Prehomogeneous zeta functions and toric periods for inner forms of $\mathrm{GL}(2)$

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Periods of automorphic representations (1/2)

- G : **reductive algebraic group over a number field** F
- π : **automorphic representation of** $G(\mathbb{A}_F)$
- H : **subgroup of** G

We say that π **is** <u>H**-distinguished**</u> **if**

$$\exists\ \phi \in \pi \quad \textbf{s.t. } \mathcal{P}_H(\phi) := \int_{H(F)\backslash H(\mathbb{A}_F)} \phi(h)\mathrm{d}h \neq 0.$$

Haar measure

This integral is called a <u>**period**</u> **with respect to** H**.**

Periods of automorphic representations (2/2)

Periods are closely related to analytic properties of automorphic L-functions of π.

In particular, there are many examples like

- $\exists$ **non-vanishing periods** $\Longrightarrow L(1/2, \pi) \neq 0$
- $\exists$ **non-vanishing periods** $\Longleftrightarrow L(s, \pi)$ **has a pole**

Toric periods (1/2)

- E/F **: quadratic extension of number fields**
- D **: quaternion algebra over** F **s.t.** $E \hookrightarrow D$
- π **: cuspidal automorphic representation of** $D^\times_{\mathbb{A}_F}$

We say that π **is** **$E^\times$-distinguished** **if**

$$\exists\ \phi \in \pi \quad \textbf{s.t. } \mathcal{P}_E(\phi) := \int_{E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times} \phi(h)\mathrm{d}h \neq 0.$$

This integral is called a **toric period** **with respect to** E.

Toric periods (2/2)

- $\chi_E : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$ **: the unique quadratic character s.t.** $\mathrm{Ker}(\chi_E) = \mathrm{N}_{E/F}(\mathbb{A}_E^\times)$**.**

Then,

$$\begin{array}{c} \pi \text{ : } E^\times\text{-distinguished} \\ \& \ \dim \pi \neq 1 \end{array} \quad \Rightarrow \quad L(1/2, \pi \otimes \chi_E) \neq 0.$$

This is a reuslt of Waldspurger.

Main result

- S **: finite set of places of** F

- π **: irreducible cuspidal automorphic representation of** $D_{\mathbb{A}}^{\times}$ **s.t.** $\dim \pi \neq 1$

Theorem

Suppose we have $L(1/2, \pi) \neq 0$.

Then $\exists\ \mathcal{E}_v$: quadratic semi-simple algebra over F_v for $v \in S$

$$\text{s.t.} \quad \# \left\{ E/F : \text{ quad. ext.} \;\middle|\; \begin{array}{l} (1)\, E \overset{\exists}{\hookrightarrow} D \\ (2)\, E_v = \mathcal{E}_v \quad \forall\, v \in S \\ (3)\, \pi \text{ is } E^{\times}\text{-distinguished} \end{array} \right\} = \infty$$

The main result is motivated by the remarkable results of Waldspurger in 1980's about:

- **Shimura correspondence in the framework of autormorphic representations of Mp_2 ;**
- **non-vanishing of central L-values ;**
- **non-vanishing of toric periods.**

Waldspurger's result (1/3)

For $\xi \in F^\times$**,**

- $E = E_\xi = F[X]/(X^2 - \xi)$

  : the associated quadratic algebra over F**.**

- $\chi_\xi = \chi_E : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$ **: the unique quadratic character s.t.** $\mathrm{Ker}(\chi_\xi) = \mathrm{N}_{E/F}(\mathbb{A}_E^\times)$**.**

In particular, $\chi_\xi = 1$ **iff** $E_\xi \simeq F \times F$**.**

Waldspurger's result (2/3)

Suppose that $D = \mathrm{Mat}_2(F)$**.**

- π **: irreducible cuspidal automorphic representation of** $D_{\mathbb{A}_F}^\times = \mathrm{GL}_2(\mathbb{A}_F)$
- $\delta_v \in \mathbb{R}_{>0}$ **for each** $v \in S$

Theorem (Waldspurger '91)

If $\varepsilon(1/2, \pi) = 1$, then

$$\exists\, \xi \in F^\times \text{ s.t. } \begin{cases} \bullet \quad |\xi - 1|_v < \delta_v \quad \forall v \in S; \\ \bullet \quad L(1/2, \pi \otimes \chi_\xi) \neq 0. \end{cases}$$

Waldspurger's result (3/3)

- **This theorem plays an important role in the description of the discrete spectrum of $L^2_{\mathrm{disc}}(\mathrm{Mp}_2(F)\backslash\mathrm{Mp}_2(\mathbb{A}_F))$.**
- **The original proof is based on the representation theory of $\widetilde{\mathrm{GL}}_2$, which was studied by Flicker using the trace formula.**
- **In 1995, Friedberg and Hoffstein obtained more general results by using the technique of analytic number theory.**

result of Friedberg-Hoffstein (1/2)

- E_0 **: quadratic extension of** F
- π **: irreducible cuspidal automorphic representation of** $\mathrm{GL}_2(\mathbb{A}_F)$

Theorem

If $\varepsilon(1/2, \pi \otimes \chi_{E_0}) = 1$, then

$$\exists \infty\ E/F : \text{quad. extension} \quad \text{s.t.} \begin{cases} \bullet \quad E_v = E_{0,v} \quad \forall v \in S \\ \bullet \quad L(1/2, \pi \otimes \chi_E) \neq 0. \end{cases}$$

result of Friedberg-Hoffstein (2/2)

For the proof, they used a multi-variable Dirichlet series (roughly) of the form :

$$\Phi(s,z,\pi) = \sum_{E} \sum_{\mathfrak{a}} \frac{L(z,\pi \otimes \chi_E)}{\mathrm{N}(\mathfrak{a})^s} \times (\textbf{some factor}).$$

- E **: quadratic extension of** F
- $\mathfrak{a}$ **: certain ideal class of** F**.**

The above theorem is obtained by analyzing poles at $s=1$**.**

- X **: a set**
- $c(x) \in \mathbb{C}$**,** $x \in X$

Q. **How can we prove** $\#\{x \in X \mid c(x) \neq 0\} = \infty$ **?**

A technique of analytic number theory

Study the poles of certain Dirichlet series

$$\Phi(X, s) = \sum_{x \in X} \sum_{\mathfrak{a}} \frac{c(x)}{\mathrm{N}(\mathfrak{a})^s} \times (\textbf{some factor}).$$

typical examples

- **Dirichlet's theorem on arithmetic progressions**
- **Chebotarev's density theorem**

The proof of our theorem uses this technique.

Friedberg-Hoffstein vs main theorem

- **By using another result of Waldspurger,**
  we can restate Friedberg-Hoffstein as
  an existence theorem of infinitely many non-vanishing
  toric periods.

- **There is NO obvious implication between**
  Friedberg-Hoffstein and our result in <u>either direction</u>.

- **There is an explicit local condition $(*)$ on π and S**

  s.t. Friedberg-Hoffstein $+$ $(*)$ $\Longrightarrow$ main theorem

Main tool

The main tool is a Prehomogeneous zeta function, roughly of the form

$$Z(s,\phi)=\zeta_F(2s-1)\sum_{E}\sum_{\mathfrak{a}}\frac{L(1,\chi_E)^2|\mathcal{P}_E(\phi)|^2}{\mathrm{N}(\mathfrak{a})^s}\times\mathcal{D}_E(s),$$

where

- $\phi\in\pi$,
- ζ_F **: Dedekind zeta function of** F,
- $\mathcal{D}_E(s)$ **: a meromorphic function with a simple pole at** $s=1$.

Compare the order of the pole at $s=1$ **of the both sides**

$\rightsquigarrow$ **the sum over** E **is an infinite sum**

Prehomogeneous vector space (1/2)

- $G = D^\times \times D^\times \times \mathrm{GL}_2$,
- $V = D \oplus D$
- ρ : $G \curvearrowright V$, F-rational representation, defined by

$$(x, y)\rho(g_1, g_2, g_3) = (g_1^{-1} x g_2, g_1^{-1} y g_2) g_3,$$

$$(g_1, g_2, g_3) \in G, \quad (x, y) \in V.$$

This action has a Zariski open orbit V_0.

$\rightsquigarrow$ **the triple (G, V, ρ) is called a <u>prehomogeneous vector space</u>.**

Prehomogeneous vector space (2/2)

There is a bijection $V^0(F)/G(F) \leftrightarrow X(D)$, **where**

- **L.H.S : the set of** F**-rational open orbits in** $V(F)$
- **R.H.S :** $X(D) = \left\{ E \;\middle|\; \begin{array}{c} \textbf{quad. semi-simple alg.}/F \\ \textbf{s.t. } E \overset{\exists}{\hookrightarrow} D \end{array} \right\}$

Let $\omega : G \to \mathbb{G}_m$ **be the character given by**

$$\omega((g_1, g_2, g_3)) = \det(g_1)^{-2} \det(g_2)^2 \det(g_3)^2$$

$$\textbf{(}g_1, g_2 \in D^\times,\ \ g_3 \in \mathrm{GL}_2\textbf{)}$$

where $\det$ **: reduced norm on** D **or determinant on** GL_2

Zeta function

We define the <u>zeta function</u> by

$$Z(\Phi, \phi, s) = \int_{G(F)(\mathrm{Ker}\,\rho)_{\mathbb{A}}\backslash G(\mathbb{A})} |\omega(g)|^s \, \phi(g_1) \, \overline{\phi(g_2)} \sum_{x \in V^0(F)} \Phi(x \cdot \rho(g)) \mathrm{d}g,$$

where

- $\phi \in \pi$, $\quad g = (g_1, g_2, g_3)$
- Φ **: Schwartz-Bruhat function on** $V(\mathbb{A})$

Basic properties

Theorem

(1) *The zeta function* $Z(\Phi, \phi, s)$ *has meromorphic continuation to the whole* s*-plane.*

(2) *The zeta function* $Z(\Phi, \phi, s)$ *satisfies a functional equation*

$$Z(\Phi, \phi, s) = Z^{\vee}(\widehat{\Phi}, \phi, 2 - s),$$

where $Z^{\vee}$ *is the "dual zeta function" and* $\widehat{\Phi}$ *is the Fourier transform of* Φ*.*

(3) *The possible poles of* $Z(\Phi, \phi, s)$ *are at* $s = 1/2$ *and* $s = 3/2$ *and both are at most simple poles.*

unfolding

By the standard unfolding process, we get

$$
\begin{aligned}
&Z(\Phi, \phi, s) \\
&\quad = \frac{1}{2} \sum_{E \in X(D)} \int_{G_{x(E)}(\mathbb{A}) \backslash G(\mathbb{A})} \mathcal{P}_E(\pi(g_1)\phi) \, \overline{\mathcal{P}_E(\pi(g_2)\phi)} \, |\omega(g)|^s \, \Phi(x(E)\rho(g)) \mathrm{d}g,
\end{aligned}
$$

where

- $g = (g_1, g_2, g_3) \in G$
- $x(E)$ **: a fixed representative of the** F**-rational open orbit which corresponds to** $E \in X(D)$
- $G_{x(E)}$ **: the stabilizer of** $x(E)$

  $\fallingdotseq \mathbb{G}_{m,E}$

Intermediate result

As a consequence, we see that

$$Z(\Phi, \phi, s) \not\equiv 0$$

$$\Downarrow$$

$$\exists\, E \in X(D), \exists\, g \in G \ \ \textbf{s.t.} \ \ \mathcal{P}_E(\pi(g)\phi) \neq 0$$

$$\Downarrow$$

$$\exists\, E \in X(D) \ \ \textbf{s.t.} \ \ \pi : E^\times\textbf{-distinguished}$$

Theorem

If $L(1/2, \pi) \neq 0$*, then*

$$\exists\, \Phi, \exists\, \phi \in \pi \ \ \textit{s.t.} \ \ Z(\Phi, \phi, s) \textit{ has a simple pole at } s = 1/2.$$

In particular, $L(1/2, \pi) \neq 0$

$$\Rightarrow \exists\, \phi \in \pi,\ \exists\, E \in X(D) \ \ \textit{s.t.} \ \ \mathcal{P}_E(\phi) \neq 0$$

In order to show non-vanishing of infinitely many periods, we need an **Euler factorization** of the contribution of each F-rational open orbit:

$$Z_E(\Phi, \phi, s) \\ := \frac{1}{2}\int_{G_{x(E)}(\mathbb{A})\backslash G(\mathbb{A})} \mathcal{P}_E(\pi(g_1)\phi)\,\overline{\mathcal{P}_E(\pi(g_2)\phi)}\,|\omega(g)|^s\,\Phi(x(E)\rho(g))\mathrm{d}g.$$

$\rightsquigarrow$ we have $Z(\Phi, \phi, s) = \displaystyle\sum_{E \in X(D)} Z_E(\Phi, \phi, s)$.

We want a factorization of $Z_E(\Phi, \phi, s)$.

Waldspurger formula

For each place v, we can take $\alpha_{E_v} \in \mathrm{Hom}_{E_v^\times \times E_v^\times}(\pi_v \boxtimes \bar{\pi}_v, \mathbb{C}^\times)$

(an $E_v^\times \times E_v^\times$-invariant hermitian pairing)

so that we have the following Euler factorization:

Theorem (Waldspurger'85)

For a factorizable $\phi = \otimes_v \phi_v$, we have

$$|\mathcal{P}_E(\phi)|^2 = \frac{1}{4} \cdot \frac{\zeta_F(2) L(1/2, \pi) L(1/2, \pi \otimes \chi_E)}{L(1, \pi, \mathrm{Ad}) L(1, \chi_E)} \prod_v \alpha_{E_v}(\phi_v, \phi_v).$$

Euler factorization

Applying this formula, we get

$$Z_E(\Phi, \phi, s) = \frac{(L\textbf{-values})}{(\textbf{constant})} \times \prod_v Z_{E,v}(\Phi_v, \phi_v, s)$$

for $\Phi = \otimes_v \Phi_v$ **and** $\phi = \otimes_v \phi_v$**.**

Here, $Z_{E_v}(\Phi_v, \phi_v, s)$ **is the** <u>**local zeta function**</u> **given by**

$$\int_{G_{x(E)}(F_v)\backslash G(F_v)} \alpha_{E_v}(\pi_v(g_1)\phi_v, \pi_v(g_2)\phi_v)\, |\omega_v(g)|^{s-2}\, \Phi_v(x(E)\rho(g))\, \mathrm{d}g$$

$$\times\, (\textbf{constant})(\textbf{local } L\textbf{-values})$$

Explicit formula

Computing the local zeta functions at unramified places, we see that

$$Z(\Phi, \phi, s) =$$

$$\sum_{\mathcal{E}_S \in X(D_S)} \left(\prod_{v \in S} (\textbf{local zeta})_v \right) \sum_{E \in X(D, \mathcal{E}_S)} \frac{L(1, \chi_E)^2 |\mathcal{P}_E(\phi)|^2}{\mathrm{N}(\mathfrak{a}_E)^{s-1}} \cdot \mathcal{D}_E^S(\pi, s)$$

$$\times \zeta_F^S(2s-1) \cdot (\textbf{constant})(L\textbf{-values})$$

where

- $X(D_S) = \prod_{v \in S} X(D_v)$ **and** $\mathcal{E}_S \in X(D_S)$ **is a quad. alg. over** $F_S = \prod_{v \in S} F_v$**.**
- $X(D, \mathcal{E}_S) = \{E \in X(D) \mid E_v = \mathcal{E}_v \ \forall v \in S\}$
- $\mathfrak{a}_E \subset F$ **: a certain ideal**
- $\mathcal{D}_E^S(\pi, s)$ **is a meromorphic function.**

Proof of the main theorem (1/2)

Assume the sum over $X(D, \mathcal{E}_S)$ is a non-empty finite sum.

Then,

- $Z(\Phi, \phi, s)$ **is holomorphic at** $s = 1$.

  $\rightsquigarrow$ **L.H.S is holomorphic at** $s = 1$
- $\zeta_F^S(2s-1)$ **has a simple pole at** $s = 1$.
- $\mathcal{D}_E^S(\pi, s)$ **has a simple pole at** $s = 1$.

  $\rightsquigarrow$ **R.H.S has a pole of order** 2 **at** $s = 1$.

This is a contradiction.

$\Rightarrow$ **The sum over** $X(D, \mathcal{E}_S)$ **is empty or an infinite sum.**

$\Rightarrow$ **If** $\exists\ E \in X(D, \mathcal{E}_S)$ **s.t.** $\mathcal{P}_E(\phi) \neq 0$, **then** $\exists\ \infty$ **such** E.

Proof of the main theorem (2/2)

Suppose $L(1/2, \pi) \neq 0$**.**

Then, by the previous theorem, $\exists\ \phi \in \pi,\ \exists\ E_0 \in X(D)$
s.t. $\mathcal{P}_{E_0}(\phi) \neq 0$**.**

Set $\mathcal{E}_S := \prod_{v \in S} E_{0,v}$**.**

From the above argument, $\exists\ \infty\ E \in X(D, \mathcal{E}_S)$ **s.t.** $\mathcal{P}_E(\phi) \neq 0$**.**

This completes the proof the main theorem. $\square$

Future works (1/4)

In the main theorem, we do not have a control on on the local components $\mathcal{E}_S$ of the quad. alg. at 'bad places'.

This is the reason why we did not obtain a generalization of Friedberg-Hoffstein.

What is required for extending the main theorem to cover Friedberg-Hoffstein is a close study on the
<u>functional equations of local zeta functions</u>
at 'bad places'.

Future works (2/4)

- **F. Sato '06 $\cdots$ proved the local F.E. for some specific representations of $\mathrm{GL}_2(\mathbb{R})$ in our setting and computed the 'gamma factors' explicitly.**
- **Wen-Wei Li '18 $\cdots$ proved the local F.E. in general case.**

$\rightsquigarrow$ **We get the following partial result:**

Theorem

Suppose that $F = \mathbb{Q}$ and $D_\infty = \mathrm{Mat}_2(\mathbb{R})$.

Assume that $L(1/2, \pi) \neq 0$.

Then, $\exists\ \infty\ E$: real quad. fields s.t. π is $E^\times$-distinguished.

Future works (3/4)

$\exists$ **another direction to refine the main theorem**

Analysis on the residue of (usual) prehomogeneous zeta functions

$\Downarrow$

an explicit formula for mean values of class numbers

Apply this technique to $Z(s, \Phi, \phi)$**, to get a quantitative result on non-vanishing of toric periods.**

Future works (4/4)

Suppose that $\displaystyle\sum_{E\in X(D,\mathcal{E}_S)} \frac{L(1,\chi_E)^2|\mathcal{P}_E(\phi)|^2}{\mathrm{N}(\mathfrak{a}_E)^{s-1}} \cdot \mathcal{D}_E^S(\pi,s)$ **has a simple pole at** $s=3/2$**.**

Applying a theorem of Tauberian type,
we might be able to get the following density theorem:

Density Theorem

$$\lim_{t\to\infty} t^{-\frac{1}{2}} \sum_{\substack{E\in X(D,\mathcal{E}_S)\\ \mathrm{N}(\mathfrak{a}_E)\le t}} L(1,\chi_E)^2|\mathcal{P}_E(\phi)|^2 = \left(\begin{array}{c}\textbf{the residue of}\\ \textbf{the above series}\\ \textbf{at } s=3/2\end{array}\right)$$

R.H.S will be written as a Godement-Jacquet zeta integral.

Thank you for your attention.