Ext-vanishing result for Gan-Gross-Prasad model

Rui Chen

June 21, 2023

Branching law

F: p-adic field, G: group over F, and H: closed subgroup of G.

Branching law: would like to study:

$$\mathrm{Hom}_H(\pi, \nu),$$

where π: irred rep of G; ν: character (or some special rep) of H.

Examples: (B = Bessel case):
$(G, H, \nu) = (\mathrm{SO}_n \times \mathrm{SO}_{n+1}, \mathrm{SO}_n, \mathbb{C})$ or $(\mathrm{U}_n \times \mathrm{U}_{n+1}, \mathrm{U}_n, \mathbb{C})$.

(FJ = Fourier-Jacobi case): $(G, H, \nu) = (\mathrm{Sp}_{2n} \times \widetilde{\mathrm{Sp}}_{2n}, \widetilde{\mathrm{Sp}}_{2n}, \omega_{\psi_F})$ or $(\mathrm{U}_n \times \mathrm{U}_n, \mathrm{U}_n, \omega_{\psi_F,\mu})$.

Local Gan-Gross-Prasad conjecture

The local GGP conjecture asserts:

(1) **multiplicity one in generic L-packet**: for generic L-parameter ϕ of G, $\exists ! \ \pi$ in Vogan L-packet Π_ϕ, s.t.

$$\mathrm{Hom}_H(\pi, \nu) \neq 0,$$

hence of dimension 1;

(2) **recipe of π given by explicit epsilon factors**.

It was proved:
(B): local trace formula (Waldspurger, Beuzart-Plessis, Zhilin Luo, Hang Xue, Cheng Chen, ...)
(FJ): theta correspondence (Gan-Ichino, Atobe, Hang Xue...)

Geometric multiplicity

Waldspurger's *geometric multiplicity formula*:

$$m_{geo}(\pi,\sigma) = \sum_{T\in\mathcal{T}} |W(H,T)|^{-1} \int_T c_\pi(t)c_\sigma(t)D^H(t)\Delta(t)dt$$

for admissible π of SO_{n+1} and σ of SO_n. He showed that:

- this integral is absolutely convergent;
- if both π and σ are irred tempered, or at least one of them is s.c., then

$$\dim \mathrm{Hom}_H(\pi \boxtimes \sigma, \nu) = m_{geo}(\pi,\sigma).$$

Question: What is $m_{geo}(\pi,\sigma)$ when π or σ is non-tempered?

Observation

Observe that the geometric multiplicity behaves nicely under the short exact sequence, in the following sense:

If there is a short exact sequence

$$0 \longrightarrow \pi' \longrightarrow \pi \longrightarrow \pi'' \longrightarrow 0,$$

then

$$m_{geo}(\pi, \sigma) = m_{geo}(\pi', \sigma) + m_{geo}(\pi'', \sigma).$$

Likewise, if there is a short exact sequence

$$0 \longrightarrow \sigma' \longrightarrow \sigma \longrightarrow \sigma'' \longrightarrow 0,$$

then

$$m_{geo}(\pi, \sigma) = m_{geo}(\pi, \sigma') + m_{geo}(\pi, \sigma'').$$

Homological branching law (a.k.a. Higher branching law)

Prasad's idea: think of the branching as a functor

$$\mathrm{Hom}_H(-,\nu) : \mathcal{R}(G) \longrightarrow \mathrm{Vect}_{\mathbb{C}},$$

consider its *derived functors* & *Euler-Poincaré characteristic*:

$$\mathrm{EP}_H(-,\nu) = \sum_i (-1)^i \dim \mathrm{Ext}^i_H(-,\nu).$$

Theorem (Bernstein)

If $i > \mathrm{rank}\, H$, then $\mathrm{Ext}^i_H(\pi,\nu) = 0$ for any smooth $\pi \in \mathcal{R}(G)$.

First task: Check that $\mathrm{EP}_H(-,\nu)$ is well-defined for π admissible rep of G, i.e. $\mathrm{Ext}^i_H(\pi,\nu)$ are finite dimensional.

Finite dimensionality

Theorem (Prasad)

For (B), all Ext-groups

$$\mathrm{Ext}^i_H(\pi, \nu)$$

of admissible reps π of G are finite dimensional. Therefore $\mathrm{EP}_H(\pi, \nu)$ is well-defined.

Relevant result:

Aizenbud-Sayag: Homological multiplicities in representation theory of p-adic groups.

Wen-Wei Li: Higher localizations and higher branching laws.

Conjectures of Prasad

Conjecture (Prasad)

1. *For any admissible reps* π *of* SO_{n+1} *and* σ *of* SO_n*, we have*

$$\mathrm{EP}_H(\pi \boxtimes \sigma, \nu) = m_{geo}(\pi, \sigma).$$

2. *If both* π *and* σ *are tempered, then for any* $i > 0$ *we have*

$$\mathrm{Ext}^i_H(\pi \boxtimes \sigma, \nu) = 0.$$

Recently in his IHES lecture note, Prasad proved: (2) implies (1).

Our result: (2) holds (arXiv:2303.12619).

Sketch of the proof of (2) implies (1)

Let π be a standard module of SO_{n+1}, namely

$$\pi = \tau_1|\cdot|^{s_1} \times \cdots \times \tau_r|\cdot|^{s_r} \rtimes \pi_0,$$

where τ_i tempered rep of GL, π_0 tempered rep of SO_{2m+1}, and

$$s_1 > \cdots > s_r > 0.$$

Likewise, let σ be a standard module of SO_{2n}. Mimic an argument of Mœglin-Waldspurger, can show:

- $m_{geo}(\pi, \sigma) = m_{geo}(\pi_0, \sigma_0)$;
- $\mathrm{Ext}^i_H(\pi, \sigma) \simeq \mathrm{Ext}^i_{Bes}(\pi_0, \sigma_0)$ for any $i \geq 0$.

Combining these get (1) for standard modules. Then use bilinear property of both sides.

Remarks for General linear groups

For GL an EP-formula has been established by Prasad himself:

Theorem (Prasad)

Let π *be an admissible rep of* GL_{n+1} *and* σ *of* GL_n. *Then*

$$\mathrm{EP}_{\mathrm{GL}_n}(\pi,\sigma) = \dim \mathrm{Wh}(\pi)\cdot \dim \mathrm{Wh}(\sigma).$$

The similar Ext-vanishing result has been established by Chan:

Theorem (K. Y. Chan)

Let π *be a generic rep of* GL_{n+1} *and* σ *of* GL_n. *Then*

$$\mathrm{Ext}^i_{\mathrm{GL}_n}(\pi,\sigma) = 0 \quad \textit{for any } i > 0.$$

Idea of the proof of (2)

Idea: Embed tempered representations into some suitably chosen "*acyclic*" representations, and then using the standard dim shifting.

In this talk we shall use an example to illustrate the proof. Let:

- $G_n = \mathrm{SO}_{n+1,n}$, and π_n the unique irred subrep of

$$|\cdot|^{n-\frac{1}{2}} \times |\cdot|^{n-\frac{3}{2}} \times \cdots \times |\cdot|^{\frac{1}{2}} \rtimes \mathbb{1}_{\mathrm{SO}_1};$$

  π_n is a d.s. (with L-parameter S_{2n});
- $H_n = \mathrm{SO}_{n,n}$, and σ_n the unique irred subrep of

$$|\cdot|^{n-1} \times |\cdot|^{n-2} \times \cdots \times |\cdot|^{1} \rtimes \sigma_1,$$

  where σ_1 is the trivial character of $\mathrm{SO}_{1,1} \simeq F^\times$. σ_n is a d.s. when $n > 1$ (with L-parameter $\mathbb{1} + S_{2n-1}$).

Goal: Show that $\mathrm{Ext}^i_{H_n}(\pi_n, \sigma_n) = 0$ for any $i > 0$.

Example

Step 1: Let χ: an unitary character of GL_1, *not unramified*. Using the Mackey theory easy to see

$$\mathrm{Ext}^i_{H_n}(\pi_n, \sigma_n) \simeq \mathrm{Ext}^i_{G_n}(\chi \rtimes \sigma_n, \pi_n) \quad \text{for any } i > 0.$$

Step 2: Note that there is an exact sequence

$$0 \longrightarrow \chi \rtimes \sigma_n \longrightarrow |\cdot|^{n-1} \rtimes (\chi \rtimes \sigma_{n-1}) \longrightarrow \chi \rtimes K_n \longrightarrow 0,$$

where K_n is the unique irred quotient of $|\cdot|^{n-1} \rtimes \sigma_{n-1}$. If one can show that

$$\mathrm{Ext}^i_{G_n}(|\cdot|^{n-1} \rtimes (\chi \rtimes \sigma_{n-1}), \pi_n) = 0$$

for any $i > 0$, then applying the functor $\mathrm{Hom}_{G_n}(-, \pi_n)$, one gets

$$\mathrm{Ext}^i_{G_n}(\chi \rtimes \sigma_n, \pi_n) \simeq \mathrm{Ext}^{i+1}_{G_n}(\chi \rtimes K_n, \pi_n) \quad \text{for any } i > 0.$$

Example

Step 2 (continued): On the other hand, using the MVW and contragredient functor one gets

$$0 \longrightarrow \chi \rtimes K_n \longrightarrow |\cdot|^{1-n} \rtimes (\chi \rtimes \sigma_{n-1}) \longrightarrow \chi \rtimes \sigma_n \longrightarrow 0.$$

If one can also show that

$$\mathrm{Ext}^i_{G_n}(|\cdot|^{1-n} \rtimes (\chi \rtimes \sigma_{n-1}), \pi_n) = 0$$

for any $i > 0$, then the functor $\mathrm{Hom}_{H_n}(-, \sigma_n)$ yeilds

$$\mathrm{Ext}^i_{G_n}(\chi \rtimes K_n, \pi_n) \simeq \mathrm{Ext}^{i+1}_{G_n}(\chi \rtimes \sigma_n, \pi_n) \quad \text{for any } i > 0.$$

These imply that $\{\mathrm{Ext}^i_{G_n}(\chi \rtimes \sigma_n, \pi_n)\}_{i>0}$ is periodic, hence vanish.

Example

Step 3: Let $x \in \{n-1, 1-n\}$. As explicated in Step 2, it suffices to show that

$$\mathrm{Ext}^i_{G_n}(|\cdot|^x \rtimes (\chi \rtimes \sigma_{n-1}), \pi_n) = 0$$

for any $i > 0$. Analyze it using the Mackey theory:

- contribution of the closed orbit:

$$\mathrm{Ext}^i_{G_n}(|\cdot|^{x+\frac{1}{2}} \rtimes (\chi \rtimes \sigma_{n-1} \mid_{\mathrm{SO}_{2n-1}}), \pi_n),$$

  which is zero by computing Jacquet modules of π_n;
- contribution of the open orbit:

$$\mathrm{Ext}^i_{H_n}(\pi_n, \chi \rtimes \sigma_{n-1}).$$

Thus the Goal is reduced to

$$\mathrm{Ext}^i_{H_n}(\pi_n, \chi \rtimes \sigma_{n-1}) = 0 \quad \text{for any } i > 0.$$

Example

Step 4: Repeating Step 1–3. Eventually the Goal is reduced to

$$\mathrm{Ext}^i_{H_n}(\pi_n, Ps_{2n,\chi}) = 0 \quad \text{for any } i > 0,$$

where $Ps_{2n,\chi} = \chi \times \cdots \times \chi \rtimes \mathbb{1}_{\mathrm{SO}_0}$ is an unitary p.s. of H_n.

Step 5: Note that there exists an exact sequence

$$0 \longrightarrow \pi_n \longrightarrow |\cdot|^{n-\frac{1}{2}} \rtimes \pi_{n-1} \longrightarrow Q_n \longrightarrow 0,$$

where Q_n is the unique irred quotient of $|\cdot|^{n-\frac{1}{2}} \rtimes \pi_{n-1}$.
Repeating Step 1–4. Eventually the Goal is reduced to

$$\mathrm{Ext}^i_{H_n}(Ps_{2n+1,\mu}, Ps_{2n,\chi}) = 0 \quad \text{for any } i > 0,$$

where $Ps_{2n+1,\mu} = \mu \times \cdots \times \mu \rtimes \mathbb{1}_{\mathrm{SO}_1}$ is an unitary p.s. of G_n.

Example

Step 6: The desired conlusion for unitary p.s. can be shown easily using the Mackey theory:

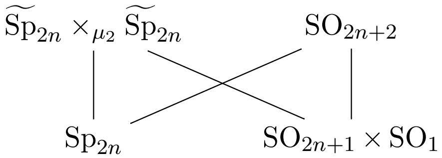
$$\begin{aligned}\mathrm{Ext}^i_{H_n}(Ps_{2n+1,\mu}, Ps_{2n,\chi}) &\simeq \mathrm{Ext}^i_{G_{n-1}}(Ps_{2n,\chi^\vee}, Ps_{2n-1,\mu^\vee})\\ &\simeq \mathrm{Ext}^i_{H_{n-1}}(Ps_{2n-1,\mu}, Ps_{2n-2,\chi})\\ &\cdots\\ &\simeq \mathrm{Ext}^i_{H_0}(Ps_{1,\mu}, Ps_{0,\chi}) = 0.\end{aligned}$$

This completes the proof of the Goal.

Fourier-Jacobi case

One can also consider the Ext-analogue for (FJ).

Recall the proof of (FJ): following Gan-Ichino, Atobe considered:



,

the associated seesaw identity reads:

$$\mathrm{Hom}_{\mathrm{Sp}_{2n}}(\Theta_{\psi_F}(\sigma) \boxtimes \omega_{\psi_F{}^{-1}}, \pi) \simeq \mathrm{Hom}_{\mathrm{SO}_{2n+1}}(\Theta_{\psi_F}(\pi), \sigma)$$

for irred reps π of Sp_{2n} and σ of SO_{2n+1}.

Fourier-Jacobi case

In the Ext-setting:

- one can replace the seesaw identity by two spectral sequences, both convergent to the same thing;
- when π and σ are *tempered*, one can show these spectral sequences degenerate at E_2-pages.

Upshot: if π and σ are tempered, then for any $i \geq 0$

$$\mathrm{Ext}^i_{\mathrm{Sp}_{2n}}(\Theta_{\psi_F}(\sigma) \boxtimes \omega_{\psi_F{}^{-1}}, \pi) \simeq \mathrm{Ext}^i_{\mathrm{SO}_{2n+1}}(\Theta_{\psi_F}(\pi), \sigma)$$

Theorem

Let $(G, H, \nu) = (\mathrm{Sp}_{2n} \times \widetilde{\mathrm{Sp}}_{2n}, \widetilde{\mathrm{Sp}}_{2n}, \omega_{\psi_F})$, and $\widetilde{\pi}$ a tempered rep of G. Then for any $i > 0$ we have

$$\mathrm{Ext}^i_H(\widetilde{\pi}, \nu) = 0.$$

Twisted Fourier-Jacobi case

One can also consider the Ext-analogue for twisted FJ.

Biquadratic extension:

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ F & \longrightarrow & E \end{array}$$

Let W be a Hermitian space w.r.t. E/F. One has

$$\mathrm{U}(W) = \mathrm{U}(W)(F) \hookrightarrow \mathrm{U}(W_K) = \mathrm{U}(W)(K).$$

The twisted GGP problem considers

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi, \omega_{\psi_F,\mu}) \quad \text{for } \pi \in \mathrm{Irr}\,\mathrm{U}(W_K). \qquad \text{(TFJ)}$$

Twisted Fourier-Jacobi case

- When $E = K$: using the similar argument, G-G-P showed that for any tempered rep Π of $\mathrm{U}(W_K) \simeq \mathrm{GL}(W)$,

$$\mathrm{Ext}^i_{\mathrm{U}(W)}(\Pi, \omega_{\psi_F,\mu}) = 0 \quad \text{for any } i > 0.$$

- When $E \neq K$: *still working in progress...*

Conjecture for spherical varieties

Let (G, H): spherical pair, $X = G/H$, and χ: character of H.

X is said to be *tempered* (resp. *strongly tempered*), if all the matrix coefficients of *d.s.* (resp. *tempered*) reps are integrable over $H/A_{G,H}$.

Conjecture (Ext-vanishing)

Suppose that X is tempered (resp. strongly tempered), and $A_{G,H} = 1$. Then for any d.s. (resp. tempered) rep π of G, we have

$$\mathrm{Ext}^i_H(\pi, \chi) = 0.$$

According to Prasad, this is some kind of "Kodaira vanishing thm" for spherical varieties.

Conjecture for spherical varieties

Moreover, in some recent work of C. Wan and Wan-Zhang, they defined the geometric multiplicity $m_{X,geo}(\pi, \chi)$, and showed that

$$\dim \mathrm{Hom}_H(\pi, \chi) = m_{X,geo}(\pi, \chi)$$

for tempered rep π in the strongly tempered case.

Conjecture (EP-formula)

One can properly extend the definition $m_{X,geo}(-, \chi)$ to all admissible rep π of G, such that

$$\mathrm{EP}_H(\pi, \chi) = m_{X,geo}(\pi, \chi).$$

According to Prasad, this is some kind of "Riemann-Roch thm" for spherical varieties.

Thank you for your attention!