On residues of some intertwining operators (associated to Heisenberg parabolic subgroups)

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Notation

- G - a reductive group over F, $[F : \mathbb{Q}_p] < \infty$.
- $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ maximal parabolic subgroup.
- (π, V_π) - irreducible unitary supercuspidal complex representation of $\mathrm{M}(F) \twoheadleftarrow \mathrm{P}(F)$.
- $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\,\pi$ - on the space

$$\left\{ f : \mathrm{G}(F) \to V_\pi \text{ smooth }, f(pg) = \delta_{\mathrm{P}}(p)^{1/2}\pi(p) \cdot f(g) \quad \forall\, p \in \mathrm{P}(F), g \in \mathrm{G}(F). \right\}$$

Question: When is $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\,\pi$ irreducible?

A necessary condition for reducibility

Recall P is a maximal parabolic subgroup.

Harish-Chandra: $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\,\pi$ is irreducible unless $\exists\, w_0 \in \mathrm{G}(F)$ such that:

- w_0 normalizes $\mathrm{M}(F)$;
- $w_0\,\mathrm{P}\,w_0^{-1} = \mathrm{P}^- =$ opposite parabolic ('P is self-associate');
- ${}^{w_0}\pi \cong \pi$, where ${}^{w_0}\pi := \pi \circ \mathrm{Int}\, w_0^{-1} =$ representation of $\mathrm{M}(F)$.

⚠ But when such a w_0 exists, $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\,\pi$ may or may not be irreducible.

Expectation: Irreducible if and only if (the conjectural in general) $L(s,\pi,r)$ has a pole at $s = 0$, where $r = {}^{L}\mathrm{M} \circlearrowright \mathrm{Lie}\,\hat{\mathrm{N}}$.

Conjecturally, should be related to π coming by functorial transfer from certain 'smaller' groups H.

Partial/crude aim: Assuming such a w_0 exists, prove statements roughly of the form: "Irreducibility $\iff$ π comes from certain 'smaller' H."

Intertwining operators

Henceforth $w_0 \, \mathsf{P} \, w_0^{-1} = \mathsf{P}^-$, ${}^{w_0}\pi \cong \pi$.

π sits as π_0 in a family of representations $\{\pi_s \mid s \in \mathbb{C}\}$:

$\pi = \pi_0 \in \{\pi_s = \pi \otimes \nu^s \mid s \in \mathbb{C}\}$, $\nu : \mathsf{M}(F) \to \mathbb{R}_{>0} \subset \mathbb{C}^\times$ an appropriate character.

$$\mathbb{C} \ni s \mapsto \underbrace{A(s, \pi, w_0)}_{\text{Intertwining operator}} : \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \pi_s \to \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \, {}^{w_0}(\pi_s),$$

$$f \mapsto \left(g \mapsto \int_{\mathsf{N}(F)} f(w_0^{-1} n g) \, dn \right)$$

(Harish-Chandra: converges for $\mathrm{Re}\, s \gg 0$).

Can identify each $\mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \pi_s \cong \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \pi, \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \, {}^{w_0}(\pi_s) \cong \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \, {}^{w_0}\pi$

$\Rightarrow$ $s \mapsto A(s, \pi, w_0) : \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \pi \to \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \, {}^{w_0}\pi$.

Harish-Chandra: meromorphic continuation to all $s \in \mathbb{C}$.

Poles of intertwining operators, $R(\tilde{\pi})$ and irreducibility

Theorem (Harish-Chandra)

(Recall $w_0 \mathrm{P} w_0^{-1} = \mathrm{P}^-$, ${}^{w_0}\pi \cong \pi$). TFAE:

- $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi$ *is irreducible.*
- $s \mapsto A(s, \pi, w_0)$ *has a (necessarily simple) pole at* $s = 0$.

$$\Rightarrow \quad \left(A_{\mathrm{res}}(\pi, w_0) = Res_{s=0} A(s, \pi, w_0) : \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi \to \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\, {}^{w_0}\pi\right) \quad = (0 \text{ or iso.}).$$

OTOH, *choosing* $\tilde{\pi}(w_0)$: ${}^{w_0}\pi \xrightarrow{\cong} \pi$, get: $\ell : \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\, {}^{w_0}\pi \xrightarrow{\cong} \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi$

$$\Rightarrow \quad (\ell \circ A_{\mathrm{res}}(\pi, w_0) : \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi \to \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi) \quad = (0 \text{ or iso.}).$$

$$\Rightarrow \quad \boxed{\ell \circ A_{\mathrm{res}}(\pi, w_0) = (\text{mult. by a scalar, } R(\tilde{\pi})).}$$

Thus, $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi$ is irreducible $\iff R(\tilde{\pi}) \neq 0$.

Aim: When does $R(\tilde{\pi})$ not vanish? What is it when it doesn't vanish?

Note: $R(\tilde{\pi})$ depends on $\tilde{\pi}(w_0)$: ${}^{w_0}\pi \cong \pi \leftrightarrow$ Extn $\tilde{\pi}$ of π to $\mathrm{M}(F) \rtimes \langle w_0 \rangle$.

Prescription for $R(\tilde{\pi})$

Theorem (Harish-Chandra)

(Recall $w_0 \, \mathrm{P} \, w_0^{-1} = \mathrm{P}^-$, ${}^{w_0}\pi \cong \pi$.) TFAE:

- $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \pi$ *is irreducible.*
- $s \mapsto A(s, \pi, w_0)$ *has a pole at* $s = 0$.

Connection to the L-function expectation: Expect $A(s, \pi, w_0)$ has a pole at 0 $\iff$ $\frac{L(s,\pi,r)}{L(1+s,\pi,r)\varepsilon(s,\pi,r,\psi)}$ $\left(\text{rather, } \prod_{i=1}^{m} \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)\varepsilon(is,\pi,r_i,\psi)}\right)$ or eq. some $\gamma(s, \pi, r_i, \psi)^{-1}$ or $L(s, \pi, r_i)$ does.

A more precise statement, called Arthur's local intertwining relation, suggests: If w_0, measure on $\mathrm{N}(F)$, $\tilde{\pi}(w_0)$ normalized well, should have:

$$R(\tilde{\pi}) = (\text{constant}) \cdot \boldsymbol{\lambda}_{w_0}(\psi) \cdot \mathrm{Res}_{s=0} \prod_{i=1}^{m} \frac{L(is,\pi,r_i,\psi)}{L(1+is,\pi,r_i)\varepsilon(is,\pi,r_i,\psi)}$$
$$= (\text{constant}) \cdot \boldsymbol{\lambda}_{w_0}(\psi) \cdot \mathrm{Res}_{s=0} \prod_{i=1}^{m} \gamma(is, \pi, r_i, \psi)^{-1}.$$

Prescription for $R(\tilde{\pi})$ (contd.)

Consequence of Arthur's local intertwining relation: If w_0, measure on $\mathrm{N}(F)$, $\tilde{\pi}(w_0)$ normalized well, should have:

$$\begin{aligned} R(\tilde{\pi}) &= (\text{constant}) \cdot \boldsymbol{\lambda}_{w_0}(\psi) \cdot \mathrm{Res}_{s=0} \prod_{i=1}^{m} \frac{L(is,\pi,r_i,\psi)}{L(1+is,\pi,r_i)\varepsilon(is,\pi,r_i,\psi)} \\ &= (\text{constant}) \cdot \boldsymbol{\lambda}_{w_0}(\psi) \cdot \mathrm{Res}_{s=0} \prod_{i=1}^{m} \gamma(is,\pi,r_i,\psi)^{-1}. \end{aligned}$$

Special case: Everything generic, 'Whittaker normalized' $\Rightarrow$ the 'constant' should be 1 (should follow from local Langlands-Shahidi theory).

Aim: Compute $R(\tilde{\pi})$, interpret in light of the above.

A 'twisted space' $\tilde{\mathrm{M}}$

(Recall $w_0 \, \mathrm{P} \, w_0^{-1} = \mathrm{P}^-$, $w_0 \, \mathrm{M} \, w_0^{-1} = \mathrm{M}$, ${}^{w_0}\pi \cong \pi \Rightarrow$)
π extends from $\mathrm{M}(F)$ to $\langle \mathrm{M}(F), w_0 \rangle \supset \mathrm{M}(F) w_0 =: \tilde{\mathrm{M}}(F)$.

'Twisted space:' $\tilde{\mathrm{M}} := \mathrm{M} \, w_0 =$ the non-identity component of $\langle \mathrm{M}, w_0 \rangle \subset \mathrm{G}$.

So π on $\mathrm{M}(F)$ satisfying ${}^{w_0}\pi \cong \pi \rightsquigarrow$ 'repn' $\tilde{\pi}$ of $\tilde{\mathrm{M}}(F) = \mathrm{M}(F) w_0$, with $\tilde{\pi}(w_0) : {}^{w_0}\pi \to \pi$.

E.g., Arthur's 2013 book studies representations of Sp_{2n} or SO_n by relating them to representations of the twisted space $\mathrm{GL}_N \rtimes \theta \subset \mathrm{GL}_N \rtimes \langle \theta \rangle$, using twisted endoscopy.

Partial aim: Prove statements *roughly* of the form "$R(\tilde{\pi}) \neq 0 \iff \tilde{\pi}$ comes from certain groups H by endoscopic transfer" (needs modification, to reflect some $L(s, \pi, r_i)$ having pole at $s = 0$).

A technique introduced by Shahidi, Duke 1992

Shahidi's idea: Express the residue of the intertwining operator in terms of harmonic analysis on $\tilde{\mathrm{M}} := \mathrm{M}\, w_0 = w_0\, \mathrm{M} = \mathrm{M}\, w_0^{-1}$.

(A lemma of Rallis $\Rightarrow$) Enough to study poles of $\langle \tilde{u}, A(s, \pi, w_0)\tilde{v}\rangle$ where:

- $\tilde{u} \in (\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\ {}^{w_0}\pi)^*$ is of the form

  $$\tilde{u}_u : f \mapsto \langle u \circ \tilde{\pi}(w_0), f(1)\rangle$$

  for some $u \in \pi^*$ (then $u \circ \tilde{\pi}(w_0) \in (\ {}^{w_0}\pi)^*$).
- $\tilde{v} \in \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\pi$ is supported on $\mathrm{P}(F)\,\mathrm{N}^-(F) \cong \mathrm{P}(F) \times \mathrm{N}^-(F)$, of the form

  $$\tilde{v}_{v,\phi^-} : pn^- \mapsto (\pi\delta_{\mathrm{P}}^{1/2})(p)\left[\phi^-(n^-)v\right],$$

  for some $v \in \pi, \phi^- \in C_c^\infty(\mathrm{N}^-(F))$.

Shahidi's technique (contd.) - from N to $\tilde{\mathrm{M}}$

This helps compute $R(\tilde{\pi})$ by:

$$\phi^-(1)R(\tilde{f}) = \operatorname{Res}_{s=0}\langle \tilde{u}_u, A(s,\pi,w_0)\tilde{v}_{v,\phi^-}\rangle = R(\tilde{\pi})\phi^-(1)\langle u, v\rangle.$$

for each $\phi^- \in C_c^\infty(\mathrm{N}^-(F))$ and each $u \otimes v \in \pi^* \otimes \pi \cong$ the space of matrix coefficients for π.

This will involve an integral of the form:

$$\int_{\mathrm{N}'(F)} (\dots)(\underbrace{w_0^{-1} n}_{=m\dot{n}n^-})\, dn$$

where $\mathrm{N} \cap w_0 \mathrm{P}\, \mathrm{N}^- =: \mathrm{N}' \overset{\text{dense}}{\subset}{}^{\text{open}} \mathrm{N}$.

$\Rightarrow$ a map $\mathrm{N}' \to w_0 \mathrm{M} = \tilde{\mathrm{M}}$ given by $n \mapsto w_0 m$, where

$$\underbrace{n}_{\in \mathrm{N}'} = \underbrace{(w_0 m)}_{\in w_0 \mathrm{M} = \tilde{\mathrm{M}}} \underbrace{\dot{n}}_{\in \mathrm{N}} \underbrace{n^-}_{\in \mathrm{N}^-}.$$

Respects M-conjugation.

Shahidi's technique (contd.) - from N to $\tilde{\mathrm{M}}$

Thus, get a map $\mathrm{N}' \to \tilde{\mathrm{M}} = w_0\,\mathrm{M}$ given by $n \mapsto w_0 m$, where

$$\underbrace{n}_{\in \mathrm{N}'} = \underbrace{(w_0 m)}_{\in w_0\,\mathrm{M} = \tilde{\mathrm{M}}} \underbrace{\dot{n}}_{\in \mathrm{N}} \underbrace{n^-}_{\in \mathrm{N}^-}$$

Respects M-conjugation.

Idea: via $\mathrm{N}' \to \tilde{\mathrm{M}}$,

$$\text{transfer } \int_{\mathrm{N}'(F)} \rightsquigarrow \qquad \int_{\text{subset of } \tilde{\mathrm{M}}(F)}$$

to relate the residue to invariant distributions on $\tilde{\mathrm{M}}(F)$

(equivalently, to twisted invariant distributions on $\mathrm{M}(F)$).

The abelian/prehomogeneous case

Simplest cases - when N is abelian (or prehomogeneous) (Shahidi, Duke 1992; Goldberg, Crelle 1994; Shahidi, Compositio 2000).

Simplifying features:

- N (or N′) has a dense open orbit N° under M-conjugation.
- The map $\mathrm{N}' \to \tilde{\mathrm{M}}$ is finite-to-one on N°.

Recall ${}^{w_0}\pi \cong \pi$, so π extends to a representation $\tilde{\pi}$ of $\tilde{\mathrm{M}}(F) \subset \langle \mathrm{M}(F), w_0 \rangle$.

The residue assoc. to $\phi^- \in C_c^\infty(\mathrm{N}^-(F))$, $u \otimes v \in \pi^* \otimes \pi$ gets computed as:

$$(\text{constant multiple of } \phi^-(1)) \cdot \underbrace{\sum_{\text{finite}} a_i \; O(\tilde{t}_i, \tilde{f})}_{R(\tilde{f})},$$

where $\tilde{f}$ is a pseudocoefficient for $\tilde{\pi}$ related to $u \otimes v$.

$$\boxed{\phi^-(1) R(\tilde{f}) = \mathrm{Res}_{s=0} \langle \tilde{u}_u, A(s, \pi, w_0) \tilde{v}_{v,\phi^-} \rangle = R(\tilde{\pi}) \phi^-(1) \langle u, v \rangle.}$$

The abelian/prehomogeneous case (contd.)

Crude aim: this residue is nonzero if and only if $\tilde{\pi}$ 'comes from' an endoscopic group H related to a pole of $L(s, \pi, r)$ at $s = 0$.

'Typically' accomplished by showing: the 'residue distribution'

$$\tilde{f} \mapsto R(\tilde{f}) = (\text{ endoscopic transfer of } \underbrace{f^{\mathrm{H}} \mapsto f^{\mathrm{H}}(1)}_{\text{or some other simple distribution}}).$$

(H = some endoscopic group of $\tilde{\mathrm{M}}$).

In other words: , $\tilde{f} \rightsquigarrow f^{\mathrm{H}}$ endoscopic transfer $\Rightarrow R(\tilde{f}) = (\text{const.})\ f^{\mathrm{H}}(1)$

Alternatively, show:

$$R(\tilde{f}) = (\text{const. depending on } \tilde{f}) \cdot (\text{some expression involving } \Pi),$$

for some 'packet' Π of representations on $\mathrm{H}(F)$ transferring to $\tilde{\pi}$.

$\mathrm{G} = \mathrm{GL}_{2n} \supset \mathrm{GL}_n \times \mathrm{GL}_n = \mathrm{M}$, Hiraga-Ichino-Ikeda: combine with $|R(\tilde{\pi})|$ given by Langlands-Shahidi theory $\Rightarrow$ the formal degree conjecture for GL_n.

Outside the abelian/prehomogeneous cases

- In general, $\mathrm{N}' \to \tilde{\mathrm{M}}$ not generically finite-to-one, fibers should be understood better.
- (Looking at dual side) multiple L-functions $L(s, \pi, r_i)$ are involved, and even more endoscopic groups.

Recall $R(\tilde{f})\phi^-(1) = \mathrm{Res}_{s=0}\langle \tilde{u}_u, A(s,\pi,w_0)\tilde{v}_{v,\phi^-}\rangle = R(\tilde{\pi})\phi^-(1)\langle u, v\rangle$. In many cases expect:

- $R(\tilde{f})$ should be a sum
$$\sum_{\tilde{\mathrm{T}} \in \mathcal{T}} R_{\tilde{\mathrm{T}}}(\tilde{f})$$
over some set of *'twisted' nonmaximal* tori $\tilde{\mathrm{T}} = \mathrm{T} w_0^{-1}$ in $\tilde{\mathrm{M}} = \mathrm{M}\, w_0^{-1}$;
- Obtain $R(\tilde{\pi})$, then rearrange, add-subtract etc. — to decompose:
$$R(\tilde{\pi}) = \sum_{\mathrm{H}} R^{\mathrm{H}}(\tilde{\pi}),$$
each H an endoscopic group 'related to poles of' some $L(s, \pi, r_i)$.

Decompositions of $R(\tilde{f})$

Parts of $R(\tilde{f})$ easier to analyze seem to be:

- $R_{\tilde{T}}$ with T 'as anisotropic as possible', call this the 'elliptic' case (nonstandard terminology).
- Endoscopic contributions $R^{H}(\tilde{\pi})$, where H is related to poles of $L(s, \pi, r_1)$ (not r_2 etc.) — often related to the refined formal degree conjecture.

Some of the work so far outside abelian/prehomogeneous cases:

- Goldberg-Shahidi — the foundation plus study of $R_{\tilde{\mathrm{T}}}(\tilde{f})$ in various 'classical group' cases.
- Spallone - in many 'classical' cases, employed techniques used for local trace formula for absolute convergence and some analysis of $R_{\tilde{\mathrm{T}}}(\tilde{f})$.
- Shahidi-Spallone - Endoscopic interpretation when $(\mathrm{G}, \mathrm{M}) = (\mathrm{SO}_6^*, \mathrm{GL}_2 \times \mathrm{SO}_2^*)$ (all T 'elliptic', but 'r_1' and 'r_2' treated).
- Wen-Wei Li - $(\mathrm{SO}_{6n}, \mathrm{GL}_{2n} \times \mathrm{SO}_{2n})$; studied 'elliptic' $R_{\tilde{\mathrm{T}}}$ to propose and analyze endoscopic contributions associated to '$L(s, \pi, r_1)$'.
- Spallone - $(\mathrm{Sp}_6, \mathrm{GL}_2 \times \mathrm{SL}_2)$ — for many ϕ^-, 'nonelliptic' $R_{\tilde{\mathrm{T}}}$ vanishes.
- L. Cai and B. Xu - $(\mathrm{U}_{3,3}, \mathrm{Res}_{E/F}\,\mathrm{GL}_2 \times \mathrm{U}_{1,1})$ - studied 'elliptic' $R_{\tilde{\mathrm{T}}}$ to analyze endoscopic contributions from '$L(s, \pi, r_1)$', plus 'nonelliptic' $R_{\tilde{\mathrm{T}}}$ vanishes.
- X. Yu - vanishing of 'nonelliptic' $R_{\tilde{\mathrm{T}}}$ for $\mathrm{G} = \mathrm{Sp}, \mathrm{SO}$.

The case of our interest

$\mathrm{G}_{\mathrm{der}}$ absolutely almost simple, of type B_n or D_n $(n \geqslant 4)$, E_6, E_7, E_8, F_4 or G_2.

$\mathrm{P} = \mathrm{M}\,\mathrm{N}$ 'Heisenberg' — throw out the unique simple root β connected to the lowest root $-\gamma$ in the extended Dynkin diagram.

e.g., $(\mathrm{GSpin}_m, \mathrm{GL}_2 \times \mathrm{GSpin}_{m-4})$, and, indicating only root system types, $(E_6, \mathrm{A}_5), (E_7, D_6), (E_8, E_7), (F_4, C_3), (G_2, A_1)$, various inner forms.

Key property: $\mathfrak{g}_{\pm\beta} + \mathfrak{g}_{\pm(\gamma-\beta)} + \mathfrak{g}_{\pm\gamma} \subset \mathfrak{g}$ generate a copy of $\mathfrak{sl}_3 \subset \mathfrak{g}$ (over $\bar{F}$), which belongs to a particularly nice conjugacy class of embeddings $\mathrm{SL}_3 \to \mathrm{G}$.

$\mathrm{M} \circlearrowright \mathrm{Lie}\,\mathrm{N} = \mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, and for $(X_1, X_2) \in \mathfrak{n}_1 \oplus \mathfrak{n}_2$ in 'general position', have unique SL_2-homomorphisms associated to X_1, X_2 adapted to $(\mathrm{N}^-, \mathrm{M}, \mathrm{N})$, and they generate a copy of a quasi-split $\mathrm{SU}_3 \hookrightarrow \mathrm{G}$ in this conjugacy class.

Why does SU_3 help?

Can ensure: the w_0 ‘is a w_0’ for each of these SU_3’s:

$$\begin{array}{ccc} M \circlearrowright N' & \longrightarrow & \tilde{M} \circlearrowleft M \\ \uparrow & & \uparrow \\ T_\varepsilon \subset T \circlearrowright (SU_3 \cap N') & \longrightarrow & (SU_3 \cap \tilde{M}) = \tilde{T} \circlearrowleft T \supset T_\varepsilon \end{array}$$

$$SU_3 \cap N = \text{(maximal unipotent subgroup in } SU_3\text{)},$$
$$SU_3 \cap M =: T = \text{(maximal torus in } SU_3\text{)},$$
$$SU_3 \cap \tilde{M} = SU_3 \cap M\, w_0^{-1} = T\, w_0^{-1} =: \tilde{T}.$$

$$T_\varepsilon := \text{(connected) fixed points of } \varepsilon := \mathrm{Int}\, w_0^{-1} \text{ in } T\ .$$

(The fiber in N' over a ‘generic’ element of $\tilde{T}$) = a torsor for $\underline{T}_\varepsilon$:= the image of T_ε in PU_3.

$R_{\tilde{\mathsf{T}}}(\tilde{f})$ for $\tilde{\mathsf{T}}$ elliptic

The sum in $R(\tilde{f}) = \sum_{\tilde{\mathsf{T}} \in \mathcal{T}} R_{\tilde{\mathsf{T}}}(\tilde{f})$ can be chosen to be over $\tilde{\mathsf{T}} = \mathsf{T}\, w_0^{-1}$ with T as above.

For $\tilde{\mathsf{T}} = \mathsf{T}\, w_0$ with T non-split ('elliptic') get an expression:

$$R_{\tilde{\mathsf{T}}}(\tilde{f}) = \text{(roughly)} \int_{\substack{\mathsf{T}\text{-conj cl} \\ \text{in } \tilde{\mathsf{T}}(F)}} \text{(disc. factor)(orbital integral)}$$

$$R_{\tilde{\mathsf{T}}}(\tilde{f}) = \text{(const.)} \int_{\mathsf{T}_\varepsilon(F)} D(\tilde{S}(t))^{1/2} \int_{z \in \sim \backslash \mathsf{Z}_\mathsf{M}(F)} \chi_\pi^{-1}(z) O(z\tilde{S}(t), \tilde{f})\, dz\, dt$$

$$= \text{(const. depending on } \tilde{f}) \int_{\mathsf{T}_\varepsilon(F)} D(\tilde{S}(t))^{1/2} c_{\tilde{\pi},0}(\tilde{S}(t))\, dt,$$

$c_{\tilde{\pi},0}(\tilde{S}(t))$ = coefficient of 0 orbit in the char. expansion of $\tilde{\pi}$ about $\tilde{S}(t)$.

The case of $\tilde{\mathsf{T}} = \mathsf{T}\,w_0$ non-elliptic (i.e., T split).

As we saw (by X. Yu) the $R_{\tilde{\mathsf{T}}}(\tilde{f})$ (T split) vanished in the Sp and SO cases.

Expression for $R_{\tilde{\mathsf{T}}}(\tilde{f})$ when T split:

$$\begin{cases} 0, & \text{'classical cases'} \\ \sim O(z\tilde{S}(t), \tilde{f}) \rightsquigarrow O^{\tilde{\mathsf{M}}}_{\tilde{\mathsf{M}}_{\mathsf{T}}}(z\tilde{S}(t), \tilde{f}) \text{ in earlier expression,} & \text{'exceptional cases'} \end{cases}$$

(weighted orbital integral in the sense of Arthur, D_4 with triality considered exceptional).

Thus, in exceptional cases, easy consequence of twisted local trace formula:

$$R_{\tilde{\mathsf{T}}}(\tilde{f}) = (\text{const depending on } \tilde{f}, \mathsf{T}) \cdot \int_{\mathsf{T}_\varepsilon(F)} D(\tilde{S}(t))^{1/2} c_{\tilde{\pi},0}(\tilde{S}(t))\, dt.$$

Endoscopic decomposition of $R(\tilde{\pi})$

In each case, can define $\omega : F^\times \to \mathbb{C}^\times$, from central character of π, via $\gamma^\vee : \mathbb{G}_m \to \mathrm{M}$.

From ${}^{w_0}\pi \cong \pi$, can get $\omega^2 = \mathbb{1}$, so $\omega = \mathbb{1}$ or $\omega = \mathrm{sgn}_{E/F}$.

Can decompose

$$R(\tilde{\pi}) = \sum_{\mathrm{H}} R^{\mathrm{H}}(\tilde{\pi}),$$

where each H is (at least in many cases) an endoscopic datum determined by a possibility for ω.

If π has central character ω corresponding to H, then $R(\tilde{\pi}) = R^{\mathrm{H}}(\tilde{\pi})$.

The example of $(\mathsf{G},\mathsf{M}) = (\mathsf{G}_2, \mathsf{GL}_2)$; $\tilde{\mathsf{M}} = \mathsf{GL}_2 \rtimes \theta$

Here $\omega : F^\times \to \mathbb{C}^\times$ is the central character of π. ${}^{w_0}\pi \cong \pi \Rightarrow \omega^2 = \mathbb{1}$.

$\tilde{\pi}$ is a transfer from a unique endoscopic group H depending on ω, and $R(\tilde{\pi}) = R^{\mathsf{H}}(\tilde{\pi})$.

- If $\omega = \mathrm{sgn}_{E/F}$, E/F quadratic, $\mathsf{H} = \mathsf{SO}_2^*$, $\mathsf{H}(F) = E^1$. Associated to $L(s, \pi, r_1)$.
- If $\omega = \mathbb{1}$, $\mathsf{H} = \mathsf{SO}_3 = \mathsf{PGL}_2$. Accounts for $L(s, \pi, r_2)$.

$R_{\tilde{\mathsf{T}}}(\tilde{f})$ equals:

$$(\text{const. depending on } \tilde{f}) \cdot (1 \text{ or } 3) \int_{\mathsf{T}_\varepsilon(F)} D(\tilde{S}(t))^{1/2} \Theta_{\tilde{\pi}}(\tilde{S}(t))\, dt.$$

Case 1. $\omega = \mathrm{sgn}_{E/F}$, so $\mathsf{H}(F) = E^1$. Have unique $\mathsf{T} \cong \mathrm{Res}_{E/F}\, \mathbb{G}_m$:

$$R(\tilde{\pi}) = R^{\mathsf{H}}(\tilde{\pi}) = (\text{const.}) \int_{\mathsf{T}_\varepsilon(F)} D(\tilde{S}(t))^{1/2} \cdot \frac{\Theta_{\tilde{\pi}}(\tilde{S}(t)) - \Theta_{\tilde{\pi}}(\tilde{S}'(t))}{2}\, dt,$$

$= (\text{const.}) \int_{E^1} \lambda(t)\, dt$, if $\tilde{\pi}$ a transfer of $\lambda : \mathsf{H}(F) = E^1 \to \mathbb{C}^\times$.

The example of $(\mathsf{G},\mathsf{M}) = (\mathsf{G}_2, \mathsf{GL}_2)$ (contd.)

$\int_{E^1} \lambda(t)\, dt = 0$ unless λ is trivial, which $\not\rightsquigarrow$ supercuspidal. Recover:

Corollary (Consequence of Shahidi's work ∼1990)

If π has nontrivial central character, $R(\tilde\pi) = 0$, so $\mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} \pi$ is reducible.

Case 2. $\omega = \mathbb{1}$, so $\mathsf{H} = \mathsf{SO}_3 = \mathsf{PGL}_2$. If $\pi =$ inflatn of σ from $\mathsf{SO}_3(F)$, get:

$$R(\tilde\pi) = (\text{const.}) \left(\sum_{\mathsf{T}} \int_{\mathsf{T}(F)} D(t)^{1/2} \Theta_\sigma(t)\, dt + 3 \int_{\mathsf{T}_s(F)} D(t)^{1/2} \Theta_\sigma(t)\, dt \right),$$

where T runs over conjugacy classes of anisotropic maximal tori of SO_3, $\mathsf{T}_s =$ split maximal torus of SO_3.

Then earlier expectation on $R(\tilde\pi)$ in terms of gamma factors suggests:

$$\gamma(0, \sigma, \mathrm{std}, \psi)^{-2} = \frac{1}{4d(\sigma)} \left(\sum_{\mathsf{T}} \int_{\mathsf{T}(F)} D(t)^{1/2} \Theta_\sigma(t)\, dt + 3 \int_{\mathsf{T}_s(F)} D(t)^{1/2} \Theta_\sigma(t)\, dt \right).$$

The case of B_3

$\mathrm{G} \cong \mathrm{Spin}_7$ (split for simplicity), so:

$$\mathrm{M} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det g_1 = \det g_2\} \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2 =: \mathrm{M}^\diamond(F).$$

Choose $\pi_1 \otimes \pi_2$ on $\mathrm{M}^\diamond(F)$ restricting to π on $\mathrm{M}(F)$, have:

$$\omega = \omega_{\pi_1}\omega_{\pi_2} : F^\times \to \mathbb{C}^\times \qquad (\omega = \mathbb{1} \text{ or some } \mathrm{sgn}_{E/F}).$$

Case 1. $\omega = \mathbb{1}$ (accounts for a pole of $L(s, \pi, r_1)$).

Schur orthogonality relations and the refined formal degree conjecture $\Rightarrow$ recover the expected formula (Shahidi) for $R(\tilde{\pi})$:

$$R(\tilde{\pi}) = \mathrm{Res}_{s=0}\left(\gamma(s, \pi_1 \times \pi_2, \psi)^{-1}\gamma(2s, \pi_1, \mathrm{Ad}, \psi)^{-1}\right).$$

In particular, $\mathrm{Ind}_\mathrm{P}^\mathrm{G}\,\pi$ irreducible if and only if $\pi_1 \cong \pi_2^\vee$.

The case of B_3 (contd.)

Case 2. $\omega = \mathrm{sgn}_{E/F}$ (accounts for a pole of $L(s, \pi, r_2)$).

Labesse-Langlands character identities, refined formal degree conjecture and the expected formula for $R(\tilde{\pi})$ together suggest:

σ supercuspidal repn. of $\mathrm{GL}_2(F)$ and $\theta : E^\times \to \mathbb{C}^\times$ with $\theta|_{F^\times} = \omega_\sigma^{-1}$, then:

$$\gamma(0, \sigma_E \otimes \theta, \psi_E)^{-1} = \frac{\omega_\sigma(-1)}{2d(\sigma)} \int_{\mathsf{T}(F)/F^\times} D(t)^{1/2} \Theta_\sigma(t) \theta(t)\, dt$$

where $\mathsf{T} \subset \mathrm{GL}_2, \mathsf{T}(F) = E^\times$.

The case of D_4

G absolutely almost simple, simply connected, of type D_4, say with triality.

$$\mathsf{M}(F) \cong \{g \in \mathsf{GL}_2(K) \mid \det g \in F^\times\} \subset \mathsf{GL}_2(K) = \mathsf{M}^\diamond(F).$$

Choosing $\pi^\diamond =$ repn of $\mathsf{GL}_2(K)$ restricting to π on $\mathsf{M}(F)$, have:

$$\omega = \omega_{\pi^\diamond}|_{F^\times} \qquad (\omega = \mathbb{1} \text{ or some } \mathrm{sgn}_{E/F}).$$

Case 1. $\omega = \mathrm{sgn}_{E/F}$ (accounts for a pole of $L(s, \pi, r_1)$).

Use Labesse-Langlands character identities and the refined formal degree conjecture to recover/get:

$$R(\tilde{\pi}) = \pm \boldsymbol{\lambda}(K/F, \psi)^2 \operatorname{Res}_{s=0}(\gamma(s, \pi, r_1, \psi)^{-1} \gamma(2s, \pi, r_2, \psi)^{-1}),$$

where the '$\pm$' is '$+$' for appropriate Whittaker normalization.

In particular (Shahidi + Henniart-Lomeli + Luo) — $\mathrm{Ind}_\mathsf{P}^\mathsf{G} \pi$ is irreducible if and only if $\pi^\diamond$ is a Weil representation associated to some $\theta : (EK)^\times \to \mathbb{C}^\times$, such that $\theta|_{E^\times} \equiv 1$.

The case of D_4 (contd.)

Case 2. $\omega = \mathbb{1}$ (accounts for a pole of $L(s, \pi, r_2)$).

Expected expression for $R(\tilde{\pi})$ suggests, for $\pi^\diamond$ a supercuspidal representation of $\mathrm{GL}_2(K)$ with $\omega_{\pi^\diamond}|_{F^\times} = \mathbb{1}$:

$$\gamma(0, \pi^\diamond, \mathrm{Asai}, \psi)^{-1} = \frac{\boldsymbol{\lambda}(K/F, \psi)^{-2}}{4d(\pi^\diamond)} \cdot$$

$$\left(\sum_{\mathsf{T}} \mathfrak{v}_{\mathsf{T}} \int_{\mathsf{T}(F)/F^\times} D(t)^{1/2} \Theta_{\pi^\diamond}(t)\, dt + 3\mathfrak{v}_{\mathsf{T}_s} \int_{\mathsf{T}_s(F)/F^\times} D(t)^{1/2} \Theta_{\pi^\diamond}(t)\, dt \right).$$