On the irrationality of certain 2-adic zeta values

Li Lai

Apr. 5, 2023

Contents

- Review the definition of p-adic zeta values
- Compare the results on the irrationality of Riemann zeta values and p-adic zeta values
- An outline of the proof that at least one of $\zeta_2(7), \zeta_2(9), \zeta_2(11), \zeta_2(13)$ is irrational
- A detail: how we prove that some Volkenborn integrals are nonzero?

Definition of p-adic zeta values

The Riemann zeta function $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$, $\Re s > 1$. Analytic continuation. $\zeta(1-n) = -\frac{B_{n+1}}{n+1}$. Bernoulli numbers.

Kummer's congruence

If $p-1 \nmid k$ and $k \equiv k' \pmod{(p-1)p^N}$, then

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \pmod{p^{N+1}}.$$

p-adic interpolation! For simplicity, let p be an odd prime number and let $\mathbb{N}_{[n_0]} := \{n \in \mathbb{N} \mid n \equiv n_0 \pmod{p-1}\}$. We can define a p-adic zeta function $\zeta_p^{[n_0]} \colon \mathbb{Z}_p \to \mathbb{Q}_p$ by

$$\zeta_p^{[n_0]}(1-s) := \lim_{\mathbb{N}_{[n_0]} \ni n \to s} (1-p^n)\zeta(1-n),$$

where the limit is taken over all $n \in \mathbb{N}_{[n_0]}$ converging p-adically to $s \in \mathbb{Z}_p$.

There are $p-1$ branches of p-adic zeta function. The ambiguity in the choices of different branches of p-adic zeta functions disappears if we introduce Kubota-Leopoldt p-adic L-functions. Then we have the interpolation formula

$$L_p(1-n,\chi\omega^n) = (1-\chi(p)p^n)L(1-n,\chi), \quad n \in \mathbb{N}.$$

This is much better, because now we see that the Teichmüller character ω, which is a character of a group of order $(p-1)$, is responsible for the $(p-1)$ different branches and by comparing the two interpolation formulas, we get

$$\zeta_p^{[n_0]}(1-s) = L_p(1-s,\omega^{[n_0]}).$$

Although there is not a single well-defined p-adic zeta function, it still makes sense to define p-adic zeta values by

$$\zeta_p(n) := L_p(n,\omega^{1-n}).$$

There are several approaches to define p-adic zeta values.

- Volkenborn integrals
- Bernoulli measures
- Iwasawa's definition
- Coleman's definition

They are essentially the same. The key point is that p-adic zeta function should be the p-adic interpolation of the Riemann zeta function.

For our purpose, we will take the Volkenborn integral approach. We will briefly review this approach. For details, see Cohen's book (GTM 240, Chapter 11).

A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is said to be *strictly differentiable* on $\mathbb{Z}_p$ – denoted by $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$ – if

$$f(x) - f(y) = (x - y)g(x, y)$$

for some continuous function $g(x, y)$ on $\mathbb{Z}_p \times \mathbb{Z}_p$.
A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is said to be *Volkenborn integrable* if the sequence

$$\frac{1}{p^n} \sum_{k=0}^{p^n-1} f(k)$$

converges p-adically as $n \to \infty$. In this case, the value

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t := \lim_{n\to\infty} \frac{1}{p^n} \sum_{k=0}^{p^n-1} f(k)$$

is called the *Volkenborn integral* of f.
A fact: every strictly differentiable function is Volkenborn integrable.

We set $q_p = p$ if p is an odd prime, and $q_2 = 4$. The units $\mathbb{Z}_p^\times$ of the p-adic integers decompose canonically as

$$\mathbb{Z}_p^\times \cong \mu_{\varphi(q_p)}(\mathbb{Z}_p) \times (1 + q_p\mathbb{Z}_p)\,.$$

Here $\mu_n(R)$ denotes the group of n-th roots of unity in a ring R and $\varphi(\cdot)$ denotes Euler's totient function. The canonical projection

$$\omega : \mathbb{Z}_p^\times \to \mu_{\varphi(q_p)}(\mathbb{Z}_p)$$

is called the *Techmüller character*. We extend ω to a map $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ by setting

$$\omega(x) := p^{v_p(x)}\omega\left(\frac{x}{p^{v_p(x)}}\right)$$

and define

$$\langle x \rangle := \frac{x}{\omega(x)}.$$

For $s \in \mathbb{C}_p \setminus \{1\}$ such that $|s|_p < q_p p^{-1/(p-1)}$ and $x \in \mathbb{Q}_p$ such that $|x|_p \geqslant q_p$, we define

$$\zeta_p(s,x) := \frac{1}{s-1} \int_{\mathbb{Z}_p} \langle t+x \rangle^{1-s} \mathrm{d}t.$$

For fixed $x \in \mathbb{Q}_p$ such that $|x|_p \geqslant q_p$, the *p-adic Hurwitz zeta function* $\zeta_p(\cdot, x)$ is a p-adic meromorphic function on $|s|_p < q_p p^{-1/(p-1)}$, which in addition is analytic, except for a simple pole at $s = 1$ with residue 1.

We have:

$$\zeta(s,x) \sim \frac{x^{1-s}}{s-1} \sum_{k=0}^{\infty} \binom{1-s}{k} B_k x^{-k} \quad \text{as } \mathbb{R} \ni x \to +\infty$$

$$\omega(x)^{1-s} \zeta_p(s,x) = \frac{x^{1-s}}{s-1} \sum_{k=0}^{\infty} \binom{1-s}{k} B_k x^{-k} \quad \text{for } x \in \mathbb{Q}_p \text{ such that } |x|_p \geqslant q_p$$

Let χ be a primitive character of conductor f. Let M be a common multiple of f and q_p. For $s \in \mathbb{C}_p \setminus \{1\}$ such that $|s|_p < q_p p^{-1/(p-1)}$, we define

$$L_p(s, \chi) := \frac{\langle M \rangle^{1-s}}{M} \sum_{\substack{a=0 \\ p \nmid a}}^{M-1} \chi(a) \zeta_p\left(s, \frac{a}{M}\right).$$

It can be shown that the above definition is independent of the choice of M. The *p-adic L-function* $L_p(\cdot, \chi)$ is a p-adic analytic function on $|s|_p < q_p p^{-1/(p-1)}$, except when $\chi = \chi_0$ is the trivial character, in which case $L_p(\cdot, \chi_0)$ has a simple pole at $s = 1$ with residue $1 - 1/p$.

The functions $L_p(\cdot, \omega^0), L_p(\cdot, \omega^1), \ldots, L_p(\cdot, \omega^{p-2})$ are exactly the $p-1$ branches interpolating the values of the Riemann zeta function at negative integers. We define the p-adic zeta value $\zeta_p(s) := L_p(s, \omega^{1-s})$ for an integer $s \geqslant 2$.

Recent results

Theorem A (2021+, Fischler)

For any sufficiently large odd integer s we have

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} (1, \zeta(3), \zeta(5), \dots, \zeta(s)) \geqslant 0.21\sqrt{\frac{s}{\log s}}.$$

Theorem A' (2020, Sprang)

Let K be a number field (in $\mathbb{C}_p$). For $\varepsilon > 0$ and a sufficiently large positive odd integer s, we have

$$\dim_K \operatorname{Span}_K (1, \zeta_p(3), \zeta_p(5), \dots, \zeta_p(s)) \geqslant \frac{1-\varepsilon}{2[K:\mathbb{Q}](1+\log 2)} \log s.$$

- S. Fischler, *Linear independence of odd zeta values using Siegel's lemma*, Preprint (2021), arXiv:2109.10136.
- J. Sprang, *Linear independence result for p-adic L-values*, Duke Math. J. 169 (2020), no. 18, 3439–3476.

Theorem B (2020, L. and Yu)

For any $\varepsilon > 0$, *there exists* $s_0(\varepsilon)$ *such that for any odd integer* $s \geqslant s_0(\varepsilon)$, *we have*

$$\#\{ \text{ odd } j \in [3, s] \mid \zeta(j) \notin \mathbb{Q}\} \geqslant (c_0 - \varepsilon)\sqrt{\frac{s}{\log s}},$$

where the constant

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}\left(1 - \log\frac{\sqrt{4e^2+1}-1}{2}\right)} \approx 1.192507\ldots.$$

L. Lai and P. Yu, *A note on the number of irrational odd zeta values.* Compos. Math. 156 (2020), no. 8, 1699-1717.

Theorem B' (2023+, L. and Sprang)

For any prime p and any $\varepsilon > 0$, there exists $s_0(p,\varepsilon)$ such that for any odd integer $s \geqslant s_0(p,\varepsilon)$, we have

$$\#\{\text{ odd } j \in [3,s] \mid \zeta_p(j) \notin \mathbb{Q}\} \geqslant (c_p - \varepsilon)\sqrt{\frac{s}{\log s}},$$

where the constant

$$c_p = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)} \cdot \frac{\left(l_p + \frac{1}{p-1}\right)\log p - 1 - \log 2}{p^{l_p-2}(p^2-p+1)}},$$

and

$$l_p = \begin{cases} 1, & \text{if } p \geqslant 5, \\ 2, & \text{if } p = 3, \\ 3, & \text{if } p = 2. \end{cases}$$

Theorem C (1979, Apéry)

$\zeta(3) \notin \mathbb{Q}$.

Theorem C' (2005, Calegari)

$\zeta_p(3) \notin \mathbb{Q}$ *for* $p = 2, 3$.

Theorem D (2001, Zudilin)

At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

Theorem D' (2020+, Calegari, Dimitrov and Tang)

$\zeta_2(5) \notin \mathbb{Q}$.

- R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, in *Journées Arithmétiques (Luminy, 1978)*, Astérisque, vol. 61 (Société Mathématique de France, Paris, 1979), 11–13.
- F. Calegari, *Irrationality of certain p-adic periods for small p*, Int. Math. Res. Not. (2005), no. 20, 1235–1249.

Theorem E (2002, Zudilin)

For every odd integer $s \geqslant 1$, the following set contains at least one irrational number:

$$\{\zeta(j) \ | \ j \in \mathbb{Z} \cap [s+2, 8s-1], j \textit{ odd}\}$$

Theorem E' (2023+, L.)

For every integer $s \geqslant 0$, the following set contains at least one irrational number:

$$\{\zeta_2(j) \ | \ j \in \mathbb{Z} \cap [s+3, 3s+5], j \textit{ odd}\}.$$

W. Zudilin, *Irrationality of values of the Riemann zeta function*, Izvestiya Ross. Akad. Nauk Ser. Mat. [Izv. Math.] 66 (2002), 49-102 [489–542].

L. Lai, *On the irrationality of certain 2-adic zeta values*, Preprint (2023), arXiv:2304.00816.

An outline of the proof of Thmeorem E'

Lemma 1

Let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{Q}_p$ *and let* $a_{n,0}, a_{n,1}, \ldots, a_{n,m} \in \mathbb{Z}$ $(n = 1, 2, 3, \ldots)$ *be* $m + 1$ *sequences of integers such that*

$$\lim_{n\to\infty} \left(\max_{0\leqslant j\leqslant m} |a_{n,j}| \cdot \left| a_{n,0} + \sum_{j=1}^{m} a_{n,j}\alpha_j \right|_p \right) = 0 \tag{1}$$

and

$$a_{n,0} + \sum_{j=1}^{m} a_{n,j}\alpha_j \neq 0 \quad \textit{infinitely often.}$$

Then at least one of $\alpha_1, \alpha_2, \ldots, \alpha_m$ *is irrational.*

Pochhammer symbol: $(\alpha)_k := \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$.
Fix a nonnegative integer s and fix a choice of $\delta \in \{0,1\}$. For every positive integer n we define the following rational function.

$$A_n(t) := 2^{(6s+12)n} \cdot (4t+2n)^{\delta} \cdot \frac{\left(t+\frac{1}{4}\right)_n^{s+2} \left(t+\frac{3}{4}\right)_n^{s+2}}{(t)_{n+1}^{2s+4}}.$$

We denote the partial fraction decomposition of $A_n(t)$ by

$$A_n(t) =: \sum_{i=1}^{2s+4} \sum_{k=0}^{n} \frac{a_{n,i,k}}{(t+k)^i}.$$

We consider the Volkenborn integral

$$S_n := \int_{\mathbb{Z}_2} A_n^{(s)}\left(t+\frac{1}{4}\right) \mathrm{d}t \ \in \mathbb{Q}_2,$$

where $A_n^{(s)}\left(t+\frac{1}{4}\right)$ is the s-th derivative of $A_n\left(t+\frac{1}{4}\right)$.

S_n is a linear form in 1 and $\zeta_2(j, 1/4)$ ($s+3 \leqslant j \leqslant 3s+5$, $j \equiv s+1+\delta \pmod 2$) with rational coefficients:

Lemma 2

For every positive integer n we have

$$S_n = \rho_{n,0} + \sum_{\substack{2 \leqslant i \leqslant 2s+4 \\ i \equiv \delta \pmod 2}} \rho_{n,i} \, \zeta_2\left(i+s+1, \frac{1}{4}\right),$$

where

$$\rho_{n,0} = (-1)^{s+1} \sum_{i=1}^{2s+4} \sum_{k=1}^{n} \sum_{\ell=0}^{k-1} \frac{(i)_{s+1} a_{n,i,k}}{\left(\ell + \frac{1}{4}\right)^{i+s+1}} \in \mathbb{Q},$$

$$\rho_{n,i} = (-1)^s (i)_{s+1} 4^{i+s} \sum_{k=0}^{n} a_{n,i,k} \in \mathbb{Q}. \quad (1 \leqslant i \leqslant 2s+4)$$

We study the common denominator of the coefficients:

Lemma 3

For all sufficiently large odd integer n we have

$$\Phi_n^{-s-2} d_n^{3s+5} \rho_{n,0} \in \mathbb{Z}, \quad \Phi_n^{-s-2} d_n^{2s+4-i} \rho_{n,i} \in \mathbb{Z}, \ \ 2 \leqslant i \leqslant 2s+4,$$

where $d_n := \mathrm{lcm}[1, 2, \ldots, n]$ *and*

$$\Phi_n := \prod_{\substack{\sqrt{10n} < q \leqslant n \\ \left\{ \frac{n}{q} \right\} > \frac{1}{2}}} q.$$

Therefore, $\Phi_n^{-s-2} d_n^{3s+5} S_n$ is a linear form in 1 and $\zeta_2(j, 1/4)$ $(s+3 \leqslant j \leqslant 3s+5,\ j \equiv s+1+\delta \pmod 2)$ with integer coefficients.

Lemma 4

As $n \to \infty$, we have

$$\max_{0 \leqslant i \leqslant 2s+4} |\rho_{n,i}| \leqslant 2^{(6s+12+o(1))n},$$
$$d_n = \exp\left((1+o(1))n\right),$$
$$\Phi_n = \exp\left((2\log 2 - 1 + o(1))n\right).$$

Lemma 5

For any integer n of the form $n = 2^m - 1$ with $m \geqslant 2$, we have

$$v_2\left(\Phi_n^{-s-2} d_n^{3s+5} S_n\right) = (10s+20)n + (s+2)m + 2s + v_2((s+2)!) + 2.$$

In particular, $\Phi_n^{-s-2} d_n^{3s+5} S_n \neq 0$ and

$$\left|\Phi_n^{-s-2} d_n^{3s+5} S_n\right|_2 = 2^{(-10s-20+o(1))n}$$

as $n = 2^m - 1 \to \infty$.

When $n = 2^m - 1 \to \infty$ we have

$$\max_{0 \leqslant i \leqslant 2s+4} \left|\Phi_n^{-s-2} d_n^{3s+5} \rho_{n,i}\right| \cdot \left|\Phi_n^{-s-2} d_n^{3s+5} S_n\right|_2$$
$$\leqslant \exp\left(\left((4 - 6\log 2)s + 7 - 12\log 2 + o(1)\right) n\right) \to 0,$$

(because $4 - 6\log 2 < 0$ and $7 - 12\log 2 < 0$) and importantly

$$\Phi_n^{-s-2} d_n^{3s+5} S_n \neq 0.$$

Applying Lemma 1, we deduce that the following set contains at least one irrational number:

$$\left\{\zeta_2\left(j, \frac{1}{4}\right) \ \middle|\ j \in \mathbb{Z} \cap [s+3, 3s+5],\ j \equiv s + 1 + \delta \pmod 2\right\}.$$

Taking $\delta \in \{0, 1\}$ such that $\delta \equiv s \pmod 2$ and noting that $\zeta_2(j, 1/4) = \zeta_2(j)/2$ for odd j, we obtain that the following set contains at least one irrational number:

$$\left\{\zeta_2(j) \mid j \in \mathbb{Z} \cap [s+3, 3s+5], j \text{ odd}\right\}.$$

Taking $s = 3$, we obtain that at least one of $\zeta_2(7), \zeta_2(9), \zeta_2(11), \zeta_2(13)$ is irrational.

How we prove that some Volkenborn integrals are nonzero?

For a nonnegative integer $k = a_0 + a_1 p + \cdots + a_l p^l$, $a_0, \ldots,$ $a_l \in \{0, 1, \ldots, p-1\}$, $a_l \neq 0$, we donote by $k_- = a_0 + a_1 p + \cdots + a_{l-1} p^{l-1}$.

Definition

Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a function. For every nonnegative integer m, we define

$$\triangle_m(f) := \inf_{k \geqslant p^m} v_p\left(\frac{f(k) - f(k_-)}{k - k_-}\right).$$

We also define

$$\triangle(f) := \inf\{1 + v_p(f(0)), \triangle_0(f)\}.$$

From the definition, it is clear that

$$-\infty \leqslant \triangle(f) \leqslant \triangle_0(f) \leqslant \triangle_1(f) \leqslant \triangle_2(f) \leqslant \triangle_3(f) \leqslant \cdots \leqslant +\infty,$$

and $\triangle(f) > -\infty$ if f is strictly differentiable on $\mathbb{Z}_p$. For a constant function f, we have $\triangle_0(f) = +\infty$.

Lemma 6

Let $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$. Then

$$v_p\left(\int_{\mathbb{Z}_p} f(t)\mathrm{d}t\right) \geqslant \triangle(f) - 1. \tag{2}$$

Moreover, for every nonnegative integer m we have

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t \equiv \frac{1}{p^m}\sum_{k=0}^{p^m-1} f(k) \pmod{p^{\triangle_m(f)-1}\mathbb{Z}_p}. \tag{3}$$

Proof. It is straightforward to check that, for any integer $M > m$ we have

$$\frac{1}{p^M}\sum_{k=0}^{p^M-1} f(k) = \frac{1}{p^m}\sum_{k=0}^{p^m-1} f(k) + \sum_{l=m}^{M-1}\sum_{k=p^l}^{p^{l+1}-1} \frac{f(k)-f(k_-)}{p^{l+1}}.$$

For any integer $k \in [p^l, p^{l+1} - 1]$ ($l \geqslant m$) we have

$$v_p\left(\frac{f(k) - f(k_-)}{p^{l+1}}\right) \geqslant \triangle_m(f) - 1$$

by the definition of $\triangle_m(f)$. Therefore,

$$\frac{1}{p^M}\sum_{k=0}^{p^M-1} f(k) \equiv \frac{1}{p^m}\sum_{k=0}^{p^m-1} f(k) \pmod{p^{\triangle_m(f)-1}\mathbb{Z}_p}.$$

Taking the limit as $M \to \infty$ we obtain (3).
In particular,

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t \equiv f(0) \pmod{p^{\triangle_0(f)-1}\mathbb{Z}_p}.$$

Now (2) follows immediately from the definition of $\triangle(f)$. □

Lemma 7

We have the following properties for the operators $\triangle$ *and* $\triangle_m$ $(m = 0, 1, 2, \dots)$.

(1) *If* $f(t) = \sum_{j=0}^{\infty} a_j t^j \in \mathbb{Z}_p[[t]]$ *and* $\lim_{j\to\infty} |a_j|_p = 0$, *then* $\triangle_m(f) \geqslant \triangle(f) \geqslant 0$.

(2) *If* $f, g \in S^1(\mathbb{Z}_p, \mathbb{Z}_p)$, *then* $\triangle(f \cdot g) \geqslant \min\{\triangle(f), \triangle(g)\}$ *and* $\triangle_m(f \cdot g) \geqslant \min\{\triangle_m(f), \triangle_m(g)\}$.

(3) *For* $n, j \in \mathbb{Z}$, $n > 0$, *and* $f(t) = \binom{t+j}{n}$, *we have* $\triangle_m(f) \geqslant \triangle(f) \geqslant -\left\lfloor \frac{\log n}{\log p} \right\rfloor$. *Moreover, for* $m > \left\lfloor \frac{\log n}{\log p} \right\rfloor$ *we have*

$$\triangle_m(f^p) \geqslant -\left\lfloor \frac{\log n}{\log p} \right\rfloor + 1.$$

A toy model: we prove that for $n = p^m - 1$,

$$\int_{\mathbb{Z}_p} \binom{t+n}{n}^p \mathrm{d}t \neq 0.$$

Proof. By Lemma 7 (3), we have

$$\triangle_m\left(\binom{t+n}{n}^p\right) \geqslant -m+2.$$

By Lemma 6, we have

$$\int_{\mathbb{Z}_p} \binom{t+n}{n}^p \mathrm{d}t \equiv \frac{1}{p^m}\sum_{k=0}^{p^m-1}\binom{k+n}{n}^p \pmod{p^{-m+1}\mathbb{Z}_p}.$$

By Lucas' theorem (or by Kummer's theorem), for $1 \leqslant k \leqslant$ we have $p \mid \binom{k+n}{n}$. Therefore,

$$\frac{1}{p^m}\sum_{k=0}^{p^m-1}\binom{k+n}{n}^p \equiv \frac{1}{p^m} \pmod{p^{-m+1}\mathbb{Z}_p}.$$

A sketch of the proof of Lemma 5.

For $n = 2^m - 1$, $m \geqslant 2$, after applying the Leibniz rule we can write

$$A_n^{(s)}\left(t + \frac{1}{4}\right) = \sum_{\underline{i} \in I} f_{\underline{i}}(t)$$

as a sum of finitely many functions. We find a particular index $\underline{i}^*$ such that $f_{\underline{i}^*}(t)$ dominates the sum in the sense that:

- $$\int_{\mathbb{Z}_2} f_{\underline{i}^*}(t)\mathrm{d}t \neq 0,$$
- $$v_2\left(\int_{\mathbb{Z}_2} f_{\underline{i}}(t)\mathrm{d}t\right) > v_2\left(\int_{\mathbb{Z}_2} f_{\underline{i}^*}(t)\mathrm{d}t\right) \quad \text{for every } \underline{i} \neq \underline{i}^*.$$

Therefore $S_n \neq 0$ when $n = 2^m - 1$, $m \geqslant 2$.

Thanks for your attention!