Hodge-Tate map

$$\mathrm{HT}_m^*\colon H^0(\overline{Sh}_m^{\mu}, \mathcal{V}_\lambda) \to \mathbb{V}_m^U[\lambda].$$

For dominant λ, the map

$$H^0(\mathrm{Ig}_{\infty,1}/U, \mathcal{V}_\lambda) \to \mathbb{V}_\infty^U[\lambda]$$

(locally) comes from the injective map ($B_L = B \cap L$)

$$\mathrm{Ind}_{B_L}^{L}(\lambda) \to \text{top-}\mathrm{Ind}_{B_L(\mathbb{Z}_p)}^{L(\mathbb{Z}_p)}(\lambda)$$

where the RHS is the space of continuous maps $f\colon L(\mathbb{Z}_p) \to \mathbb{Z}_p$ such that $f(gb) = \lambda(b)^{-1} f(g)$ for $b \in B_L(\mathbb{Z}_p)$.

Each $f \in \text{top-}\mathrm{Ind}_{B_L(\mathbb{Z}_p)}^{L(\mathbb{Z}_p)}(\lambda)$ is determined by its restriction to the dense subset $B_L(\mathbb{Z}_p)B_L^{\mathrm{opp}}(\mathbb{Z}_p)$ of $L(\mathbb{Z}_p)$.

Moreover the conjugation by $(\prod_{\alpha \text{ dom}} \alpha(p))^{n!}$ contracts $B_L(\mathbb{Z}_p)^{\mathrm{opp}}$ into $B_L(\mathbb{Z}_p)$.

Thus if $f(1) = 0$, then applying e to f shows that $e(f) = 0$. As a result we get isomorphisms

$$e \cdot \mathrm{Ind}_{B_L}^{L}(\lambda) \simeq e \cdot \text{top-}\mathrm{Ind}_{B_L(\mathbb{Z}_p)}^{L(\mathbb{Z}_p)}(\lambda) \simeq \mathbb{Z}_p[\lambda].$$

Then one can construct modified Hecke operators $\widetilde{U}_\alpha$ taking functions on $\mathrm{Ig}_{\infty,1}$ to functions on $\overline{Sh}^{\mu}_{\infty}$.
For λ sufficiently regular, the difference $\widetilde{U}_\alpha - U_\alpha$ is divisible by p, and thus applying the projector e gives the isomorphisms $eH^0(\overline{Sh}^{\mu}_{\infty}, \mathcal{V}_\lambda) \simeq e\mathbb{V}^{U}_{\infty}[\lambda]$.
We have the following commutative diagram ($m \geq 1$)

$$\begin{array}{ccc} \overline{Sh}_m & \xhookrightarrow{i_m} & \overline{Sh} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Sh^{\mathrm{min}}_m & \xhookrightarrow{i_m} & Sh^{\mathrm{min}} \end{array}$$

Proposition

We have the following isomorphism

$$i_m^* \pi_* \mathcal{V}_{\lambda,\mathrm{cusp}} \simeq \pi_* i_m^* \mathcal{V}_{\lambda,\mathrm{cusp}}.$$

The minimal compactification $Sh^{\min}$ is affine, thus the reduction mod p^m map is an isomorphism ($i_m \colon Sh_m^{\min,\mu} \hookrightarrow Sh^{\min,\mu}$)

$$\begin{aligned} H^0(\overline{Sh}^{\mu}, \mathcal{V}_{\lambda,\mathrm{cusp}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m &= H^0(Sh^{\min,\mu}, \pi^* \mathcal{V}_{\lambda,\mathrm{cusp}}) \otimes \mathbb{Z}/p^m \\ &= H^0(Sh^{\min,\mu}, i_m^* \pi_* \mathcal{V}_{\lambda,\mathrm{cusp}}) \\ &= H^0(Sh^{\min,\mu}, \pi_* i_m^* \mathcal{V}_{\lambda,\mathrm{cusp}}) \\ &= H^0(\overline{Sh}^{\mu}, i_m^* \mathcal{V}_{\lambda,\mathrm{cusp}}). \end{aligned}$$

Similarly we have $\mathbb{V}^U_{\infty,\mathrm{cusp}}[\lambda] \otimes \mathbb{Z}/p^m \simeq \mathbb{V}^U_{m,\mathrm{cusp}}[\lambda]$.
Moreover, the Hodge line bundle ω over $Sh^{\min}$ is ample over, so for $k \gg 0$, $H^1(Sh^{\min}, \pi_* \mathcal{V}_{\lambda+\underline{k},\mathrm{cusp}}) = 0$ and therefore

$$H^0(Sh^{\min}, \pi_* \mathcal{V}_{\lambda+\underline{k},\mathrm{cusp}}) \otimes \mathbb{Z}/p^m = H^0(Sh^{\min}, i_m^* \pi_* \mathcal{V}_{\lambda+\underline{k},\mathrm{cusp}}).$$

$$S_{\lambda+\underline{k}}(K, \mathbb{Z}_p) \otimes \mathbb{Z}/p^m = S_{\lambda+\underline{k}}(K, \mathbb{Z}/p^m).$$

One can show that the multiplication by the Hasse invariant Ha on $S_\lambda(K, \mathbb{Z}/p)$ gives rise to isomorphisms ($k \gg 0$)

$$eS_{\lambda+\underline{k}}(K, \mathbb{Z}/p) \simeq eS_{\lambda+\underline{k+N_G}}(K, \mathbb{Z}/p).$$

Since

$$H^0(\overline{Sh}_1^\mu, \mathcal{V}_{\lambda,\mathrm{cusp}}) = \bigcup_{r\in\mathbb{N}} \frac{S_{\lambda+\underline{rN_G}}(K, \mathbb{Z}/p)}{\mathrm{Ha}^r},$$

applying e, we get $eH^0(\overline{Sh}_1^\mu, \mathcal{V}_{\lambda+\underline{k}}) = eS_{\lambda+\underline{k}}(K, \mathbb{Z}/p)$ for $k \gg 0$.
Using the reduction mod p^m map, we get, for λ sufficiently regular,

$$eH^0(\overline{Sh}^\mu, \mathcal{V}_\lambda) = eS_\lambda(K, \mathbb{Z}_p),$$

which also shows that $e\mathbb{V}^U_{\infty,\mathrm{cusp}}[\lambda]$ is free of finite rank over $\mathbb{Z}_p$, bounded independently of λ.
We deduce the density of the classical modular forms in $\mathbb{V}_{\infty,\mathrm{cusp}}$:

$$\mathrm{HT}^*_\infty\left(\oplus_{\lambda\in X^*(T)} S_\lambda(K, \mathbb{Z}_p)[\frac{1}{p}]\right) \bigcap \mathbb{V}_{\infty,\mathrm{cusp}}$$

We write $\mathbb{T} \subset \mathrm{End}_\Lambda(\mathbb{V}^U_{\mathrm{cusp}})$ for the Hecke algebra generated by the Hecke operators U_α and the spherical ones outside p.
The μ-ordinary Hecke algebra $e\mathbb{T}$ is finite flat Λ-algebra. Then each irreducible component of $\mathrm{Spec}(e\mathbb{T})$ is called a Hida family.
By construction, the $\overline{\mathbb{Z}_p}$-points of $\mathrm{Spec}(e\mathbb{T})$ correspond to eigenforms of $e\mathbb{T}$ in $e\mathbb{V}^U_{\mathrm{cusp}}$. For an eigenform $f \in e\mathbb{V}^U_{\mathrm{cusp}}[\lambda]$ and any other weight $\lambda' \equiv \lambda (\mathrm{mod}\, N_G)$, there is an eigenform $f' \in e\mathbb{V}^U_{\mathrm{cusp}}[\lambda']$ such that $f' \equiv f(\mathfrak{m}_{\overline{\mathbb{Z}_p}})$. So we can choose a sequence of sufficiently regular $\lambda_k \equiv \lambda (\mathrm{mod}\, N_G)$ which p-adically converge to λ such that the eigenforms $f_k \in e\mathbb{V}^U_{\mathrm{cusp}}[\lambda_k]$ are all classical and congruent to f.

Recall the adjoint group $G' = G^{\mathrm{ad}} = \mathrm{SO}(L, Q)$. This is a central extension

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}(L, Q) \to \mathrm{SO}(L, Q) \to 1$$

We can deduce from the Hida theory on $G = \mathrm{GSpin}(L, Q)$ the Hida theory on $G' = \mathrm{SO}(L, Q)$. This relies on the construction of the integral model Sh' of the Shimura variety (G', X') of abelian type, which is obtained from Sh by quotient by a finite abelian group Δ (by Kisin).
Hida theories on orthogonal groups make it possible to construct p-adic L-functions of (μ-)ordinary families of automorphic representations of $G'(\mathbb{A}_{\mathbb{Q}})$ using doubling method, as in the case of Sp_{2n} by Liu, in the case of U_n by Eischen-Harris-Li-Skinner and many other cases.
The rough idea is as follows: write $(\widetilde{L}, \widetilde{Q}) = (L, Q) \oplus (L, -Q)$ for the quadratic space of signature $(n+2, n+2)$ and $H = \mathrm{SO}(\widetilde{L}, \widetilde{Q})$, which contains the diagonal image of $G' \times G'$. For automorphic representations $\pi, \pi^\vee$ of $G'(\mathbb{A}_{\mathbb{Q}})$ and Eisenstein series $E(F(\xi, s), h)$ on $H(\mathbb{A})$ for some section $F(\xi, s) \in \text{n-Ind}_{B_H(\mathbb{A})}^{H(\mathbb{A})}(\xi_s)$, the doubling method gives

$(f_1 \in \pi, f_2 \in \pi^\vee)$

$$\begin{aligned} &\langle E(F(\xi, s), \cdot)|_{G' \times G'}, f_1 \otimes f_2 \rangle \\ =&L^S(s + \frac{1}{2}, \pi \times \xi) \langle f_1, f_2 \rangle \prod_{v \in S} Z_v(F_v(\xi, s), f_{1,v}, f_{2,v}) \end{aligned}$$

Now one can try to apply differential operators to the Eisenstein series to construct an explicit p-adic family of Eisenstein series (and show its restriction to $G' \times G'$ is μ-ordinary cuspidal modular forms). This gives us a p-adic family $\mathcal{E}(F(\xi))$ with values in the μ-ordinary families of modular forms (on $G'(\mathbb{A}) \times G'(\mathbb{A})$). Hida theory then applies to show that for any sufficiently regular weight λ, for any μ-ordinary automorphic representation $\pi \subset \mathcal{A}(G'(\mathbb{A}))$ corresponding to this weight λ,
$\mathcal{E}(F(\xi))(\pi) = L^S(s_0 + \frac{1}{2}, \pi \times \xi) \times *$.

Thank you for your attention!