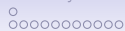


# Uniformly bounded multiplicities in the branching problem and $\mathcal{D}$ -modules

Masatoshi Kitagawa

Waseda University

Aug. 24, 2022 / Online



This talk is a summary of

1. [arXiv:2109.05556](#),
2. [arXiv:2109.05555](#).

# Outline

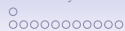
Introduction

Review:  $\mathcal{D}$ -module

Uniformly bounded family of  $\mathcal{D}$ -modules

Uniformly bounded family of  $\mathfrak{g}$ -modules

Application



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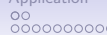
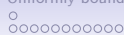
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## Finiteness property

In the representation theory of reductive Lie algebras/groups, there are many fundamental finiteness properties, e.g.

1. Length of a Verma module  $< \infty$
2. Length of a principal series representation  $< \infty$
3. Harish-Chandra's admissibility theorem  
 $\dim \operatorname{Hom}_K(F, V) < \infty$  ( $V$ : irreducible  $(\mathfrak{g}, K)$ -module,  $F \in \widehat{K}$ )
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### Topic in this talk

Each quantity is bounded with respect to the parameter (1. highest weight, 2.  $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*$ , 3.  $V$ , 4. infinitesimal character  $\lambda$ ).



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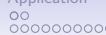
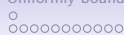
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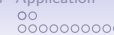
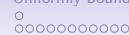
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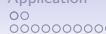
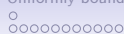


## Motivation: Boundedness property

1.  $\sup_{\text{highest weight}} (\text{Length of a Verma module}) < \infty$
2.  $\sup_{(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*} (\text{Length of a principal series representation}) < \infty$
3.  $\sup_V \dim \text{Hom}_K(F, V) < \infty$   
( $V$ : irr.  $(\mathfrak{g}, K)$ -module,  $F \in \widehat{K}$ )
4.  $\sup_{\lambda} |\{\text{irr. } (\mathfrak{g}, K)\text{-modules with inf. char. } \lambda\} / \simeq| < \infty$

cf.

1. W. Soergel's study on blocks of the BGG category  $\mathcal{O}$
2. Kobayashi–Oshima '13 Appendix
3. Harish-Chandra's subquotient theorem
4. Langlands' and Knapp–Zuckerman's classifications,  
Beilinson–Bernstein's classification of  $K$ -equivariant  $\mathcal{D}$ -modules



## Motivation: $\mathcal{U}(\mathfrak{g})^K$ -module

- $G$ : connected real reductive Lie group
- $K$ : maximal compact subgroup of  $G$
- $\mathfrak{g} := \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$

Then

- $\mathcal{U}(\mathfrak{g})^K$ -module  $\text{Hom}_K(F, V)$  is irreducible or zero.  
( $F \in \widehat{K}$ ,  $V$ : irr.  $(\mathfrak{g}, K)$ -module)

(Application: Harish-Chandra's subquotient theorem, theta lift for compact dual pair)

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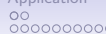
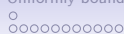
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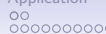
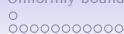
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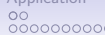
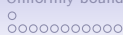
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# Goal

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Want to find a good framework that can handle these boundedness properties.

Application: uniformly bounded multiplicity theorem

- branching problem of unitary highest weight module (T. Kobayashi '97, '08)
- Kobayashi's conjecture ('11) for  $A_q(\lambda)$  ( $q$ : 'virtually symmetric type')
- Kobayashi–Oshima's uniformly bounded theorem ('13)

## What we want

- $\mathfrak{g}$ : complex reductive Lie algebra

Want to define

- $\text{Mod}_{ub}(\mathfrak{g}_I) \subset \prod_{i \in I} \text{Mod}(\mathfrak{g})$ : category of uniformly bounded families of  $\mathfrak{g}$ -modules ( $I$ : index set)

satisfying the following conditions:

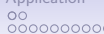
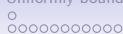
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3. Any family of Harish-Chandra modules (or objects in the BGG category  $\mathcal{O}$ ) with bounded lengths is uniformly bounded.
4. The parabolic induction functor  $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\cdot): \text{Mod}(\mathfrak{l}) \rightarrow \text{Mod}(\mathfrak{g})$  and the Zuckerman derived functors  $\mathbb{D}^j \Gamma_M^K(\cdot)$  preserve uniform boundedness.
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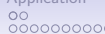
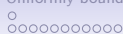
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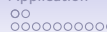
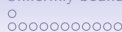
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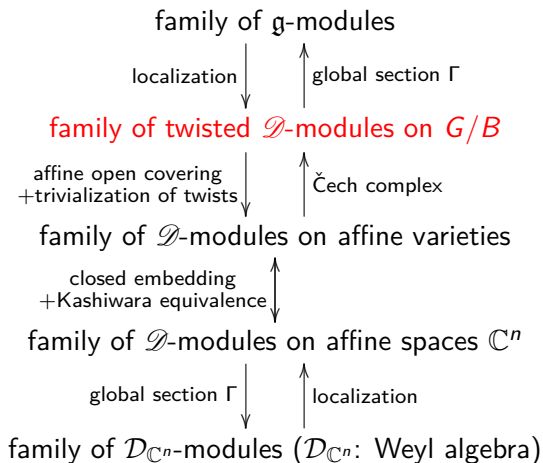
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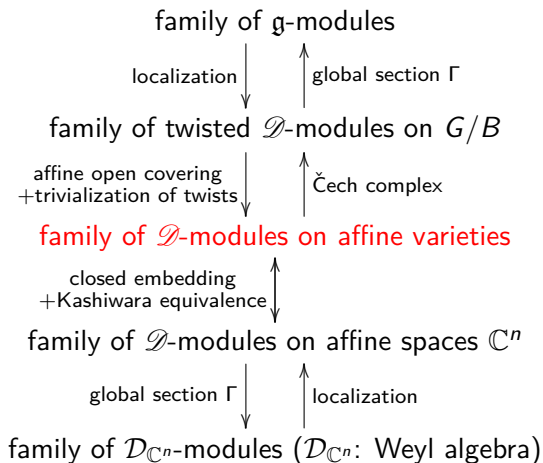
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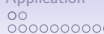
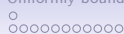
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## TDO

Recall the notion of TDOs (see e.g. Kashiwara '89).

For a (complex quasi-projective) smooth variety  $U$ ,

- $\mathcal{O}_U$ : structure sheaf of  $U$
- $\mathcal{D}_U$ : sheaf of algebras of (non-twisted) differential operators

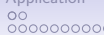
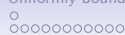
### Definition

Let  $\mathcal{A}_X$  be a sheaf of algebras on a smooth variety  $X$ . We say that  $\mathcal{A}_X$  is an algebra of twisted differential operators (TDO) if

1. a monomorphism  $\mathcal{O}_X \hookrightarrow \mathcal{A}_X$  is given,
2. there are an open covering  $X = \bigcup_i U_i$  and isomorphisms  $\varphi_i: \mathcal{A}_X|_{U_i} \simeq \mathcal{D}_{U_i}$  with  $\varphi_i|_{\mathcal{O}_{U_i}} = \text{id}$ .

### Remark

In many literatures, TDO is not necessarily assumed to be locally trivial in the Zariski/étale topology. For the definition of uniformly bounded families, we need some local triviality (Zariski/étale).



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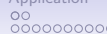
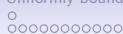
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### Remark

In many literatures, TDO is not necessarily assumed to be locally trivial in the Zariski/étale topology. For the definition of uniformly bounded families, we need some local triviality (Zariski/étale).



## TDO

Recall the notion of TDOs (see e.g. Kashiwara '89).

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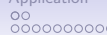
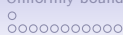
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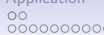
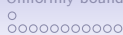
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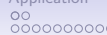
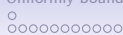
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$$\mathrm{gr}\mathcal{A}_X \simeq \pi_*\mathcal{O}_{T^*X}, \quad \pi: T^*X \rightarrow X \text{ projection}$$

For an  $\mathcal{A}_X$ -module  $\mathcal{M}$  with a good filtration ( $\Rightarrow$  coherent),

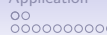
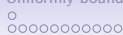
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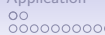
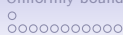
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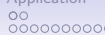
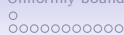
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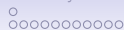
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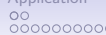
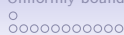
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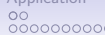
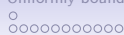
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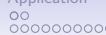
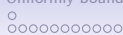
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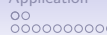
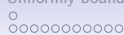
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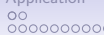
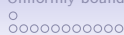
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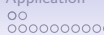
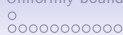
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## Multiplicities

Review Bernstein's work ('71, '72).

- $\mathcal{D}_{\mathbb{C}^n} := \Gamma(\mathcal{D}_{\mathbb{C}^n})$ : algebra of differential operators with polynomial coefficients

For a finitely generated  $\mathcal{D}_{\mathbb{C}^n}$ -module  $M$ , the multiplicity  $m(M)$  and the dimension  $d(M)$  are defined by

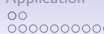
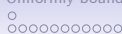
$$\dim(F^i M) \sim \frac{m(M)}{d(M)!} i^{d(M)} \quad (i \rightarrow \infty),$$

where  $(F^i M)_{i \geq 0}$  is a good filtration of  $M$  with respect to the Bernstein filtration of  $\mathcal{D}_{\mathbb{C}^n}$ .

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Bernstein filtration:

$$F^i \mathcal{D}_{\mathbb{C}^n} = \bigoplus_{|\alpha|+|\beta| \leq i} \mathbb{C} z^\alpha \frac{\partial^\beta}{\partial z^\beta}$$



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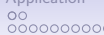
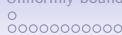
$$\dim(F^i M) \sim \frac{m(M)}{d(M)!} i^{d(M)} \quad (i \rightarrow \infty),$$

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Bernstein filtration:

$$F^i \mathcal{D}_{\mathbb{C}^n} = \bigoplus_{|\alpha|+|\beta| \leq i} \mathbb{C} z^\alpha \frac{\partial^\beta}{\partial z^\beta}$$



## Multiplicities

Review Bernstein's work ('71, '72).

- $\mathcal{D}_{\mathbb{C}^n} := \Gamma(\mathcal{D}_{\mathbb{C}^n})$ : algebra of differential operators with polynomial coefficients

For a finitely generated  $\mathcal{D}_{\mathbb{C}^n}$ -module  $M$ , the multiplicity  $m(M)$  and the dimension  $d(M)$  are defined by

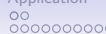
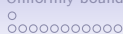
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## Multiplicity and length

### Fact

Let  $L, M$  and  $N$  be finitely generated  $\mathcal{D}_{\mathbb{C}^n}$ -module.

1. If  $M \neq 0$ , then  $d(M) \geq n$ .

$$d(M) \leq n \Leftrightarrow \mathcal{D}_{\mathbb{C}^n} \otimes_{\mathcal{D}_{\mathbb{C}^n}} M \text{ is holonomic.}$$

2.  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  (exact)

$$d(M) = \max(d(L), d(N)),$$

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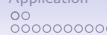
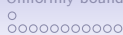
3.  $d(M) \leq n \Rightarrow \text{Len}_{\mathcal{D}_{\mathbb{C}^n}}(M) \leq m(M)$ .

For  $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_{\mathbb{C}^n})$ , set

$$m(\mathcal{M}^\bullet) := \sum_i m(\Gamma(H^i(\mathcal{M}^\bullet))).$$

Then we have

$$\text{Len}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}) \leq m(\mathcal{M}) \quad (\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^n})).$$



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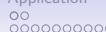
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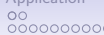
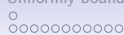
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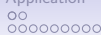
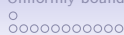
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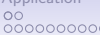
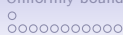
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## Multiplicity and functors

### Proposition (Derived version of Bernstein's estimate)

Let  $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$  be a morphism of varieties. Set  $d := \max(1, \deg(f))$ . For  $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_{\mathbb{C}^n})$ ,  $\mathcal{N}^\bullet \in D_h^b(\mathcal{D}_{\mathbb{C}^m})$ , we have

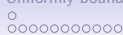
$$\begin{aligned} m(Df_+(\mathcal{M}^\bullet)) &\leq d^{n+m} m(\mathcal{M}^\bullet), \\ m(Lf^*(\mathcal{N}^\bullet)) &\leq d^{n+m} m(\mathcal{N}^\bullet). \end{aligned}$$

$f$  is decomposed as

$$\begin{aligned} \mathbb{C}^n &\xrightarrow{i} \mathbb{C}^n \oplus \mathbb{C}^m \xrightarrow{f'} \mathbb{C}^n \oplus \mathbb{C}^m \xrightarrow{p} \mathbb{C}^m, \\ i(x) &= (x, 0), \quad f'(x, y) = (x, f(x) + y), \quad p(x, y) = y. \end{aligned}$$

If  $m = 1$ ,

$$\begin{aligned} \Gamma(D^0 i_+(\mathcal{M})) &\simeq \Gamma(\mathcal{M}) \boxtimes \mathcal{D}_{\mathbb{C}}/z_{n+1}\mathcal{D}_{\mathbb{C}} \quad (\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^n})), \\ \Gamma(L_0 i^*(\mathcal{M})) &\simeq \Gamma(\mathcal{M})/z_{n+1}\Gamma(\mathcal{M}) \quad (\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^{n+1}})), \\ \Gamma(D^0 p_+(\mathcal{M})) &\simeq \Gamma(\mathcal{M})/\frac{\partial}{\partial z_{n+1}}\Gamma(\mathcal{M}) \quad (\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^{n+1}})), \\ \Gamma(L_0 i^*(\mathcal{M})) &\simeq \Gamma(\mathcal{M}) \boxtimes \Gamma(\mathcal{O}_{\mathbb{C}}) \quad (\mathcal{M} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^n})). \end{aligned}$$



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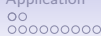
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# Outline

Introduction

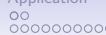
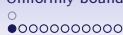
Review:  $\mathcal{D}$ -module

**Uniformly bounded family of  $\mathcal{D}$ -modules**

Uniformly bounded family of  $\mathfrak{g}$ -modules

Application





## Affine case: multiplicity

- $X$ : smooth affine variety
- $\iota: X \hookrightarrow \mathbb{C}^n$ : closed embedding

For  $\mathcal{M}^* \in D_h^b(\mathcal{D}_X)$ , set

$$m_\iota(\mathcal{M}^*) := m(D\iota_+(\mathcal{M}^*)).$$

By Kashiwara's equivalence,

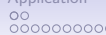
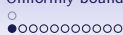
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$(\mathcal{N} \in \text{Mod}_h^{\iota(X)}(\mathcal{D}_{\mathbb{C}^n}) \Leftrightarrow \text{supp}(\mathcal{N}) \subset \iota(X) \text{ and } \mathcal{N} \in \text{Mod}_h(\mathcal{D}_{\mathbb{C}^n}))$

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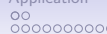
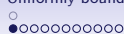
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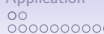
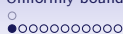
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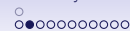
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$$\forall \mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X), \mathcal{N}^\bullet \in D_h^b(\mathcal{D}_Y),$$

$$m_{\iota'}(Df_+(\mathcal{M}^\bullet)) \leq C \cdot m_\iota(\mathcal{M}^\bullet),$$

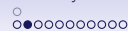
$$m_\iota(Lf^*(\mathcal{N}^\bullet)) \leq C \cdot m_{\iota'}(\mathcal{N}^\bullet).$$

Note that  $f$  extends to a morphism  $\tilde{f}: \mathbb{C}^n \rightarrow \mathbb{C}^m$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \iota & & \downarrow \iota' \\ \mathbb{C}^n & \xrightarrow{\tilde{f}} & \mathbb{C}^m \end{array}$$

If  $X = Y$  and  $f = \text{id}$ , then

$$C^{-1} \cdot m_{\iota'}(\mathcal{M}^\bullet) \leq m_\iota(\mathcal{M}^\bullet) \leq C \cdot m_{\iota'}(\mathcal{M}^\bullet).$$



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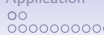
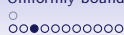
$$m_\iota(Lf^*(\mathcal{N}^\bullet)) \leq C \cdot m_{\iota'}(\mathcal{N}^\bullet).$$

Note that  $f$  extends to a morphism  $\tilde{f}: \mathbb{C}^n \rightarrow \mathbb{C}^m$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \iota & & \downarrow \iota' \\ \mathbb{C}^n & \xrightarrow{\tilde{f}} & \mathbb{C}^m \end{array}$$

If  $X = Y$  and  $f = \text{id}$ , then

$$C^{-1} \cdot m_{\iota'}(\mathcal{M}^\bullet) \leq m_\iota(\mathcal{M}^\bullet) \leq C \cdot m_{\iota'}(\mathcal{M}^\bullet).$$



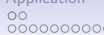
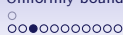
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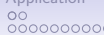
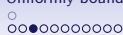
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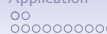
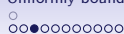
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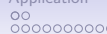
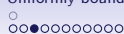
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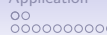
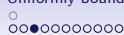
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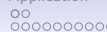
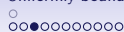
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## Twist and multiplicity

Let  $X$  be an affine smooth variety. For  $\omega \in \mathcal{Z}(X) (\simeq \text{Aut}(\mathcal{D}_X))$  and  $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X)$ ,

- $(\mathcal{M}^\bullet)^\omega$ : complex twisted by  $A_\omega \in \text{Aut}(\mathcal{D}_X)$

### Proposition

Let  $\iota: X \hookrightarrow \mathbb{C}^n$  be a closed embedding and  $W \subset \mathcal{Z}(X)$  a finite-dimensional subspace. Then  $\exists C > 0$  s.t.  $\forall \mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X)$ ,  $\omega \in W$

$$m_i((\mathcal{M}^\bullet)^\omega) \leq C \cdot m_i(\mathcal{M}^\bullet).$$

### Remark

$\dim(W) < \infty$  is essential. In fact, for  $\omega \in \mathcal{Z}(\mathbb{C})$  with  $A_\omega(d/dz) = d/dz - z^n$ , we have  $m(\mathbb{C}[z]^\omega) = \max(n, 1)$ .



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## Definition: bounded trivialization

- $\mathcal{A}_{X,\Lambda} := (\mathcal{A}_{X,\lambda})_{\lambda \in \Lambda}$ : family of TDOs on a smooth variety  $X$

### Definition

$(\mathcal{U}, \Phi)$ : trivialization of  $\mathcal{A}_{X,\Lambda} \xLeftrightarrow{\text{def}}$

- $\mathcal{U}$  is an open covering of  $X$
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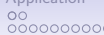
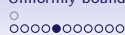
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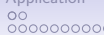
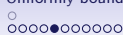
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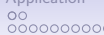
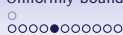
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## Definition: uniformly bounded family

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An equivalence class of bounded trivialization is called a bornology of  $\mathcal{A}_{X,\Lambda}$ .

### Definition

Let  $\mathcal{B}$  be a bornology of  $\mathcal{A}_{X,\Lambda}$  and fix  $(\mathcal{U}, \Phi) \in \mathcal{B}$  such that  $\mathcal{U}$  is an affine open covering. For  $\mathcal{M} \in \prod_{\lambda} \text{Mod}_h(\mathcal{A}_{X,\lambda})$ ,

$\mathcal{M}$  is uniformly bounded w.r.t.  $\mathcal{B} \stackrel{\text{def}}{\iff}$

$m_{\iota}(\mathcal{M}_{\lambda}|_U)$  is bounded on  $\Lambda$  ( $\forall U \in \mathcal{U}$ , closed embedding  $\iota: U \hookrightarrow \mathbb{C}^n$ ).

The definition does not depend on the choice of  $(\mathcal{U}, \Phi)$  and  $\iota$ .

- $\text{Mod}_{ub}(\mathcal{A}_{X,\Lambda}, \mathcal{B})$ : full subcategory of  $\prod_{\lambda} \text{Mod}_h(\mathcal{A}_{X,\lambda})$  consisting of uniformly bounded families
- $D_{ub}^b(\mathcal{A}_{X,\Lambda}, \mathcal{B})$ : full subcategory of  $\prod_{\lambda} D_h^b(\mathcal{A}_{X,\lambda})$  consisting of complexes with  $H^i(\mathcal{M}^{\bullet}) \in \text{Mod}_{ub}(\mathcal{A}_{X,\Lambda}, \mathcal{B})$ ,  $H^i(\mathcal{M}^{\bullet}) = 0$  ( $|i| \gg 0$ )





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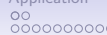
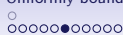
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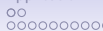
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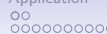
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Similarly, corresponding to the operations of  $\mathcal{A}_{X,\Lambda}$  (product  $\#$ , exterior tensor  $\boxtimes$ ,  $(\cdot)^{\mathcal{L}}$  twisted by an invertible sheaf  $\mathcal{L}$ , opposite  $(\cdot)^{\text{op}}$  of algebras), one can define operations of bornology.



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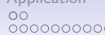
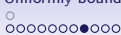
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## Fundamental properties

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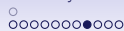
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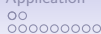
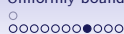
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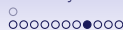
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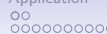
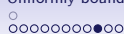
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- $G$ : affine algebraic group
- $X$ : smooth  $G$ -variety

A TDO  $\mathcal{A}_X$  on  $X$  is said to be  $G$ -equivariant if

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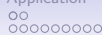
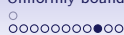
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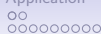
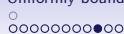
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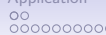
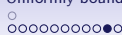
where  $X \xleftarrow[\text{projection}]{\pi} G \times X \xrightarrow[\text{multiplication}]{m} X$ .

### Definition

Let  $\mathcal{A}_{X,\Lambda}$  be a family of  $G$ -equivariant TDOs on  $X$ . A bornology  $\mathcal{B}$  is said to be  $G$ -equivariant if

$$\pi^\# \mathcal{B} = m^\# \mathcal{B}$$

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## Homogeneous variety

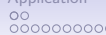
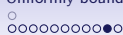
### Theorem

*If  $X$  is a homogeneous  $G$ -variety  $G/H$ , there is a unique  $G$ -equivariant bornology of  $\mathcal{A}_{X,\Lambda}$ .*

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- If  $X = G$ , any  $G$ -equivariant TDO is canonically isomorphic to  $\mathcal{D}_G$ .
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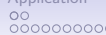
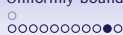
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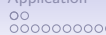
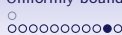
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## $G$ -equivariant $\mathcal{D}$ -module

- Assume  $|G \backslash X| < \infty$ . ( $X$  is not necessarily homogeneous.)

By Beilinson–Bernstein’s classification of equivariant  $\mathcal{D}$ -modules ('81), any irreducible  $G$ -equivariant  $\mathcal{A}_{X,\lambda}$ -module can be obtained by taking direct images, cohomologies and subquotients.

$$G \rightarrow G/G_x \hookrightarrow X$$

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# Outline

Introduction

Review:  $\mathcal{D}$ -module

Uniformly bounded family of  $\mathcal{D}$ -modules

Uniformly bounded family of  $\mathfrak{g}$ -modules

Application



## Beilinson–Bernstein correspondence

- $G$ : connected reductive algebraic group
- $B = TU$ : Borel subgroup of  $G$
- $\mathcal{D}_{G/B,\lambda} := (p_*(\mathcal{D}_{G/U})/p_*(\mathcal{D}_{G/U})\text{Ker}(-\lambda))^T$  ( $\lambda \in \mathfrak{t}^*$ )
- $I_\lambda$ : minimal primitive ideal of  $\mathcal{U}(\mathfrak{g})$  with infinitesimal character  $\lambda - \rho$

Then we have  $\mathcal{U}(\mathfrak{g})/I_\lambda \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/B,\lambda})$ .

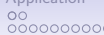
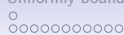
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*gives an equivalence of categories. If  $\lambda - \rho$  is anti-dominant and not regular, this is true for some full subcategory of  $\text{Mod}_{\text{qc}}(\mathcal{D}_{G/B,\lambda})$ .*

Hereafter any twist  $\lambda$  is assumed to satisfy the anti-dominant condition.



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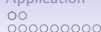
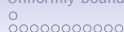
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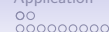
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## Uniformly bounded family

### Definition

We say that a family  $(V_i)_{i \in I}$  of  $\mathfrak{g}$ -modules is uniformly bounded if

1.  $\sup_j \text{Len}_{\mathfrak{g}}(V_j) < \infty$
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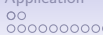
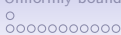
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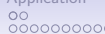
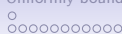
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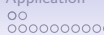
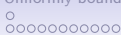
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## Family of $(\mathfrak{g}, K)$ -modules

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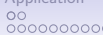
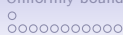
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**Proof.**

This theorem follows from the similar result of  $K$ -equivariant  $\mathcal{D}$ -modules. □



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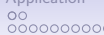
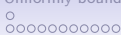
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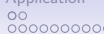
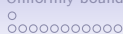
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## $\mathcal{U}(\mathfrak{g})^{G'}$ -module

- $G'$ : connected reductive subgroup of  $G$
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Let  $(V_i)_{i \in I}$  (resp.  $(V'_i)_{i \in I}$ ) be a uniformly bounded family of  $(\mathfrak{g}, K')$ -modules (resp.  $(\mathfrak{g}', K')$ -modules). Then  $\exists C > 0$  s.t.  $\forall i \in I, j \in \mathbb{N}$

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2.  $H_j(\mathfrak{g}', K'; V_i \otimes V'_i) \simeq \mathbb{D}^{n-j} \Gamma_{\Delta(K')}^{\Delta(G')}(V_i \otimes V'_i)^{\Delta(G')}$
3.  $(\cdot)^{\Delta(G')}: \text{Mod}(\mathfrak{g} \oplus \mathfrak{g}', \Delta(G')) \rightarrow \text{Mod}(\mathcal{U}(\mathfrak{g})^{G'})$  is exact and sends irreducible objects to irreducible objects or zero.



## Theta lift

### Example

- $G = \mathrm{Sp}(n, \mathbb{C})$
- $V = (\text{Harish-Chandra module of Segal–Shale–Weil rep.})$
- $((\mathfrak{g}', K'), (\mathfrak{g}'', K''))$ : reductive dual pair

For an irreducible  $(\mathfrak{g}', K')$ -module  $V'$ , set

$$\Theta_i(V') := H_i(\mathfrak{g}', K'; V \otimes (V')_{K'}^*) \quad ((\mathfrak{g}'', K'')\text{-module}).$$

Then  $\exists C > 0$  (independent of  $V'$ ) s.t.

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R. Howe ('89) has proved that  $\Theta_0(V')$  has finite length and has a unique irreducible quotient. (We can not prove the later from uniformly bounded family.)

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# Outline

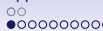
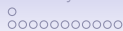
Introduction

Review:  $\mathcal{D}$ -module

Uniformly bounded family of  $\mathcal{D}$ -modules

Uniformly bounded family of  $\mathfrak{g}$ -modules

Application



## Uniformly bounded multiplicities

- $G'_{\mathbb{R}} \subset G_{\mathbb{R}}$ : Lie group and its closed subgroup

Consider when

$$\dim(\mathrm{Hom}_{G'_{\mathbb{R}}}(V|_{G'_{\mathbb{R}}}, V')) < \infty,$$

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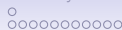
where  $V$  and  $V'$  belong to some classes of (irreducible) representations of  $G_{\mathbb{R}}$  and  $G'_{\mathbb{R}}$ , respectively.

### Example

- finite-dimensional representations
- principal series representations on a partial flag variety
- holomorphic discrete series representations
- cohomologically induced representations

Suppose  $G_{\mathbb{R}}$  is reductive.





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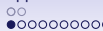
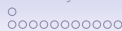
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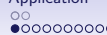
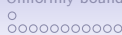
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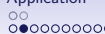
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# Known results 1

## Multiplicity-free

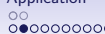
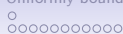
- T. Kobayashi '97-, visible action (for unitary representations)
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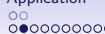
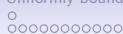
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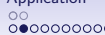
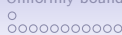
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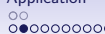
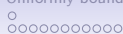
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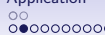
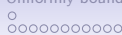
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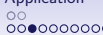
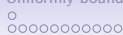
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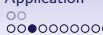
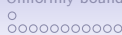
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Complex (finite) orbit, holonomicity of  $\mathcal{D}$ -modules

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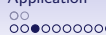
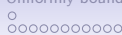
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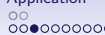
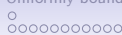
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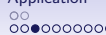
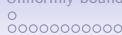
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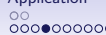
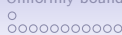
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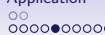
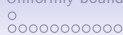
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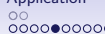
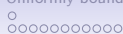
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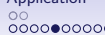
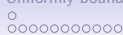
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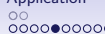
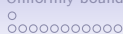
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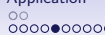
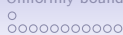
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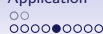
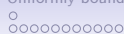
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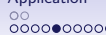
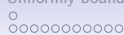
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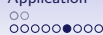
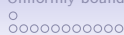
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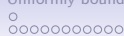
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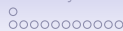
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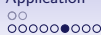
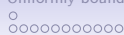
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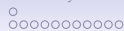
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# Characterization of uniformly bounded multiplicities

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Let  $V$  be an irreducible  $(\mathfrak{g}, K)$ -module. Set  $I := \text{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ . Then the following conditions on  $V$  are equivalent:

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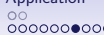
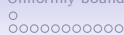
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## PI degree

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Let  $\mathcal{A}$  be a (unital associative)  $\mathbb{C}$ -algebra. A  $\mathbb{C}$ -coefficient non-commutative polynomial  $f$  is a *polynomial identity* of  $\mathcal{A}$  if

$$f(X_1, X_2, \dots, X_n) = 0 \quad (\forall X_i \in \mathcal{A}).$$

- If  $\mathcal{A}$  is commutative,  $f(X, Y) = XY - YX$  is a polynomial identity of  $\mathcal{A}$ .
- If there is a surjection  $\mathcal{A} \twoheadrightarrow \mathcal{B}$  of  $\mathbb{C}$ -algebras, then

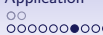
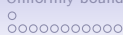
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$\text{PI.deg}(\mathcal{A})$  is the supremum of  $n$  satisfying

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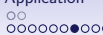
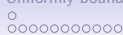
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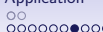
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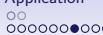
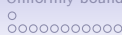
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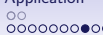
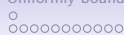
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- $\mathcal{A} \xrightarrow{\exists} M_n(\mathbb{C}) \Rightarrow \text{PI.deg}(\mathcal{A}) \geq n$
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Let  $\{V_i\}_{i \in I}$  be a family of  $\mathcal{A}$ -modules. If  $\mathcal{A} \curvearrowright \bigoplus_{i \in I} V_i$  is faithful,

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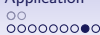
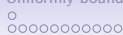
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If  $\mathcal{A}$  is noetherian and has at most countable dimension, then

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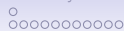
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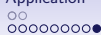
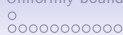
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## Bounds

### Theorem (K-)

Let  $V$  be an irreducible  $(\mathfrak{g}, K)$ -module. Set  $I := \text{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ . Then  $\exists C > 0$  independent of  $V$  and  $I$  s.t.

$$\begin{aligned} \text{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) &\leq \sup_{V'} \dim(\text{Hom}_{\mathfrak{g}', K'}(V, V')) \\ &\leq C \cdot \text{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}). \end{aligned}$$

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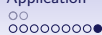
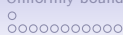
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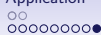
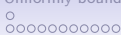
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## Corollaries

### Corollary

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### Corollary

The uniform boundedness of multiplicities depends only on complexifications of Lie algebras.

### Example

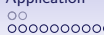
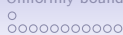
$(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (GL(n, \mathbb{R}), O(n)), (GL(n, \mathbb{R}), O(p, q))$  ( $p + q = n$ )

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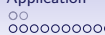
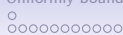
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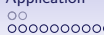
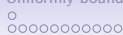
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The pairs have the same complexification  $(\text{GL}(n, \mathbb{C}), \text{O}(n, \mathbb{C}))$  modulo inner automorphisms.

$V|_{\text{O}(n)}$  has uniformly bounded multiplicities

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We can not replace 'uniformly bounded multiplicities' by 'multiplicity-free'. (e.g. Kobayashi–Ørsted–Pevzner '11)



## Corollaries

### Corollary

$V|_{\mathfrak{g}', K'}$  is multiplicity-free only if  $\text{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) = 1$ . (If  $V$  is unitarizable,  $\text{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) = 1 \Leftrightarrow (\mathcal{U}(\mathfrak{g})/I)^{G'}$  is commutative.)

### Corollary

The uniform boundedness of multiplicities depends only on complexifications of Lie algebras.

### Example

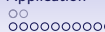
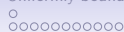
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## Reduction to fiber

- $L_{\mathbb{R}} \rtimes M_{\mathbb{R}} = P_{\mathbb{R}} \subset G_{\mathbb{R}}$ : parabolic subgroup

### Theorem (K-)

Assume that  $G/P$  is a spherical  $G'$ -variety. Then  $\exists$  reductive subgroup  $L'_{\mathbb{R}} \subset L_{\mathbb{R}} \cap G'_{\mathbb{R}}$  (concretely computable) s.t.

$\text{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(V_L)|_{G'_{\mathbb{R}}}$  has uniformly bounded multiplicities

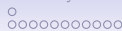
$\Leftrightarrow V_L|_{L'_{\mathbb{R}}}$  has uniformly bounded multiplicities

for any irreducible smooth admissible Fréchet rep.  $V_L$  of  $L_{\mathbb{R}}$ .

This is an analogue of

- algebraic representation case of reductive algebraic groups /  $\mathbb{C}$  (F. Sato '93, K- '14),
- the propagation theorem in the theory of visible actions (T. Kobayashi '97, '13).

The 'if' part of the theorem (+ some modification) is also true for cohomologically parabolic inductions.



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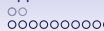
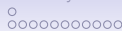
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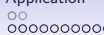
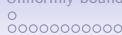
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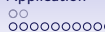
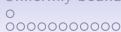
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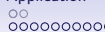
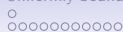


## Induction

We can treat a similar problem for

$$\sup_V \dim(\mathrm{Hom}_{\mathfrak{g}', K'}(V|_{\mathfrak{g}', K'}, V')) \cong \sup_V \dim(\mathrm{Hom}_{G_{\mathbb{R}}}(V, \mathrm{Ind}_{G'_{\mathbb{R}}}^{G_{\mathbb{R}}}(V'))).$$

In fact, it can be viewed as the branching problem from  $\mathcal{O}(G/G') \otimes \mathcal{U}(\mathfrak{g})$  to  $\mathcal{U}(\mathfrak{g})$ .



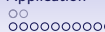
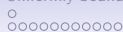
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## Summary

### Can

- Boundedness of lengths of modules
- Boundedness of the number of irreducible modules
- Boundedness of multiplicities in restrictions and inductions

### Can not

- Represent the constants  $C$  explicitly.
- Multiplicity-free?
- Irreducible?
- Unique irreducible quotient?
- Applicable only to algebraic groups (modulo connected components and covering).

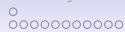
Introduction



Review:  $\mathcal{D}$ -module



Uniformly bounded family of  $\mathcal{D}$ -modules



Uniformly bounded family of  $\mathfrak{g}$ -modules



Application



Thank you for listening!

Introduction

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Review:  $\mathcal{D}$ -module

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Uniformly bounded family of  $\mathcal{D}$ -modules

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Uniformly bounded family of  $\mathfrak{g}$ -modules

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Application

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# White board