# Examples related to the Sakellaridis-Venkatesh conjecture

$\mathrm{U}(2)\backslash\mathrm{PGSp}_4$

Wan Xiaolei

National University of Singapore

08.07.2020

# Table of contents

# Sakellaridis-Venkatesh conjecture

Let spherical variety $Y = H\backslash G$ be defined over a number field $k$ with ring of adèles $\mathbb{A}$ and $\Pi$ a cuspidal automorphic representation of $G(\mathbb{A})$ with a fixing decomposition

$$\Pi = \bigotimes{}' \Pi_v$$

as a restricted tensor product of irreducible unitary representations of $G_v := G(k_v)$. With $[H] := H(k)\backslash H(\mathbb{A})$ and the Tamagawa measure $\mathrm{d}h$ on $[H]$, one may consider the global $H$-period

$$P_H : \mathcal{A}_0 \longrightarrow \mathbb{C}$$

given by

$$P_H(\phi) = \int_{[H]} \phi(h)\mathrm{d}h,$$

where $\mathcal{A}_0$ is the set of all cuspidal automorphic forms of $G$ and $\phi \in \mathcal{A}_0$.

# Sakellaridis-Venkatesh conjecture

If we know that

$$P_H \in \mathrm{Hom}_{H(\mathbb{A})}(\Pi, \mathbb{C}) = \bigotimes' \mathrm{Hom}_{H(k_v)}(\Pi_v, \mathbb{C})$$

is a pure tensor in $\mathrm{Hom}_{H(\mathbb{A})}(\Pi, \mathbb{C})$ for some reason, say $\dim \mathrm{Hom}_{H(k_v)}(\Pi_v, \mathbb{C}) \leq 1$ for all $v$, then the conjecture predicts that $P_H$ has a decomposition

$$P_H = q \cdot \prod_v^{*} \ell_{\Pi_v}^{Y(k_v)}$$

where $|q|^2$ is a rational number, $\prod^*$ is a normalization product and $\ell_{\Pi_v}^{Y_v} \in \mathrm{Hom}_{H(k_v)}(\Pi_v, \mathbb{C})$. This factorization generalizes a conjecture of Ichino and Ikeda for the Gross-Prasad periods. This conjecture is based on the local Langlands conjecture for $Y(k_v)$, since the key idea of the Sakellaridis-Venkatesh conjecture is that the local functionals $\ell_{\Pi_v}^{Y_v}$ are determined by the Plancherel decomposition of $L^2(Y(k_v))$.

# Sakellaridis-Venkatesh conjecture

Give a local spherical variety $Y(k_v)$. Motivated by and refining the work of Gaitsgory-Nadler in the geometric Langlands program, Sakellaridis and Venkatesh associated two data to $Y(k_v)$: a dual group $Y(k_v)^\vee$ and a distinguished map

$$\iota : Y(k_v)^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_v^\vee. \quad (1)$$

One might assume $\iota$ induces a map

$$\iota_* : \widehat{G_Y} \longrightarrow \widehat{G_v}$$

for simplicity, where $G_Y$ is a reductive group defined over $k_v$ such that $G_Y^\vee = Y(k_v)^\vee$, and $\widehat{G_v}$ denotes the set of unitary representations of $G_v$.

# Sakellaridis-Venkatesh conjecture

## Conjecture

*One has a spectral decomposition:*

$$L^2(Y(k_v)) = \int_{\widehat{G}_{Y_v}^{\mathrm{temp}}} m(\pi_v)\iota_*(\pi_v)\mathrm{d}\mu_{G_{Y_v}}(\pi_v)$$

*where* $\mu_{G_Y}$ *is the Plancherel measure of* $G_Y$ *and* $m(\pi_v) = \mathrm{Hom}_{G_v}(C_c^\infty(Y(k_v)), \iota_*(\pi_v))^*$ *is a multiplicity space.*

# Sakellaridis-Venkatesh conjecture

By the Bernstein's explanation of the direct integral decomposition, the conjecture is equivalent to

## Conjecture

*One has*

$$\langle \varphi_1, \varphi_2 \rangle_{Y(k_v)} = \int_{\widehat{G}_{Y_v}^{\mathrm{temp}}} J_{\pi_v}^{Y(k_v)}(\varphi_1, \varphi_2) \mathrm{d}\mu_{G_{Y_v}}(\pi_v),$$

*where* $\varphi_1, \varphi_2 \in C_c^\infty(Y(k_v))$, $\langle \cdot, \cdot \rangle_{Y(k_v)}$ *is the inner product of* $L^2(Y(k_v))$, $\mu_{G_{Y_v}}$ *is the Plancherel measure of* $G_{Y_v}$ *and* $\{J_{\pi_v}^{Y(k_v)} \mid \pi_v\}$ *is a family of positive semi-definite Hermitian forms on* $L^2(Y(k_v))$.

# Sakellaridis-Venkatesh conjecture

One may understand the Plancherel decomposition via the Bernstein's viewpoint, namely the spectral decomposition of $L^2(Y(k_v))$ gives rise to a family of linear functional $\{\ell_{\pi_v}^{Y(k_v)} \mid \pi_v\}$, which contribute to the decomposition of global periods as Sakellaridis-Venkatesh conjecture predicts. To summarize:

## Conjecture

*Assume relative LLC conjecture holds for $Y(k_v)$ for any $v$, then for a factorizable element $\phi = \otimes_v \phi_v$, one has*

$$P_H(\phi)\overline{P_H(\phi)} = q \cdot \prod_v^{*} \ell_{\pi_v}^{Y(k_v)}(\phi_v)\overline{\ell_{\pi_v}^{Y(k_v)}(\phi_v)},$$

*where $q$ is a rational number and $\ell_{\Pi_v}^{Y(k_v)}$ comes from the Plancherel decomposition of $L^2(Y(k_v))$.*

# Notations

$$\mathrm{SO}_5 \cong \mathrm{PGSp}_4 \supset (\mathrm{GL}_2 \times \mathrm{GL}_2)_{\mathrm{det}} / \Delta F^\times$$

where

$$(\mathrm{GL}_2 \times \mathrm{GL}_2)_{\mathrm{det}} := \{(g_1, g_2) \mid \det(g_1) = \det(g_2)\}$$

Let $E$ be an étale quadratic field, which gives a maximal torus $T \subset \mathrm{GL}_2$. Then

$$\mathrm{PGSp}_4 \supset (\mathrm{GL}_2 \times T)_{\mathrm{det}} / \Delta F^\times \cong \mathrm{U}(2)$$

# Weil representation

Let $V$ be a split quadratic space of four dimension. We will fix the complete polarization $V = V_1 \oplus V_2$ and a basis

$$\boldsymbol{e} := \{e_1, e_2, e_1', e_2'\}$$

for $V$ such that $V_1$ is the span of $e_1, e_2$, $V_2$ is the span of $e_1', e_2'$, and $\langle e_i, e_j' \rangle_V = \delta_{ij}$. Let $W$ be a symplectic space of four dimension. Then the Weil representation $\omega_\psi$ of $\mathrm{SO}_4 \times \mathrm{Sp}_4$ can be realized on the space $S\left(V_2 \otimes W\right)$ of Bruhat functions, and the actions are given by

1. $\omega_\psi(h)\Phi(T) = \Phi(h^{-1} \cdot T)$, for $h \in \mathrm{Sp}(W)$;
2. $\omega_\psi\left(m(A)\right)\Phi(T) = \left|\det(A)\right|^2 \cdot \Phi\left(TA\right)$, for $A \in \mathrm{GL}_2$;
3. $\omega_\psi\left(n(x)\right)\Phi(T) = \psi\left(xQ(T)\right)\Phi(T)$, for $x \in F$.

where $Q(T) = a_1 b_4 + a_2 b_3 - a_3 b_2 - a_4 b_1$ for

$$T = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}^T$$

# Weil representation

Let $R_0 := \{(h, g) \in \mathrm{GSO}_4 \times \mathrm{GSp}_4 \mid \lambda_V(h) \cdot \lambda_W(g) = 1\}$. The Weil representation $\omega_\psi$ extends naturally to the group $R_0$ via

$$\omega_\psi(h, g)\Phi = |\lambda_W(g)|^{-2} \omega_\psi(h_1, 1)(\Phi \circ g^{-1})$$

where

$$h_1 = h \cdot \begin{pmatrix} \lambda_V(h)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}_4.$$

# Weil representation

Let $\omega_\psi^+$ be the 1-eigen-space of $Z_{\mathrm{SO}_4}$ in $\omega_\psi$. Then define the Weil representation of $\mathrm{PGSO}_4 \times \mathrm{PGSp}_4$

$$\Omega := \mathrm{ind}_{Z_V \cdot R_0}^{\mathrm{GSO}_4 \times \mathrm{GSp}_4} \mathbb{C} \boxtimes \omega_\psi^+ = \bigoplus_{a \in F^{\times 2} \backslash F^\times} \omega_{\psi_a}^+.$$

The Weil representation of $\mathrm{GSO}_4 \times \mathrm{GSp}_4$ is defined as $\mathrm{ind}_{R_0}^{\mathrm{GSO}_4 \times \mathrm{GSp}_4} \omega_\psi$.

**Lemma**

$$\left(\mathrm{ind}_{R_0}^{\mathrm{GSO}_4 \times \mathrm{GSp}_4} \omega_\psi\right)_{Z_V} = \Omega.$$

# Theta correspondence

There exists an elemnt $\mathbf{t} \in \mathrm{GO}_4 \setminus \mathrm{GSO}_4$ such that

$$\sigma_1 \boxtimes \sigma_2 \circ \mathrm{Ad}(\mathbf{t}) \cong \sigma_2 \boxtimes \sigma_1$$

for irreducible representations $\sigma_1 \boxtimes \sigma_2, \sigma_2 \boxtimes \sigma_1$ of $\mathrm{PGSO}_4$. If $\sigma_1 \ncong \sigma_2$, then $(\sigma_1 \boxtimes \sigma_2)^+ := \sigma_1 \boxtimes \sigma_2 \oplus \sigma_2 \boxtimes \sigma_1$. If $\sigma_1 \cong \sigma_2 \cong \sigma$, then there are two extensions to $\mathrm{GO}_4$. Exactly one of them participates in the theta correspondence with $\mathrm{GSp}_2$, and we denote this distinguished extension by $(\sigma \boxtimes \sigma)^+$. Then define the theta correspondence of representation of $\mathrm{PGSO}_4$

$$\theta(\sigma_1 \boxtimes \sigma_2) = \theta\left((\sigma_1 \boxtimes \sigma_2)^+\right).$$

Thus this theta lifting is generically 2-to-1.

# The maps $A_\sigma$, $\theta_\sigma$ and $B_\sigma$

For every irreducible tempered representation $\sigma$ of $\mathrm{PGSO}_4$, we fix

$$A_\sigma : \omega_\psi \otimes \sigma^\vee \cong \theta(\sigma).$$

The $\theta_\sigma$ is given by duality of $A_\sigma$:

$$\theta_\sigma : \omega_\psi \longrightarrow \sigma \boxtimes \theta(\sigma)$$

Let the local doubling theta integral

$$Z_\sigma : \omega_\psi \otimes \overline{\omega_\psi} \otimes \overline{\sigma} \otimes \sigma \longrightarrow \mathbb{C}$$

given by

$$Z_\sigma(\Phi_1, \Phi_2, v_1, v_2) = \frac{1}{n} \cdot \int_{\mathrm{SO}_4} \langle \omega_\psi(h)\Phi_1, \Phi_2 \rangle_{\omega_\psi} \cdot \overline{\langle \sigma(h) v_1, v_2 \rangle_\sigma} \mathrm{d}h$$

where $n = \#F^{\times 2} \backslash F^\times$.

# The maps $A_\sigma$, $\theta_\sigma$ and $B_\sigma$

Then $Z_\sigma$ factor through the projection map:

$$A_\sigma \otimes \overline{A_\sigma} : \omega_\psi \otimes \overline{\omega_\psi} \otimes \sigma^\vee \otimes \overline{\sigma^\vee} \longrightarrow \theta(\sigma) \otimes \overline{\theta(\sigma)}$$

so that

$$Z_\sigma\left(\Phi_1, \Phi_2, v_1, v_2\right) = \langle A_\sigma(\Phi_1, v_1), A_{\sigma(\Phi_2, v_2)}\rangle_{\theta(\sigma)}.$$

Let

$$B_\sigma(\Phi, w) = \langle \theta_\sigma(\Phi), w\rangle_{\theta(\sigma)}.$$

# Spectral decomposition of Weil representation

## Spectral decomposition of Weil representation

For all tempered representations $\sigma$ of $\mathrm{PGSO}_4$, consider the Hermitian forms

$$J_\sigma^\theta(\Phi_1, \Phi_2) := \sum_{v \in \mathrm{ONB}(\sigma)} \int_{\mathrm{SO}_4} \langle \Omega(h)\Phi_1, \Phi_2 \rangle_\Omega \overline{\langle \sigma(h)v, v \rangle}_\sigma \mathrm{d}h.$$

Then $J_\sigma^\theta$ is positive semi-definite for every $\sigma \in \widehat{\mathrm{PGSO}_4}^{\mathrm{temp}}$, and

$$\langle \Phi_1, \Phi_2 \rangle_\Omega = \int_{\widehat{\mathrm{PGSO}_4}^{\mathrm{temp}}} J_\sigma^\theta(\Phi_1, \Phi_2)\, \mathrm{d}\mu_{\mathrm{PGSO}_4}(\sigma).$$

Note that the Plancherel measure $\mu_{\mathrm{PGSO}_4}$ is determined by the Haar measure $\mathrm{d}h$ on $\mathrm{PGSO}_4$.

# Spectral decomposition à la Bernstein

Let $G$ be a reductive group over a local field $F$ acting transitively on a homogeneous $G$-space $X$. We fix a base point $x_0 \in X$ with stabilizer $H \subset G$, so that $g \to g^{-1} \cdot x_0$ gives an identification $H \backslash G \cong X$. Consider the unitary representation $L^2(X)$ with the inner product

$$\langle \phi_1, \phi_2 \rangle_X = \int_X \phi_1(x) \overline{\phi_2(x)} \mathrm{d}x.$$

Such a unitary representation admits a direct integral decomposition:

$$L^2(X) \cong \int_Z \sigma(z) \mathrm{d}\nu(z),$$

where

1. $Z$ is a measurable space, equipped with measure $\nu(z)$.
2. $\sigma : z \mapsto \sigma(z)$ is a measurable map from $Z$ to the unitary dual $\widehat{G}$ equipped with the Fell topology.

# Spectral decomposition à la Bernstein

The natural inclusion $\mathcal{C}(X) \hookrightarrow L^2(X)$ is point-wise defined, i.e. there exists a family of maps $\{\alpha_z^X \in \mathrm{Hom}_G(\mathcal{S}(X), \sigma(z)) \mid z\}$ such that $\alpha_z^X(\varphi)$ represents $\varphi$.
If $\alpha_z^X$ is non-zero, then by duality, one obtains a $G$-equivariant embedding

$$\overline{\beta}_z^X : \sigma(z)^\vee \cong \overline{\sigma(z)} \longrightarrow C^\infty(X).$$

Take complex conjugate, then we get a family of maps $\{\beta_z^X \mid z\}$. These maps are uniquely determined for $\nu$-almost all $z$. Let

$$\ell_z^X := \mathrm{ev}_{x_0} \circ \beta_z^X \in \mathrm{Hom}_H(\sigma_z, \mathbb{C}).$$

# Strongly tempered variety

Assume $X$ is strongly tempered, i.e. for any irreducible tempered representation $\pi$, and any smooth vector $v_1, v_2 \in \pi$, one has $\int_H |\langle h \cdot v_1, v_2 \rangle_\pi| \mathrm{d}h < \infty$. Then

## Plancherel decomposition of strongly tempered variety

$$L^2(X) \cong \int_{\widehat{G}^{\mathrm{temp}}} \mathrm{Hom}_H(\pi, \mathbb{C}) \cdot \pi \mathrm{d}\mu_G(\pi)$$

where $\mu_G$ is the Plancherel measure of $G$. Moreover, one has

$$\ell_\pi^X(v_1) \cdot \overline{\ell_\pi^X(v_2)} = \int_H \langle h \cdot v_1, v_2 \rangle_\pi \mathrm{d}h$$

# Whittaker period

Let $N$ be the nilpotent radical of a Borel subgroup of $G$ and $\psi$ a non-degenerate character of $N$.

## Whittaker-Plancherel theorem

$$L^2(N\backslash G, \psi) \cong \int_{\widehat{G}^{\mathrm{temp}}} \mathrm{Hom}_N(\pi, \psi) \cdot \pi \mathrm{d}\mu_G(\pi)$$

Moreover, one has

$$\ell_\pi^{N,\psi\backslash G}(v_1) \cdot \overline{\ell_\pi^{N,\psi\backslash G}(v_2)} = \int_N \langle n \cdot v_1, v_2 \rangle_\pi \cdot \overline{\psi(n)} \mathrm{d}n$$

# Relative characters

For every $H$-distinguished irreducible representation $\pi$ of $G$, and $\ell_1, \ell_2 \in \mathrm{Hom}_H(\pi, \mathbb{C})$, then one has

## Definition

*The relative characters*

$$\mathcal{B}_{\pi,\ell_1,\ell_2} : C_c^\infty(G) \longrightarrow \mathbb{C}$$

*is defined by*

$$\mathcal{B}_{\pi,\ell_1,\ell_2}(f) = \sum_{v \in \mathrm{ONB}(\pi)} \overline{\ell_1\left(\pi(\overline{f})v\right)} \cdot \ell_2(v)$$

Note that $\mathcal{B}_{\pi,\ell_1,\ell_2}$ factors through

$$C_c^\infty(G) \twoheadrightarrow C_c^\infty(X) \longrightarrow \mathbb{C}.$$

# Relative characters

**Lemma**

*For every* $\varphi \in C_c^\infty(X)$, *one has*

$$\mathcal{B}_{\pi,\ell_\pi^X,\ell_\pi^X}(\varphi) = \ell_\pi^X \circ \alpha_\pi^X(\varphi)$$

Note that $\ell_\pi^X \circ \alpha_\pi^X$ can extend to the Harish-Chandra Schwartz space $\mathcal{C}(X)$.

# Transfer

Let

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^T.$$

Let $\mathrm{pr}_1 : \Omega \cong \bigoplus_{a \in F^{\times 2} \backslash F^\times} \omega_{\psi_a}^+ \longrightarrow \omega_\psi^+$ be the projection map. Let $p$ be the map

$$p : \Omega \cong \bigoplus_{a \in F^{\times 2} \backslash F^\times} \omega_{\psi_a}^+ \longrightarrow C^\infty \left( N \times T \backslash \mathrm{PGSO}_4, \psi \right)$$

given by

$$p(\Phi)(h) := \int_T \mathrm{pr}_1 \left( (1,t)h \cdot \Phi \right) (T_1) \mathrm{d}t.$$

# Transfer

Let $q$ be the map

$$q : \Omega \cong \bigoplus_{a \in F^{\times 2} \backslash F^{\times}} \omega_{\psi_a}^{+} \longrightarrow C^{\infty}\left(\mathrm{U}(2) \backslash \mathrm{PGSp}_4\right)$$

given by

$$q(\Phi)(g) := \int_{T} \mathrm{pr}_1\left((t,t)g \cdot \Phi\right)(T_1)\mathrm{d}t.$$

**Lemma**

1. *The image of $p$ $\mathcal{S}\left(N \times T \backslash \mathrm{PGSO}_4, \psi\right) \subset \mathcal{C}\left(N \times T \backslash \mathrm{PGSO}_4, \psi\right)$;*
2. *The image of $q$ equals $S\left(\mathrm{U}(2) \backslash \mathrm{PGSp}_4\right)$.*

# Local results

## Proposition

$$L^2\left(\mathrm{U}(2)\backslash\mathrm{PGSp}_4\right) = \int_{\widehat{\mathrm{PGSO}_4}^{\mathrm{temp}}} \mathrm{Hom}_{N\times T}\left(\sigma, \psi\boxtimes\mathbb{C}\right)\cdot\theta(\sigma)\mathrm{d}\mu_{\mathrm{PGSO}_4}(\sigma),$$

*where* $\mu_{\mathrm{PGSO}_4}$ *is the Plancherel measure of* $\mathrm{PGSO}_4$*. Moreover, for* $\Phi_1, \Phi_2 \in \Omega$*, then one has*

$$\langle q(\Phi_1), q(\Phi_2)\rangle_{\mathrm{U}(2)\backslash\mathrm{PGSp}_4} = \int_{\widehat{\mathrm{PGSO}_4}^{\mathrm{temp}}} J\left(q(\Phi_1), q(\Phi_2)\right)\mathrm{d}\mu_{\mathrm{PGSO}_4}(\sigma),$$

*where*

$$J\left(q(\Phi_1), q(\Phi_2)\right) := \int_T \int_N^* J_\sigma^\theta\left(t\cdot n\cdot\Phi_1, \Phi_2\right)\overline{\psi(n)}\mathrm{d}n\mathrm{d}t$$

*is a positive semi-definite Hermitian form.*

# Local results

Let $X := (N, \psi) \times T \backslash \mathrm{PGSO}_4$ and $Y := \mathrm{U}(2) \backslash \mathrm{PGSp}_4$.

**Proposition (Commutative diagram)**

*For $\mu_{\mathrm{PGSO}_4}$-almost all $\sigma$ such that*

$$\mathrm{Hom}_{N \times T}(\sigma, \psi) \neq 0,$$

*one has*

$$\alpha_\sigma^Y \circ q(\Phi) = \ell_\sigma^X \circ \theta_\sigma(\Phi)$$

**Proposition**

$$\ell_\sigma^X (B_\sigma(\Phi, w)) = \langle q(\Phi), \beta_\sigma^Y(w) \rangle_Y$$

# Local results

## Theorem

*Suppose that*

- $f_1 \in \mathcal{S}(X)$ *and* $f_2 \in S(Y)$ *are in correspondence, i.e. there exists a* $\Phi \in \Omega$ *such that* $p(\Phi) = f_1$ *and* $q(\Phi) = f_2$*;*
- $\ell_\sigma^X \in \mathrm{Hom}_{N\times T}(\sigma, \psi)$ *is determined by the Bernstein's method and* $\ell_\sigma^Y \in \mathrm{Hom}_{\mathrm{U}(2)}(\theta(\sigma_1 \boxtimes \sigma_2), \mathbb{C})$ *is uniquely determined by the commutative diagram.*

*Then one has the relative character identity*

$$\mathcal{B}_\sigma^X(f_1) = \mathcal{B}_\sigma^Y(f_2).$$

# Global results

Let $\Sigma \cong \otimes_v \Sigma_v \subset \mathcal{A}_0(\mathrm{GSO}(V))$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A})$ with trivial central character. Let $f \in \Sigma$ and $\Phi \in \omega_\psi$. For $g \in \mathrm{GSp}(W)(\mathbb{A})$, choose $h \in \mathrm{GSO}(V)(\mathbb{A})$ such that $\lambda_V(h) = \lambda_W(g)$, and define a $(\mathrm{GSO}(V) \times \mathrm{GSp}(W))_{\lambda_V=\lambda_W}$-equivariant map

$$A_\Sigma^{\mathrm{Aut}} : \omega_\psi \otimes \overline{\Sigma} \longrightarrow \mathcal{A}(\mathrm{GSp}(W))$$

by

$$A_\Sigma^{\mathrm{Aut}}(\Phi, f)(g) = \int_{[\mathrm{SO}(V)]} \theta(\Phi)(h_1 h, g)\overline{f(h_1 h)}\mathrm{d}h_1,$$

where $\mathrm{d}h_1$ is the tamagawa measure of $\mathrm{SO}(V)(\mathbb{A})$.
The image of $A_\Sigma^{\mathrm{Aut}}$ is the global theta lift of $\Sigma$, which we will denote by

$$\Pi = \Theta^{\mathrm{Aut}}(\Sigma) \subset \mathcal{A}(\mathrm{GSp}(W)).$$

Under some mild assumptions, one has $\Pi$ is cuspidal and non-zero, hence $\Pi$ is irreducible cuspidal automorphic representation of $\mathrm{GSp}(W)$ with trivial central character.

# Global results

Conversely, let $\Pi \cong \otimes_v \Sigma_v \subset \mathcal{A}_0\left(\mathrm{GSp}(W)\right)$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GSp}(W)(\mathbb{A})$ with trivial central character. Let $\varphi \in \Pi$ and $\Phi \in \omega_\psi$. For $h \in \mathrm{GSO}(V)(\mathbb{A})$, choose $g \in \mathrm{GSp}(W)(\mathbb{A})$ such that $\lambda_W(g) = \lambda_V(h)$, and define a $(\mathrm{GSO}(V) \times \mathrm{GSp}(W))_{\lambda_V = \lambda_W}$-equivariant map

$$B_\Pi^{\mathrm{Aut}} : \omega_\psi \otimes \overline{\Pi} \longrightarrow \mathcal{A}\left(\mathrm{GSO}(V)\right)$$

by

$$B_\Pi^{\mathrm{Aut}}(\Phi, \varphi)(h) = \int_{[\mathrm{Sp}(W)]} \theta(\Phi)(h, g_1 g)\overline{\varphi(g_1 g)}\mathrm{d}g_1,$$

where $\mathrm{d}g_1$ is the tamagawa measure of $\mathrm{Sp}(W)(\mathbb{A})$.
The image of $B_\Pi^{\mathrm{Aut}}$ is the global theta lift of $\Pi$, which we will denote by

$$\Sigma = \Theta^{\mathrm{Aut}}(\Pi) \subset \mathcal{A}\left(\mathrm{GSO}(V)\right).$$

If $\Sigma$ is cuspidal and non-zero, then it follows $\Sigma$ is irreducible.

# Global results

Consider the map

$$p : \omega_\psi \longrightarrow \mathcal{S}\left(N\backslash \mathrm{SO}(V), \psi\right)$$

given by

$$p(\Phi) = \omega_\psi(h)\Phi(T_1);$$

the map

$$q : \omega_\psi \longrightarrow S\left(\mathrm{SL}_2 \times 1\backslash \mathrm{Sp}(W)\right)$$

given by

$$q(\Phi) = \omega_\psi(g)\Phi(T_1).$$

# Global results

## Proposition

*For $\Phi \in \omega_\psi$ and $f \in \Pi$, then one has*

$$P_{(N,\psi)\times T}(B_\Pi^{\mathrm{Aut}}(\Phi, f)) = \langle q(\Phi), \beta_\Pi^{\mathrm{Y},\mathrm{Aut}}(f)\rangle_{Y_1(\mathbb{A})}.$$

Here,

$$\beta_\Pi^{\mathrm{Y},\mathrm{Aut}}(f) = \int_{[\mathrm{U}(2)]} f(\tau g)\mathrm{d}\tau$$

and

$$Y_1 := \{v \in V_2 \otimes W \mid Q(x) = 1\}.$$

## Proposition

$$P_{\mathrm{U}(2),\Pi}(f) = c_{12}\ell_{\Sigma_{12}}^{Y,\mathbb{A}}(f) + c_{21}\ell_{\Sigma_{21}}^{Y,\mathbb{A}}(f).$$

where $|c_{12}| = |c_{21}| = \frac{1}{2}$.

**Thank you!**