Quantum Geometry of Moduli Spaces of Local Systems

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Joint work with Alexander Goncharov
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Outline

Notations

- G: a semisimple algebraic group over $\mathbb{C}$ with trivial center.
- $\mathcal{B} = \{$Borel subgroups of G$\}$.
- We say a pair $\mathrm{B}, \mathrm{B}' \in \mathcal{B}$ is **generic** if $\mathrm{B} \cap \mathrm{B}' := \mathrm{T}$ is abelian. e.g.,
  $\mathrm{G} = \mathrm{PGL}_n$,
  B=$\{$upper triangular matrices$\}$,
  B′=$\{$lower triangular matrices$\}$.
- Let $I = \{1, \ldots, r\}$ parametrize the simple coroots $\alpha_1^\vee, \ldots, \alpha_r^\vee$.
- The datum $p = (\mathrm{B}, \mathrm{B}', x_i, y_i; i \in I)$ is called a **pinning** over a generic pair $(\mathrm{B}, \mathrm{B}')$ if it assigns to every $i \in I$ a homomorphism $\gamma_i : \mathrm{SL}_2 \to \mathrm{G}$ such that

$$\gamma_i \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_i(a) \in \mathrm{B},$$

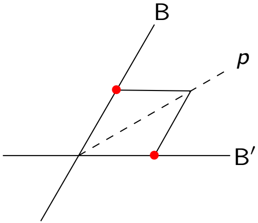
$$\gamma_i \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = y_i(a) \in \mathrm{B}',$$

$$\gamma_i \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \alpha_i^\vee(a) \in \mathrm{T}.$$

Examples

- $\mathrm{G} = \mathrm{PGL}_2$, $\mathcal{B} \overset{\sim}{=} \mathbb{P}^1$={1-dimensional subspaces in $\mathbb{C}^2$ }

  a pinning over the lines $\mathrm{B}, \mathrm{B}' \in \mathcal{B}$ is a linear isomorphism from B to B′



Examples

- $\mathrm{G} = \mathrm{PGL}_n$, $\mathcal{B} \stackrel{\sim}{=} \{V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n \mid \dim V_i = i\}$

  A pair $\mathrm{B}, \mathrm{B}' \in \mathcal{B}$ is generic $\Longleftrightarrow$ there exists a decomposition $\mathbb{C}^n = l_1 \oplus l_2 \oplus \ldots \oplus l_n$, and

$$\mathrm{B} = (l_1 \subset l_1 \oplus l_2 \subset \ldots \subset l_1 \oplus \cdots \oplus l_n)$$
$$\mathrm{B}' = (l_n \subset l_n \oplus l_{n-1} \subset \ldots \subset l_n \oplus \cdots \oplus l_1)\,.$$

  A pinning over $(\mathrm{B}, \mathrm{B}')$ is equivalent to a choice of a one dimensional space $l \subset \mathbb{C}^n$ such that l is not contained in any hyperplane spanned by a subset of $\{l_1, \ldots, l_n\}$, i.e.,

$$l \not\subset l_1 \oplus \ldots \oplus \hat{l}_i \oplus \ldots \oplus l_n, \qquad \forall i.$$

  Hence, the pinnings over $(\mathrm{B}, \mathrm{B}')$ are parametrized by the the torus $(\mathbb{C}^\times)^{n-1} \subset \mathbb{P}^{n-1}$.

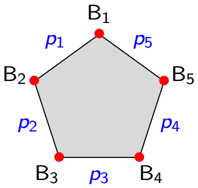
A toy example

Definition

Let D_n be a regular n-gon. The moduli space $\mathcal{P}_{\mathrm{G},D_n}$ parametrizes G-orbits of the tuples $(\mathrm{B}_1, \ldots, \mathrm{B}_n; p_1, \ldots, p_n)$, where

- $\mathrm{B}_1, \ldots, \mathrm{B}_n$ are Borel subgroups of G such that the pairs $(\mathrm{B}_i, \mathrm{B}_{i+1})$, $i \in \mathbb{Z}/n\mathbb{Z}$, are generic
- p_i is a pinning over $(\mathrm{B}_i, \mathrm{B}_{i+1})$.

Elements of $\mathcal{P}_{\mathrm{G},D_5}$ are pictured as follows:



Remark. The dimension of $\mathcal{P}_{\mathrm{G},D_n}$ is

$$m = n \dim \mathrm{B} - \dim \mathrm{G}.$$

The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathcal{P}_{\mathrm{G},D_n}$ by rotation.

A toy example

Theorem (Goncharov-S, 2019)

The space $\mathcal{P}_{\mathrm{G},D_n}$ is a rational smooth affine variety. It admits a natural cluster Poisson structure invariant under $\mathbb{Z}/n\mathbb{Z}$-action, i.e.,

1. *$\mathcal{P}_{\mathrm{G},D_n}$ is a Poisson variety,*
2. *$\mathcal{P}_{\mathrm{G},D_n}$ admits an exceptional collection $\mathfrak{C}$ of local charts $\alpha = \{X_{\alpha,1}, \ldots, X_{\alpha,m}\}$, The transition maps between these charts are given by sequences of cluster mutations.*
3. *the coordinate ring*
$$\mathcal{O}(\mathcal{P}_{\mathrm{G},D_n}) = \bigcap_{\alpha \in \mathfrak{C}} \mathbb{C}[X_{\alpha,1}^{\pm}, \ldots, X_{\alpha,m}^{\pm}]$$

The group $\mathbb{Z}/n\mathbb{Z}$ acts by Poisson automorphisms that permute the cluster charts.

Cluster theory (background)

- **Cluster algebras** are a class of commutative algebras introduced by Fomin and Zelevinsky in 2000. They first appeared in the context of representation theory, but have since appeared in many other contexts, from Discrete dynamical systems to Poisson geometry, Teichmüller theory, and Donaldson-Thomas theory.
- Cluster varieties are **log Calabi-Yau** varieties whose coordinate rings are cluster algebras. Many natural geometric objects are cluster varieties, e.g., Grassmannians, Schubert varieties, double Bruhat cells, double Bott-Samuelson cells etc.
- Every cluster Poisson variety admits a Poisson bracket which can be **quantized**. For example, there is a $\mathbb{C}[q^{\pm 1}]$-linear algebra $\mathcal{O}_q(\mathcal{P}_{\mathsf{G},D_n})$ whose semiclassical limit at $q \mapsto 1$ recovers $\mathcal{O}(\mathcal{P}_{\mathsf{G},D_n})$ and its Poisson bracket.

Cluster theory (background)

- Most cluster algebras admit **natural bases** (called the theta bases) with non-negative integer structure coefficients [Gross-Hacking-Keel-Kontsevich].

Example. The theta basis of the coordinate ring $\mathcal{O}(\mathcal{P}_{\mathrm{G},D_n})$ gives rise to a natural basis of the tensor product invariants

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mathrm{G}} \tag{1}$$

of finite dimensional irreducible representations of G .

Remarks.

1. The Θ bases are constructed by counting broken lines in scattering diagrams. The scatter diagrams are introduced by Kontsevich-Soibelman and Gross-Siebert to describe the *wall-crossing structures*.
2. It is an interesting direction for future research to compare the theta basis of (1) with other natural bases of representations (e.g. Lusztig's canonical bases for quantum groups, Mirkovic-Vilonen bases arising from geometric Satake correspondence). In particular, evidence shows that the Θ bases may coincide(?) with the MV bases.

Basics on cluster Poisson algebras

Definition (Seeds)

A seed $\mathbf{i}$ consists of the data $\{(X_1, \ldots, X_m), W\}$, where

- $X_1, \ldots, X_m$ are algebraically independent variables;
- $W = \sum a_{ij} X_i \frac{\partial}{\partial X_i} \wedge X_j \frac{\partial}{\partial X_j}$ is a bi-vector encoded by an integer $m \times m$ skew-symmetric matrix (a_{ij}).

Denote by $\mathbb{L}_{\mathbf{i}} := \mathbb{C}[X_1^{\pm}, \ldots, X_n^{\pm}]$ the ring of Laurent polynomials in $X_1, \ldots, X_n$.

Definition (Mutations)

Fix a seed $\mathbf{i}$ as above. For each $k \in \{1, \ldots, m\}$, we get a mutated seed

$$\mu_k(\mathbf{i}) := \{(X_1', \ldots, X_m'),\ W'\}$$

where $W' = W$ and

$$X_i' = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i \left(1 + X_k^{-\operatorname{sgn}(a_{ik})}\right)^{-a_{ik}} & \text{if } i \neq k \end{cases} \tag{2}$$

The process of obtaining the new seed $\mu_k(\mathbf{i})$ is called a *seed mutation* in the direction k.

Basics on cluster Poisson algebras

Remarks.

1. The bi-vector W' can be presented in terms of the new variables X_i' as follows

$$W' = \sum a_{ij}' X_i' \frac{\partial}{\partial X_i'} \wedge X_j' \frac{\partial}{\partial X_j'}, \qquad \text{where } a_{ij}' \in \mathbb{Z}.$$

   Hence one can further mutate $\mu_k(\mathbf{i})$ using (a_{ij}').

2. We say a seed $\mathbf{i}'$ is equivalent to $\mathbf{i}$ and denote by $\mathbf{i}' \sim \mathbf{i}$ if it can be obtained from $\mathbf{i}$ by a sequence of seed mutations.

Definition (Cluster Poisson algebra)

The cluster Poisson algebra associated with a seed $\mathbf{i}$ is the intersection of Laurent polynomial rings for all seeds $\mathbf{i}'$ that are mutation equivalent to $\mathbf{i}$:

$$\mathcal{O}(\mathbf{i}) := \bigcap_{\mathbf{i}' \sim \mathbf{i}} \mathbb{L}_{\mathbf{i}'}.$$

Basics on cluster Poisson algebras

Note that the bi-vector W is global. Hence it induces a Poisson algebra structure on $\mathcal{O}(\mathbf{i})$, that is, the algebra $\mathcal{O}(\mathbf{i})$ carries a bilinear map $\{\ ,\ \}: \mathcal{O}(\mathbf{i}) \times \mathcal{O}(\mathbf{i}) \rightarrow \mathcal{O}(\mathbf{i})$, which is skew-symmetric and satisfies the Jacobi identity and the Leibniz identity.

Definition

A *cluster Poisson transformation* of $\mathcal{O}(\mathbf{i})$ is a Poisson automorphism of $\mathcal{O}(\mathbf{i})$ that can be obtained by a sequence of mutations followed by a seed isomorphism.
The set $\mathcal{G}_{\mathbf{i}}$ of cluster Poisson transformations under composition forms a group, called the cluster modular group associated to $\mathbf{i}$.

Remark. The Poisson algebra $\mathcal{O}(\mathbf{i})$ admits a quantization-deformation $\mathcal{O}_q(\mathbf{i})$ defined in a similar way. The group $\mathcal{G}_{\mathbf{i}}$ naturally acts on $\mathcal{O}_q(\mathbf{i})$ as well.

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A toy example

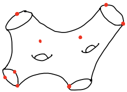
In terms of the aforementioned definitions, this Theorem asserts that

Theorem

The coordinate ring $\mathcal{O}(\mathcal{P}_{\mathsf{G},D_n})$ is a cluster Poisson algebra. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathcal{O}(\mathcal{P}_{\mathsf{G},D_n})$ by cluster Poisson transformations.

Moduli space of G-local systems

Let S be an oriented topological surface with punctures and special boundary points.



A G-local system $\mathcal{L}$ is a principal G-bundle over S with flat connections. Let $\mathcal{L}_{\mathcal{B}} := \mathcal{L} \times_{\mathrm{G}} \mathcal{B}$ be its associated $\mathcal{B}$-bundle.

Definition (Moduli space of decorated G-local systems)

The moduli space $\mathcal{P}_{\mathrm{G},S}$ parametrizes G-orbits of the data $(\mathcal{L}, \{\mathrm{B}_p\}, \{\mathrm{B}_i\}, \{p_e\})$, where

- $\mathcal{L}$ is a G-local system over S;
- for every puncture p, the data B_p is a flat section of $\mathcal{L}_{\mathcal{B}}$ over the circle around p;
- for every special boundary point i, the data B_i is a section of $\mathcal{L}_{\mathcal{B}}$ over i;
- for every boundary interval e connecting special points i, j, the associated pair $(\mathrm{B}_i, \mathrm{B}_j)$ is generic, and p_e is a pinning over $(\mathrm{B}_i, \mathrm{B}_j)$.

Examples

Let $n \geq 1$ and $2g + n \geq 3$. Let S be a genus g surface with n-many punctures. Its fundamental group $\pi_1(S)$ is a free group.
The moduli space of G-local systems over S is isomorphic to the character variety

$$\mathcal{L}_{\mathrm{G},S} = \mathrm{Hom}\left(\pi_1(S), \mathrm{G}\right)/\mathrm{G}.$$

By forgetting the flat section B_p associated to each puncture p, we get a projection

$$\pi : \mathcal{P}_{\mathrm{G},S} \longrightarrow \mathcal{L}_{\mathrm{G},S}.$$

Generically, this is a $|W|^n$-to-1 map, where W is the Weyl group of G.

Examples

Let S be once-punctured disk with 2 special boundary points. Elements of $\mathcal{P}_{\mathrm{G},S}$ are illustrated by the following figure. – Note that B is a Borel subgroup containing the monodromy g.

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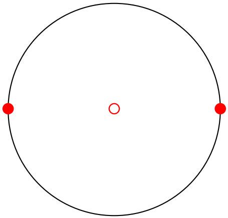

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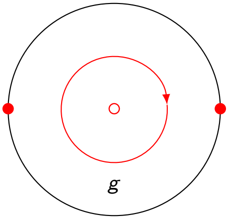

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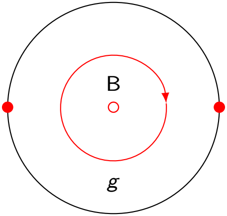

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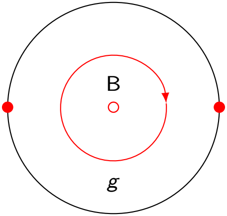

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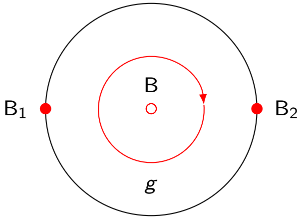

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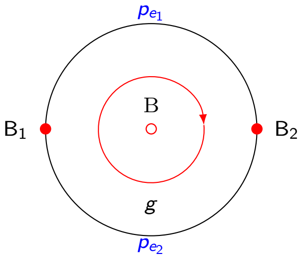

Natural actions on $\mathcal{P}_{G,S}$.

The following four groups naturally acts on $\mathcal{P}_{G,S}$:

- the **mapping class group** of S,
- the group of **outer automorphisms** of G,
- the product of **Weyl groups** over punctures of S,
- the product of **braid groups** $\mathbb{B}_G$ over boundary circles of S.

Example: braid group actions.

Definition

Let $C = (c_{ij})$ be the Cartan matrix associated to G. For any $i \neq j$, we set $m_{ij} = 2, 3, 4$, or 6 according to whether $c_{ij}c_{ji}$ is $0, 1, 2$, or 3. The braid group $\mathbb{B}_{\mathrm{G}}$ is generated by σ_i, and satisfies the relations

$$\sigma_i\sigma_j\sigma_i \ldots = \sigma_j\sigma_i\sigma_j \ldots,$$

with both sides have m_{ij} factors.

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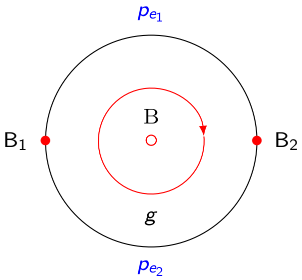

Recall that there is a Tits-distance map $d: \mathcal{B} \times \mathcal{B} \to W$.
Let B_1' be the unique Borel subgroup such that

$$d(\mathrm{B}_1, \mathrm{B}_1') = s_i, \qquad d(\mathrm{B}_1', \mathrm{B}_2) = s_i w_0.$$

Let B_2' be the unique Borel subgroup such that

$$d(\mathrm{B}_2, \mathrm{B}_2') = s_{i^*}, \qquad d(\mathrm{B}_2', \mathrm{B}_1) = s_{i^*} w_0.$$

Here $s_{i^*} = w_0 s_i w_0^{-1}$.

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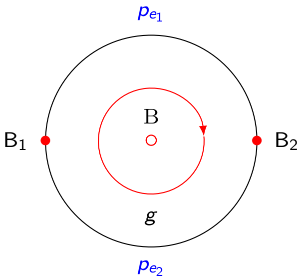

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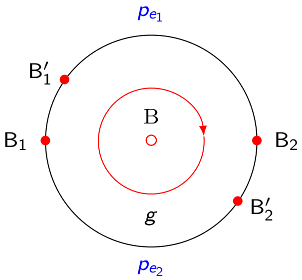

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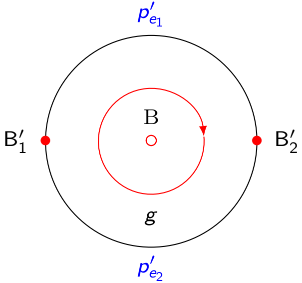

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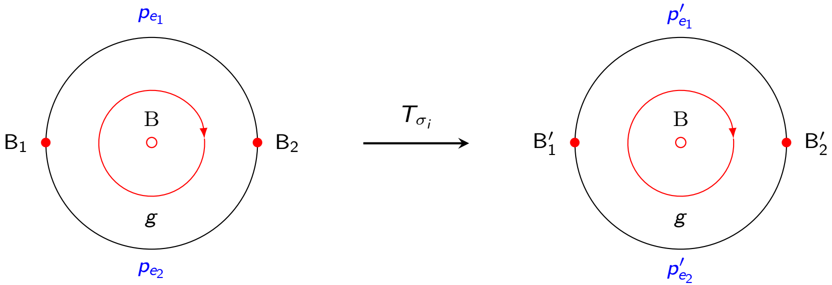

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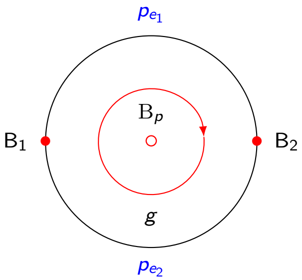

For the monodromy g surrounding a puncture p, a choice of B_p is equivalent to a choice of a Borel subgroup B_p invariant under g-conjugation.
If g is generic, then the set of Borel subgroups containing g is a W-torsor.
For example, if $g \in \mathrm{PGL}_n$ is generic, then g acts on $\mathbb{C}^n$ with distinct eigenlines $l_1, \ldots, l_n$.
Every permutation σ of $\{1, \ldots, n\}$ corresponds to a flag invariant under the action of g:

$$B_\sigma = l_{\sigma(1)} \subset l_{\sigma(1)} \oplus l_{\sigma(2)} \subset \ldots \subset l_{\sigma(1)} \oplus \cdots \oplus l_{\sigma(n)}.$$

In general, there is a birational W-action by alternating B_p.

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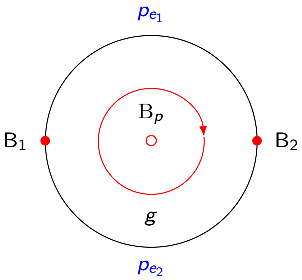

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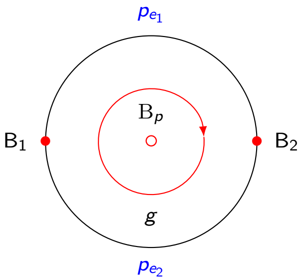

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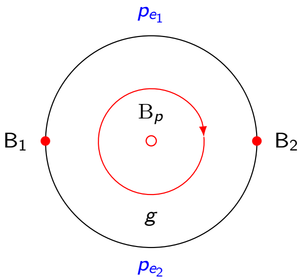

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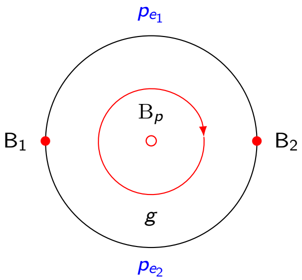

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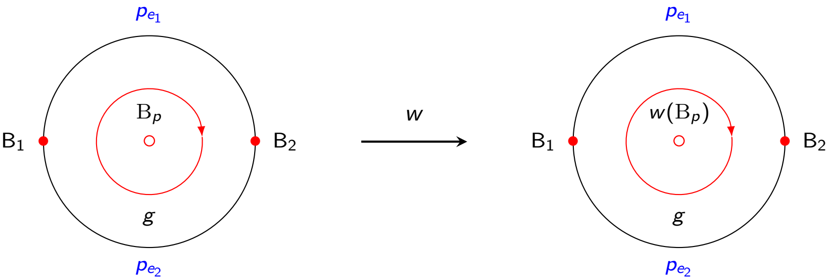
Example: Weyl group action.



For the monodromy g surrounding a puncture p, a choice of B_p is equivalent to a choice of a Borel subgroup B_p invariant under g-conjugation.
If g is generic, then the set of Borel subgroups containing g is a W-torsor.
For example, if $g \in \mathrm{PGL}_n$ is generic, then g acts on $\mathbb{C}^n$ with distinct eigenlines $l_1, \ldots, l_n$.
Every permutation σ of $\{1, \ldots, n\}$ corresponds to a flag invariant under the action of g:

$$\mathrm{B}_\sigma = l_{\sigma(1)} \subset l_{\sigma(1)} \oplus l_{\sigma(2)} \subset \ldots \subset l_{\sigma(1)} \oplus \cdots \oplus l_{\sigma(n)}.$$

In general, there is a birational W-action by alternating B_p.

Main result

Theorem (Goncharov-S)

The coordinate ring of $\mathcal{P}_{G,S}$ is a cluster Poisson algebra. The aforementioned four groups act on $\mathcal{O}(\mathcal{P}_{G,S})$ as cluster Poisson transformations.

1. Let $* \in \mathrm{Out}(G)$ be the Cartan involution. Let T_{w_0} be the lift of the longest $w_0 \in W$ to the braid group $\mathbb{B}$. We consider the following automorphism of $\mathcal{P}_{G,S}$:

$$\mathrm{C} = * \circ \prod_{\text{boundary components}} T(w_0) \circ \prod_{\text{punctures}} w_0$$

   If S is not a once punctured surface, then C is the Donaldson-Thomas transformation of $\mathcal{P}_{G,S}$.
   – For $G = \mathrm{PGL}_n$, this is proved in [Goncharov-S, 2016]; for general G, this is a work in progress.

2. As a consequence, we prove that the coordinate ring $\mathcal{O}(\mathcal{P}_{G,S})$ admits a natural basis which is invariant under the above group actions

3. The above four group actions can be lifted to actions on the quantized $\mathbb{C}[q, q^{-1}]$-algebra $\mathcal{O}_q(\mathcal{P}_{G,S})$. In particular, the algebra $\mathcal{O}_q(\mathcal{P}_{G,S})$ is expected to admit a natural $\mathbb{C}[q, q^{-1}]$-linear basis which is invariant under the above group actions.

Quantization

- For every special boundary point s of S, we introduce two collections of regular functions:
  1. the potential functions ([Goncharov-S, 2015]):
  $$\mathcal{W}_{s,1}, \ldots \mathcal{W}_{s,r}$$
  2. the h-distance functions measuring nearby pinnings:
  $$\mathcal{K}_{s,1}, \ldots \mathcal{K}_{s,r}$$
- All of them can be naturally lifted to $\mathcal{O}_q(\mathcal{P}_{\mathrm{G},S})$ respecting the underlying cluster structure of $\mathcal{P}_{\mathrm{G},S}$:
$$\mathcal{W}_{s,i} \longmapsto \mathbb{W}_{s,i} \qquad \mathcal{K}_{s,i} \longmapsto \mathbb{K}_{s,i}.$$
- Recall the Weyl group actions associated to punctures. As a consequence of our nature lift, we have
$$\mathbb{W}_{s,i},\ \mathbb{K}_{s,i} \in \mathcal{O}_q(\mathcal{P}_{\mathrm{G},S})^{W^n}$$

Quantum groups (an application)

- Let $U_q(\mathfrak{g})$ be the quantum group associated to $\mathfrak{g}$. It is a Hopf algebra generated by

$$\{\mathbf{E}_i,\ \mathbf{F}_i,\ \mathbf{K}_i^{\pm} \mid i = 1, \ldots, r\}$$

satisfying a set of relations, e.g. the quantum Serre relation for $c_{ij} = c_{ji} = -1$:

$$\mathbf{E}_i^2\mathbf{E}_j - (q + q^{-1})\mathbf{E}_i\mathbf{E}_j\mathbf{E}_i + \mathbf{E}_j\mathbf{E}_i^2 = 0$$

- Let $U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$ be the subalgebra generated by

$$\{\mathbf{E}_i,\ \mathbf{K}_i^{\pm} \mid i \in I\}$$

Theorem (Goncharov-S, 2019)

For every special boundary point s, we obtain a natural embedding

$$e_s:\ U_q(\mathfrak{b}) \longrightarrow \mathcal{O}_q(\mathcal{P}_{\mathrm{G},S})^{W^n}$$

$$\mathbf{E}_i \mapsto \mathbb{W}_{s,i}, \qquad \mathbf{K}_i \mapsto \mathbb{K}_{s,i}.$$

Quantum groups (an application)

Suppose S has a boundary component which contains exactly 2 special points s_1, s_2.

Lemma

The product $\mathbb{K}_{s_1,i}\mathbb{K}_{s_2,i^*}$ is a central element of $\mathcal{O}_q(\mathcal{P}_{\mathsf{G},S})^{W^n}$.

Let $\mathcal{I}$ be the ideal generated by $\mathbb{K}_{s_1,i}\mathbb{K}_{s_2,i^*} - 1$ for $i = 1, \ldots, r$

Theorem (Goncharov-S, 2019)

There is a natural embedding

$$U_q(\mathfrak{g}) \longrightarrow \mathcal{O}_q(\mathcal{P}_{\mathsf{G},S})^{W^n} \Big/ \mathcal{I}. \tag{3}$$

Conjecture

When S is a punctured disk with 2 special points on its boundary, the map (3) is an isomorphism.

Remarks

- Assume that (3) is an isomorphism. Then the quantized theta basis of $\mathcal{O}_q(\mathcal{P}_{G,S})$ should descend to a natural basis of $\mathcal{U}_q(\mathfrak{g})$.
- **Example.** The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is a $\mathbb{C}[q^{\pm 1}]$-algebra generated by $\mathbf{E}, \mathbf{F}, \mathbf{K}^{\pm 1}$ and satisfies the relations

$$\mathbf{KE} = q^2\mathbf{EK}, \quad \mathbf{KF} = q^{-2}\mathbf{FK}, \qquad \mathbf{EF} - \mathbf{FE} = (q - q^{-1})(\mathbf{K}^{-1} - \mathbf{K}).$$

By the last relation, we get a central element

$$\mathbf{C} := \mathbf{EF} - q\mathbf{K}^{-1} - q^{-1}\mathbf{K} = \mathbf{FE} - q^{-1}\mathbf{K}^{-1} - q\mathbf{K}.$$

Let $T_0, T_1, \ldots$ be the sequence of Chebyshev polynomials given $T_0 = 1$ and $T_n(t + t^{-1}) = t^n + t^{-n}$ for $n > 0$. The set

$$\Theta = \left\{ q^{lm}\mathbf{E}^l\mathbf{K}^m T_n(\mathbf{C}) \;\middle|\; l \geq 0, n \geq 0, m \in \mathbb{Z} \right\} \bigsqcup \left\{ q^{ml}\mathbf{K}^l\mathbf{F}^m T_n(\mathbf{C}) \;\middle|\; l \in \mathbb{Z}, m > 0, n \geq 0 \right\}$$

form a linear basis of $\mathcal{U}_q(\mathfrak{sl}_2)$ with structure coefficients in $\mathbb{N}[q, q^{-1}]$.

Remarks

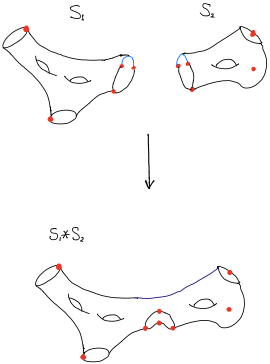
- Lusztig studied the braid group $\mathbb{B}_{\mathsf{G}}$ action on $U_q(\mathfrak{g})$. We prove that under the above embedding (3), Lusztig's braid group action coincides with our braid group action on $\mathcal{P}_{\mathsf{G},S}$.
- The braid group action, and the Weyl group action on $\mathcal{P}_{\mathsf{G},S}$ are the first step of a deeper theory connecting cluster theory and representation theory. Eventually, we aim to study the principal series representations of $\mathcal{O}_q(\mathcal{P}_{\mathsf{G},S})$ for arbitrary surfaces.
- We will develop a TQFT theory regarding the aforementioned principal series representations, which should lead to new quantum invariants of 3 dimensional manifolds.

Remarks

- Amalgamation of surfaces gives rise to maps between moduli spaces:

$$\textbf{glue}: \ \mathcal{P}_{G,S_1} \times \mathcal{P}_{G,S_2} \longrightarrow \mathcal{P}_{G,S_1 * S_2}$$

The map **glue** is Poisson and can be quantized.



Thank you!