Uniformly bounded multiplicities in the branching problem and $\mathscr{D}$-modules

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This talk is a summary of

1. arXiv:2109.05556,
2. arXiv:2109.05555.

Outline

Introduction

Review: $\mathscr{D}$-module

Uniformly bounded family of $\mathscr{D}$-modules

Uniformly bounded family of $\mathfrak{g}$-modules

Application

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Finiteness property

In the representation theory of reductive Lie algebras/groups, there are many fundamental finiteness properties, e.g.

1. Length of a Verma module $< \infty$
2. Length of a principal series representation $< \infty$
3. Harish-Chandra's admissibility theorem $\dim \mathrm{Hom}_K(F, V) < \infty$ (V: irreducible $(\mathfrak{g}, K)$-module, $F \in \widehat{K}$)
4. $|\{\text{irreducible } (\mathfrak{g}, K)\text{-modules with infinitesimal character } \lambda\} / \simeq | < \infty$

Topic in this talk

Each quantity is bounded with respect to the parameter (1. highest weight, 2. $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*$, 3. V, 4. infinitesimal character λ).

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Motivation: Boundedness property

1. $\sup_{\text{highest weight}}(\text{Length of a Verma module}) < \infty$
2. $\sup_{(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}^*}(\text{Length of a principal series representation}) < \infty$
3. $\sup_V \dim \mathrm{Hom}_K(F, V) < \infty$
   (V: irr. $(\mathfrak{g}, K)$-module, $F \in \widehat{K}$)
4. $\sup_\lambda |\{\text{irr. } (\mathfrak{g}, K)\text{-modules with inf. char. } \lambda\} / \simeq | < \infty$

cf.

1. W. Soergel's study on blocks of the BGG category $\mathcal{O}$
2. Kobayashi–Oshima '13 Appendix
3. Harish-Chandra's subquotient theorem
4. Langlands' and Knapp–Zuckerman's classifications,
Beilinson–Bernstein's classification of K-equivariant $\mathscr{D}$-modules

Motivation: $\mathcal{U}(\mathfrak{g})^K$-module

- G: connected real reductive Lie group
- K: maximal compact subgroup of G
- $\mathfrak{g} := \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$

Then

- $\mathcal{U}(\mathfrak{g})^K$-module $\mathrm{Hom}_K(F, V)$ is irreducible or zero.
  ($F \in \widehat{K}$, V: irr. $(\mathfrak{g}, K)$-module)

(Application: Harish-Chandra's subquotient theorem, theta lift for compact dual pair)

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Goal

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Want to find a good framework that can handle these boundedness properties.

Application: uniformly bounded multiplicity theorem

- branching problem of unitary highest weight module (T. Kobayashi '97, '08)
- Kobayashi's conjecture ('11) for $A_{\mathfrak{q}}(\lambda)$ ($\mathfrak{q}$: 'virtually symmetric type')
- Kobayashi–Oshima's uniformly bounded theorem ('13)

What we want

- $\mathfrak{g}$: complex reductive Lie algebra

Want to define

- $\mathrm{Mod}_{ub}(\mathfrak{g}_I) \subset \prod_{i\in I} \mathrm{Mod}(\mathfrak{g})$: category of uniformly bounded families of $\mathfrak{g}$-modules (I: index set)

satisfying the following conditions:

1. $(V_i)_{i\in I} \in \mathrm{Mod}_{ub}(\mathfrak{g}_I) \Rightarrow \sup_i \mathrm{Len}_{\mathfrak{g}}(V_i) < \infty$.
2. For $0 \to L \to M \to N \to 0$ (exact sequence in $\prod_{i\in I} \mathrm{Mod}(\mathfrak{g})$),
$$L, N \in \mathrm{Mod}_{ub}(\mathfrak{g}_I) \Leftrightarrow M \in \mathrm{Mod}_{ub}(\mathfrak{g}_I).$$
3. Any family of Harish-Chandra modules (or objects in the BGG category $\mathcal{O}$) with bounded lengths is uniformly bounded.
4. The parabolic induction functor $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\cdot) \colon \mathrm{Mod}(\mathfrak{l}) \to \mathrm{Mod}(\mathfrak{g})$ and the Zuckerman derived functors $\mathbb{D}^j\Gamma^K_M(\cdot)$ preserve uniform boundedness.
5. If $M \in \mathrm{Mod}_{ub}(\mathfrak{g}_I)$ and $(F_i)_{i\in I} \in \prod_{i\in I} \mathrm{Mod}(\mathfrak{g})$ with $\sup_i \dim(F_i) < \infty$, then $(M_i \otimes F_i)_{i\in I} \in \mathrm{Mod}_{ub}(\mathfrak{g}_I)$.

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localization $\downarrow$ $\uparrow$ global section Γ

family of twisted $\mathscr{D}$-modules on G/B

affine open covering
+trivialization of twists $\downarrow$ $\uparrow$ Čech complex

family of $\mathscr{D}$-modules on affine varieties

closed embedding
+Kashiwara equivalence $\updownarrow$

family of $\mathscr{D}$-modules on affine spaces $\mathbb{C}^n$

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TDO

Recall the notion of TDOs (see e.g. Kashiwara '89).
For a (complex quasi-projective) smooth variety U,

- $\mathscr{O}_U$: structure sheaf of U
- $\mathscr{D}_U$: sheaf of algebras of (non-twisted) differential operators

Definition

Let $\mathscr{A}_X$ be a sheaf of algebras on a smooth variety X. We say that $\mathscr{A}_X$ is an algebra of twisted differential operators (TDO) if

1. a monomorphism $\mathscr{O}_X \hookrightarrow \mathscr{A}_X$ is given,
2. there are an open covering $X = \bigcup_i U_i$ and isomorphisms $\varphi_i \colon \mathscr{A}_X|_{U_i} \simeq \mathscr{D}_{U_i}$ with $\varphi_i|_{\mathscr{O}_{U_i}} = \mathrm{id}$.

Remark

In many literatures, TDO is not necessarily assumed to be locally trivial in the Zariski/étale topology. For the definition of uniformly bounded families, we need some local triviality (Zariski/étale).

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Categories of twisted $\mathscr{D}$-modules

$\mathscr{A}_X$ has a canonical order filtration induced from the isomorphisms $\mathscr{A}_X|_{U_i} \simeq \mathscr{D}_{U_i}$.

$$\mathrm{gr}\mathscr{A}_X \simeq \pi_*\mathcal{O}_{T^*X}, \quad \pi\colon T^*X \to X \text{ projection}$$

For an $\mathscr{A}_X$-module $\mathcal{M}$ with a good filtration ($\Rightarrow$ coherent),

$$\mathrm{Ch}(\mathcal{M}) := \mathrm{supp}(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \pi^{-1}\mathrm{gr}(\mathcal{M})) \subset T^*X.$$

Definition

$\mathcal{M}$ is said to be holonomic if $\dim \mathrm{Ch}(\mathcal{M}) \leq \dim(X)$ (i.e. $\mathcal{M} = 0$ or $\dim \mathrm{Ch}(\mathcal{M}) = \dim(X)$)

(A holonomic $\mathscr{A}_X$-module has finite length.)

- $\mathrm{Mod}_h(\mathscr{A}_X)$: category of holonomic $\mathscr{A}_X$-modules
- $D^b_h(\mathscr{A}_X) \subset D^b(\mathscr{A}_X)$: full subcategory consisting of complexes $\mathcal{M}^\bullet$ such that $H^i(\mathcal{M}^\bullet) \in \mathrm{Mod}_h(\mathscr{A}_X)$

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$\mathscr{A}_X$ has a canonical order filtration induced from the isomorphisms $\mathscr{A}_X|_{U_i} \simeq \mathscr{D}_{U_i}$.

$$\mathrm{gr}\mathscr{A}_X \simeq \pi_*\mathcal{O}_{T^*X}, \quad \pi\colon T^*X \to X \text{ projection}$$

For an $\mathscr{A}_X$-module $\mathcal{M}$ with a good filtration ($\Rightarrow$ coherent),

$$\mathrm{Ch}(\mathcal{M}) := \mathrm{supp}(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \pi^{-1}\mathrm{gr}(\mathcal{M})) \subset T^*X.$$

Definition

$\mathcal{M}$ is said to be holonomic if $\dim \mathrm{Ch}(\mathcal{M}) \leq \dim(X)$ (i.e. $\mathcal{M} = 0$ or $\dim \mathrm{Ch}(\mathcal{M}) = \dim(X)$)

(A holonomic $\mathscr{A}_X$-module has finite length.)

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Direct image and inverse image

- $\mathrm{Aut}(\mathscr{D}_X) \simeq \mathcal{Z}(X)$ (the space of closed 1-forms)

For $\omega \in \mathcal{Z}(X)$, set

$$A_\omega(T) = T - \omega(T) \quad (T \in \mathcal{T}_X).$$

A_ω extends uniquely to an automorphism of $\mathscr{D}_X$.
Let $f\colon Y \to X$ be a morphism between smooth varieties.

$$\text{(pull back) } f^\#\colon \mathrm{Aut}(\mathscr{D}_U) \to \mathrm{Aut}(\mathscr{D}_{f^{-1}(U)}) \quad (U \subset X \text{ open})$$

- $f^\#\mathscr{A}_X$: TDO on Y ('pull-back' of $\mathscr{A}_X$)
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Multiplicities

Review Bernstein's work ('71, '72).

- $\mathcal{D}_{\mathbb{C}^n} := \Gamma(\mathscr{D}_{\mathbb{C}^n})$: algebra of differential operators with polynomial coefficients

For a finitely generated $\mathcal{D}_{\mathbb{C}^n}$-module M, the multiplicity $m(M)$ and the dimension $d(M)$ are defined by

$$\dim(F^i M) \sim \frac{m(M)}{d(M)!} i^{d(M)} \quad (i \to \infty),$$

where $(F^i M)_{i \geq 0}$ is a good filtration of M with respect to the Bernstein filtration of $\mathcal{D}_{\mathbb{C}^n}$.

Bernstein filtration:

$$F^i \mathcal{D}_{\mathbb{C}^n} = \bigoplus_{|\alpha|+|\beta| \leq i} \mathbb{C} z^\alpha \frac{\partial^\beta}{\partial z^\beta}$$

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Multiplicity and length

Fact

Let L, M and N be finitely generated $\mathcal{D}_{\mathbb{C}^n}$-module.

1. *If $M \neq 0$, then $d(M) \geq n$.*

$$d(M) \leq n \Leftrightarrow \mathscr{D}_{\mathbb{C}^n} \otimes_{\mathcal{D}_{\mathbb{C}^n}} M \text{ is holonomic.}$$

2. *$0 \to L \to M \to N \to 0$ (exact)*

$$d(M) = \max(d(L), d(N)),$$
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$$\mathrm{Len}_{\mathscr{D}_{\mathbb{C}^n}}(\mathcal{M}) \leq m(\mathcal{M}) \quad (\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n})).$$

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Multiplicity and functors

Proposition (Derived version of Bernstein's estimate)

Let $f\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism of varieties. Set $d := \max(1, \deg(f))$. For $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_{\mathbb{C}^n})$, $\mathcal{N}^\bullet \in D^b_h(\mathscr{D}_{\mathbb{C}^m})$, we have

$$m(Df_+(\mathcal{M}^\bullet)) \leq d^{n+m} m(\mathcal{M}^\bullet),$$
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f is decomposed as

$$\mathbb{C}^n \xrightarrow{i} \mathbb{C}^n \oplus \mathbb{C}^m \xrightarrow{f'} \mathbb{C}^n \oplus \mathbb{C}^m \xrightarrow{p} \mathbb{C}^m,$$
$$i(x) = (x, 0), \quad f'(x, y) = (x, f(x) + y), \quad p(x, y) = y.$$

If $m = 1$,

$$\Gamma(D^0 i_+(\mathcal{M})) \simeq \Gamma(\mathcal{M}) \boxtimes \mathcal{D}_{\mathbb{C}}/z_{n+1}\mathcal{D}_{\mathbb{C}} \quad (\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n})),$$
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Outline

Introduction

Review: $\mathscr{D}$-module

Uniformly bounded family of $\mathscr{D}$-modules

Uniformly bounded family of $\mathfrak{g}$-modules

Application

Affine case: multiplicity

- X: smooth affine variety
- $\iota \colon X \hookrightarrow \mathbb{C}^n$: closed embedding

For $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$, set

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$(\mathcal{N} \in \mathrm{Mod}_h^{\iota(X)}(\mathscr{D}_{\mathbb{C}^n}) \Leftrightarrow \mathrm{supp}(\mathcal{N}) \subset \iota(X)$ and $\mathcal{N} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n}))$
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$(\mathcal{N} \in \mathrm{Mod}_h^{\iota(X)}(\mathscr{D}_{\mathbb{C}^n}) \Leftrightarrow \mathrm{supp}(\mathcal{N}) \subset \iota(X)$ and $\mathcal{N} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n}))$
Then we have

$$\mathrm{Len}_{\mathscr{D}_X}(\mathcal{M}) \leq m_\iota(\mathcal{M}).$$

Affine case: functors

Proposition

Let $f\colon X \to Y$ be a morphism between affine smooth varieties. Fix closed embeddings $\iota\colon X \hookrightarrow \mathbb{C}^n$ and $\iota'\colon Y \hookrightarrow \mathbb{C}^m$. Then $\exists C > 0$ s.t. $\forall \mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$, $\mathcal{N}^\bullet \in D^b_h(\mathscr{D}_Y)$,

$$
\begin{aligned}
m_{\iota'}(Df_+(\mathcal{M}^\bullet)) &\leq C \cdot m_\iota(\mathcal{M}^\bullet),\\
m_\iota(Lf^*(\mathcal{N}^\bullet)) &\leq C \cdot m_{\iota'}(\mathcal{N}^\bullet).
\end{aligned}
$$

Note that f extends to a morphism $\widetilde{f}\colon \mathbb{C}^n \to \mathbb{C}^m$:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\
\mathbb{C}^n & \overset{\widetilde{f}}{\dashrightarrow} & \mathbb{C}^m.
\end{array}
$$

If $X = Y$ and $f = \mathrm{id}$, then

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C^{-1} \cdot m_{\iota'}(\mathcal{M}^\bullet) \leq m_\iota(\mathcal{M}^\bullet) \leq C \cdot m_{\iota'}(\mathcal{M}^\bullet).
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Naive solution

- $\mathscr{A}_{X,\Lambda} := (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$: family of TDOs on a smooth variety X

1. finite affine open covering $X = \bigcup_i U_i$
2. closed embeddings $\iota_i \colon U_i \to \mathbb{C}^{n_i}$
3. local trivializations $\Phi_{i,\lambda} \colon \mathscr{A}_{X,\lambda}|_{U_i} \xrightarrow{\simeq} \mathscr{D}_{U_i}$

If the above data is given, we can define 'multiplicities'

$$m(\mathcal{M}_\lambda) := \sum_i m_{\iota_i}(\mathcal{M}_\lambda|_{U_i}) \quad ((\mathcal{M}_\lambda)_{\lambda\in\Lambda} \in \prod_\lambda \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})).$$

The values depend on the data 1, 2, 3. The boundedness of $m(\mathcal{M}_\lambda)$ ($\lambda \in \Lambda$) does not depend on the data 1, 2, and does depend on the data 3.

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Twist and multiplicity

Let X be an affine smooth variety. For $\omega \in \mathcal{Z}(X)(\simeq \mathrm{Aut}(\mathscr{D}_X))$ and $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$,

- $(\mathcal{M}^\bullet)^\omega$: complex twisted by $A_\omega \in \mathrm{Aut}(\mathscr{D}_X)$

Proposition

Let $\iota\colon X \hookrightarrow \mathbb{C}^n$ be a closed embedding and $W \subset \mathcal{Z}(X)$ a finite-dimensional subspace. Then $\exists C > 0$ s.t. $\forall \mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$, $\omega \in W$

$$m_\iota((\mathcal{M}^\bullet)^\omega) \leq C \cdot m_\iota(\mathcal{M}).$$

Remark

$\dim(W) < \infty$ is essential. In fact, for $\omega \in \mathcal{Z}(\mathbb{C})$ with $A_\omega(d/dz) = d/dz - z^n$, we have $m(\mathbb{C}[z]^\omega) = \max(n, 1)$.

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Definition: bounded trivialization

- $\mathscr{A}_{X,\Lambda} := (\mathscr{A}_{X,\lambda})_{\lambda \in \Lambda}$: family of TDOs on a smooth variety X

Definition

$(\mathcal{U}, \Phi)$: trivialization of $\mathscr{A}_{X,\Lambda} \overset{\text{def}}{\iff}$

- $\mathcal{U}$ is an open covering of X
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$(\mathcal{U}, \Phi)$, $(\mathcal{V}, \Psi)$: trivializations of $\mathscr{A}_{X,\Lambda}$,

$$(\mathcal{U}, \Phi) \sim (\mathcal{V}, \Psi) \overset{\text{def}}{\iff}$$
$$\left\{\Phi_\lambda^U \circ (\Psi_\lambda^V)^{-1} \in \mathrm{Aut}(\mathscr{D}_{U\cap V}) \colon \lambda \in \Lambda\right\} \subset \mathcal{Z}(U \cap V)$$
is contained in a finite-dimensional subspace $(\forall U \in \mathcal{U}, V \in \mathcal{U})$.

$(\mathcal{U}, \Phi)$ is said to be bounded if $(\mathcal{U}, \Phi) \sim (\mathcal{U}, \Phi)$.

$\sim$ is not an equivalence relation of trivializations.

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Definition: uniformly bounded family

Definition

An equivalence class of bounded trivialization is called a bornology of $\mathscr{A}_{X,\Lambda}$.

Definition

Let $\mathcal{B}$ be a bornology of $\mathscr{A}_{X,\Lambda}$ and fix $(\mathcal{U}, \Phi) \in \mathcal{B}$ such that $\mathcal{U}$ is an affine open covering. For $\mathcal{M} \in \prod_\lambda \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$,

$\mathcal{M}$ is uniformly bounded w.r.t. $\mathcal{B} \overset{\text{def}}{\Longleftrightarrow}$
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- $\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$: full subcategory of $\prod_\lambda \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$ consisting of uniformly bounded families
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$\mathcal{M}$ is uniformly bounded w.r.t. $\mathcal{B} \overset{\text{def}}{\Longleftrightarrow}$
$m_\iota(\mathcal{M}_\lambda|_U)$ is bounded on Λ ($\forall U \in \mathcal{U}$, closed embedding $\iota \colon U \hookrightarrow \mathbb{C}^n$).

The definition does not depend on the choice of $(\mathcal{U}, \Phi)$ and ι.

- $\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$: full subcategory of $\prod_\lambda \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$ consisting of uniformly bounded families
- $D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$: full subcategory of $\prod_\lambda D^b_h(\mathscr{A}_{X,\lambda})$ consisting of complexes with $H^i(\mathcal{M}^\bullet) \in \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$, $H^i(\mathcal{M}^\bullet) = 0$ ($|i| >> 0$)

Operations of bornology

Definition

Let $f\colon Y \to X$ be a morphism between smooth varieties and $\mathcal{B}$ a bornology of $\mathscr{A}_{X,\Lambda}$. Fix $(\mathcal{U}, \Phi) \in \mathcal{B}$ and set

$$f^{\#}\mathcal{B} := (\text{equivalence class of } (f^{-1}\mathcal{U}, f^{\#}\Phi)).$$

$f^{\#}\mathcal{B}$ does not depend on the choice of $(\mathcal{U}, \Phi)$.

Similarly, corresponding to the operations of $\mathscr{A}_{X,\Lambda}$ (product $\#$, exterior tensor $\boxtimes$, $(\cdot)^{\mathcal{L}}$ twisted by an invertible sheaf $\mathcal{L}$, opposite $(\cdot)^{\mathrm{op}}$ of algebras), one can define operations of bornology.

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Fundamental properties

- X, Y: smooth varieties
- $\mathscr{A}_{X,\Lambda}$, $\mathscr{A}_{Y,\Lambda}$: families of TDOs on X and Y
- $\mathcal{B}_X, \mathcal{B}_Y$: bornologies of $\mathscr{A}_{X,\Lambda}$ and $\mathscr{A}_{Y,\Lambda}$, respectively

Theorem (K-)

1. $\mathcal{M} \in \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}) \Rightarrow \mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathcal{M}_\lambda)$ *is bounded on* $\lambda \in \Lambda$.
2. *For* $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ *(exact) in* $\prod_\lambda \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$,
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G-equivariant bornology

- G: affine algebraic group
- X: smooth G-variety

A TDO $\mathscr{A}_X$ on X is said to be G-equivariant if

1. a homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathscr{A}_X$ and
2. an isomorphism $\pi^\# \mathscr{A}_X \simeq m^\# \mathscr{A}_X$ (+ associative law etc.) are given,

where $X \xleftarrow[\text{projection}]{\pi} G \times X \xrightarrow[\text{multiplication}]{m} X$.

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Homogeneous variety

Theorem

If X is a homogeneous G-variety G/H, there is a unique G-equivariant bornology of $\mathscr{A}_{X,\Lambda}$.

Idea of proof.

- If $X = G$, any G-equivariant TDO is canonically isomorphic to $\mathscr{D}_G$.
- Then there is a unique G-equivariant bornology on G.
- A bornology on G/H is determined by its pull-back by the quotient map $G \to G/H$.

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- Assume $|G \backslash X| < \infty$. ($X$ is not necessarily homogeneous.)

By Beilinson–Bernstein's classification of equivariant $\mathscr{D}$-modules ('81), any irreducible G-equivariant $\mathscr{A}_{X,\lambda}$-module can be obtained by taking direct images, cohomologies and subquotients.

$$G \to G/G_x \hookrightarrow X$$

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Let $\mathcal{B}$ be a G-equivariant bornology of $\mathscr{A}_{X,\Lambda}$. Any family of G-equivariant $\mathscr{A}_{X,\lambda}$-modules with bounded lengths is uniformly bounded.

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Outline

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Review: $\mathscr{D}$-module

Uniformly bounded family of $\mathscr{D}$-modules

Uniformly bounded family of $\mathfrak{g}$-modules

Application

Beilinson–Bernstein correspondence

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- $B = TU$: Borel subgroup of G
- $\mathscr{D}_{G/B,\lambda} := (p_*(\mathscr{D}_{G/U})/p_*(\mathscr{D}_{G/U})\mathrm{Ker}(-\lambda))^T$ $(\lambda \in \mathfrak{t}^*)$
- I_λ: minimal primitive ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character $\lambda - \rho$

Then we have $\mathcal{U}(\mathfrak{g})/I_\lambda \xrightarrow{\simeq} \Gamma(\mathscr{D}_{G/B,\lambda})$.

Fact (Beilinson–Bernstein '81)

If $\lambda - \rho$ is regular anti-dominant,

$$\Gamma : \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda}) \to \mathrm{Mod}(\mathcal{U}(\mathfrak{g})/I_\lambda)$$

gives an equivalence of categories. If $\lambda - \rho$ is anti-dominant and not regular, this is true for some full subcategory of $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$.

Hereafter any twist λ is assumed to satisfy the anti-dominant condition.

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Uniformly bounded family

Definition

We say that a family $(V_i)_{i \in I}$ of $\mathfrak{g}$-modules is uniformly bounded if

1. $\sup_i \mathrm{Len}_{\mathfrak{g}}(V_i) < \infty$
2. the family of all composition factors of all V_i is isomorphic to $(\Gamma(\mathcal{M}_j))_{j \in J}$ for some uniformly bounded family $(\mathcal{M}_j)_{j \in J}$ of twisted $\mathscr{D}$-modules on G/B.

A family of $(\mathfrak{g}, K)$-modules is said to be uniformly bounded if it is uniformly bounded as a family of $\mathfrak{g}$-modules.

Then the uniform boundedness satisfies the properties stated in the introduction.
For example, the Zuckerman derived functors are realized as a composition of direct images and inverse images of $\mathscr{D}$-modules (F. V. Bien '90).

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Uniformly bounded family

Definition

We say that a family $(V_i)_{i\in I}$ of $\mathfrak{g}$-modules is uniformly bounded if

1. $\sup_i \mathrm{Len}_{\mathfrak{g}}(V_i) < \infty$
2. the family of all composition factors of all V_i is isomorphic to $(\Gamma(\mathcal{M}_j))_{j\in J}$ for some uniformly bounded family $(\mathcal{M}_j)_{j\in J}$ of twisted $\mathscr{D}$-modules on G/B.

A family of $(\mathfrak{g}, K)$-modules is said to be uniformly bounded if it is uniformly bounded as a family of $\mathfrak{g}$-modules.

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- K: a closed subgroup G

Theorem (K-)

Assume that $|K\backslash G/B| < \infty$. Then any family of $(\mathfrak{g}, K)$-modules with bounded lengths is uniformly bounded. In particular, a family of irreducible $(\mathfrak{g}, K)$-modules is uniformly bounded.

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$\mathcal{U}(\mathfrak{g})^{G'}$-module

- G': connected reductive subgroup of G
- K': (finite covering of) reductive subgroup of G'

Theorem

Let $(V_i)_{i\in I}$ (resp. $(V'_i)_{i\in I}$) be a uniformly bounded family of $(\mathfrak{g}, K')$-modules (resp. $(\mathfrak{g}', K')$-modules). Then $\exists C > 0$ s.t. $\forall i \in I, j \in \mathbb{N}$

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(H_j(\mathfrak{g}', K'; V_i \otimes V'_i)) \leq C$$

Remark

$H_0(\mathfrak{g}', K'; V_i \otimes V'_i)^* \simeq \mathrm{Hom}_{\mathfrak{g}', K'}(V_i, (V'_i)^*_{K'})$

Proof.

1. $(\mathbb{D}^{n-j}\Gamma^{\Delta(G')}_{\Delta(K')}(V_i \otimes V'_i))_{i\in I} \in \mathrm{Mod}_{ub}(\mathfrak{g} \oplus \mathfrak{g}', \Delta(G'))$
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3. $(\cdot)^{\Delta(G')}$: $\mathrm{Mod}(\mathfrak{g} \oplus \mathfrak{g}', \Delta(G')) \to \mathrm{Mod}(\mathcal{U}(\mathfrak{g})^{G'})$ is exact and sends irreducible objects to irreducible objects or zero.

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Theta lift

Example

- $G = \mathrm{Sp}(n, \mathbb{C})$
- V =(Harish-Chandra module of Segal–Shale–Weil rep.)
- $((\mathfrak{g}', K'), (\mathfrak{g}'', K''))$: reductive dual pair

For an irreducible $(\mathfrak{g}', K')$-module V', set

$$\Theta_i(V') := H_i(\mathfrak{g}', K'; V \otimes (V')^*_{K'}) \quad ((\mathfrak{g}'', K'')\text{-module}).$$

Then $\exists C > 0$ (independent of V') s.t.

$$\mathrm{Len}_{\mathfrak{g}'', K''}(\Theta_i(V')) \leq C.$$

R. Howe ('89) has proved that $\Theta_0(V')$ has finite length and has a unique irreducible quotient. (We can not prove the later from uniformly bounded family.)
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Outline

Introduction

Review: $\mathscr{D}$-module

Uniformly bounded family of $\mathscr{D}$-modules

Uniformly bounded family of $\mathfrak{g}$-modules

Application

Uniformly bounded multiplicities

- $G'_{\mathbb{R}} \subset G_{\mathbb{R}}$: Lie group and its closed subgroup

Consider when

$$\dim(\mathrm{Hom}_{G'_{\mathbb{R}}}(V|_{G'_{\mathbb{R}}}, V')) < \infty,$$
$$\sup_{V,V'} \dim(\mathrm{Hom}_{G'_{\mathbb{R}}}(V|_{G'_{\mathbb{R}}}, V')) < \infty,$$

where V and V' belong to some classes of (irreducible) representations of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$, respectively.

Example

- finite-dimensional representations
- principal series representations on a partial flag variety
- holomorphic discrete series representations
- cohomologically induced representations

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- T. Kobayashi '97-, visible action (for unitary representations)
- Aizenbud–Gourevitch '09, Sun–Zhu '12 (multiplicity one theorem) Gelfand–Kazhdan criterion (and variations)

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- $\mathcal{V}(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))$: subvariety of $\mathfrak{g}^*$ determined by $\mathrm{gr}(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))$

Known results 2

G and G' are some complexification of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$, respectively.

Complex (finite) orbit, holonomicity of $\mathscr{D}$-modules

- Kobayashi–Oshima '13 (uniformly bounded theorem) $|G' \backslash G/B| < \infty$
- Aizenbud–Gourevitch–Minchenko '16 $|G' \backslash G/Q| < \infty$
- T. Tauchi '21 $|G' \backslash G/Q| < \infty$
- Aizenbud–Gourevitch (arXiv:2109.00204) $\dim(\mathcal{O} \cap (\mathfrak{g}')^{\perp}) \leq \dim(\mathcal{O})/2$ ($\forall$ G-orbit $\mathcal{O} \subset \mathcal{V}(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))$)
- T. Kobayashi '21 $|G' \backslash G/R| < \infty$
- Wen-Wei Li (arXiv:2207.08994) $|G' \backslash G/B| < \infty$, cohomological multiplicity

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- B: Borel subgroup of G
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Uniformly bounded family

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Using uniformly bounded family, we can treat uniformly the results about uniformly bounded multiplicity theorems.

Remark

The methods in the papers and ours are different, so it does not mean that all results in the papers are generalized.

Hereafter, we concentrate on more concrete setting.

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Setting

Consider the branching problem of real reductive Lie groups.

- G: (connected) reductive algebraic group / $\mathbb{C}$
- $G' \subset G$: reductive subgroup
- $G'_{\mathbb{R}} \subset G_{\mathbb{R}}$: real forms of G' and G
- $K'_{\mathbb{R}} \subset K_{\mathbb{R}}$: maximal compact subgroups of $G'_{\mathbb{R}}$ and $G_{\mathbb{R}}$
- $K' \subset K$: the complexifications of $K'_{\mathbb{R}}$ and $K_{\mathbb{R}}$

(modulo connected components and covering)

Question

Let V be an irreducible $(\mathfrak{g}, K)$-module. When does the restriction $V|_{\mathfrak{g}',K'}$ have uniformly bounded multiplicities:

$$\sup_{V'} \dim(\mathrm{Hom}_{\mathfrak{g}',K'}(V|_{\mathfrak{g}',K'}, V')) < \infty$$

$$(V' : \text{ irreducible } (\mathfrak{g}', K')\text{-module})$$

We can treat smooth or unitary representation cases and induction setting, but omit them in this talk.

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Characterization of uniformly bounded multiplicities

Theorem (K-)

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1. *$V|_{\mathfrak{g}', K'}$ has uniformly bounded multiplicities*
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The same is true for smooth or unitary representations.

$3 \Leftrightarrow 4 \Rightarrow 5$ follows from the study of Hamiltonian Poisson G-varieties by I. Losev ('09).
To show $1 \Leftrightarrow 2$, we need the action

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PI degree

Definition

Let $\mathcal{A}$ be a (unital associative) $\mathbb{C}$-algebra. A $\mathbb{C}$-coefficient non-commutative polynomial f is a *polynomial identity* of $\mathcal{A}$ if

$$f(X_1, X_2, \ldots, X_n) = 0 \quad (\forall X_i \in \mathcal{A}).$$

- If $\mathcal{A}$ is commutative, $f(X, Y) = XY - YX$ is a polynomial identity of $\mathcal{A}$.
- If there is a surjection $\mathcal{A} \twoheadrightarrow \mathcal{B}$ of $\mathbb{C}$-algebras, then

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$\mathrm{PI.deg}(\mathcal{A})$ and multiplicity

- $\mathcal{A} \overset{\exists}{\twoheadrightarrow} M_n(\mathbb{C}) \Rightarrow \mathrm{PI.deg}(\mathcal{A}) \geq n$
- $\mathcal{A} \overset{\exists}{\twoheadrightarrow} \mathcal{B}$ or $\mathcal{B} \overset{\exists}{\hookrightarrow} \mathcal{A} \Rightarrow \mathrm{PI.deg}(\mathcal{A}) \geq \mathrm{PI.deg}(\mathcal{B})$

Proposition

Let $\{V_i\}_{i \in I}$ be a family of $\mathcal{A}$-modules. If $\mathcal{A} \curvearrowright \bigoplus_{i \in I} V_i$ is faithful,

$$\mathrm{PI.deg}(\mathcal{A}) \leq \sup \{\dim(V_i) : i \in I\}.$$

Proof.

It is obvious from $\mathcal{A} \hookrightarrow \prod_{i \in I} \mathrm{End}_{\mathbb{C}}(V_i)$. □

Proposition

If $\mathcal{A}$ is noetherian and has at most countable dimension, then

$$\mathrm{PI.deg}(\mathcal{A}) = \sup \{\dim(V) : V \text{ irreducible } \mathcal{A}\text{-module}\}.$$

$\mathrm{PI.deg}(\mathcal{A})$ and multiplicity

- $\mathcal{A} \overset{\exists}{\twoheadrightarrow} M_n(\mathbb{C}) \Rightarrow \mathrm{PI.deg}(\mathcal{A}) \geq n$
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Let $\{V_i\}_{i \in I}$ be a family of $\mathcal{A}$-modules. If $\mathcal{A} \curvearrowright \bigoplus_{i \in I} V_i$ is faithful,

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Bounds

Theorem (K-)

Let V be an irreducible $(\mathfrak{g}, K)$-module. Set $I := \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$. Then $\exists C > 0$ independent of V and I s.t.

$$\begin{aligned}\mathrm{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) &\leq \sup_{V'} \dim(\mathrm{Hom}_{\mathfrak{g}',K'}(V, V')) \\ &\leq C \cdot \mathrm{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}).\end{aligned}$$

Proof of the upper bound.

We have seen that $\exists C > 0$ independent of V and I s.t.

$$\mathrm{Len}_{(\mathcal{U}(\mathfrak{g})/I)^{G'}}((V \otimes_{\mathcal{U}(\mathfrak{g}')} V')^{K'}) \leq C.$$

This shows

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$V|_{\mathfrak{g}',K'}$ is multiplicity-free only if $\mathrm{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) = 1$. (If V is unitarizable, $\mathrm{PI.deg}((\mathcal{U}(\mathfrak{g})/I)^{G'}) = 1 \Leftrightarrow (\mathcal{U}(\mathfrak{g})/I)^{G'}$ is commutative.)

Corollary

The uniform boundedness of multiplicities depends only on complexifications of Lie algebras.

Example

$(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(n)), (\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(p,q))$ $(p+q=n)$
The pairs have the same complexification $(\mathrm{GL}(n,\mathbb{C}), \mathrm{O}(n,\mathbb{C}))$ modulo inner automorphisms.

$$\begin{aligned} &V|_{\mathrm{O}(n)} \text{ has uniformly bounded multiplicities} \\ \Leftrightarrow &V|_{\mathfrak{o}(p,q),\mathrm{O}(p)\times\mathrm{O}(q)} \text{ has uniformly bounded multiplicities} \end{aligned}$$

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Reduction to fiber

- $L_{\mathbb{R}} \ltimes N_{\mathbb{R}} = P_{\mathbb{R}} \subset G_{\mathbb{R}}$: parabolic subgroup

Theorem (K-)

Assume that G/P is a spherical G'-variety. Then $\exists$ reductive subgroup $L'_{\mathbb{R}} \subset L_{\mathbb{R}} \cap G'_{\mathbb{R}}$ (concretely computable) s.t.

$$\mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(V_L)|_{G'_{\mathbb{R}}} \text{ has uniformly bounded multiplicities}$$
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for any irreducible smooth admissible Fréchet rep. V_L of $L_{\mathbb{R}}$.

This is an analogue of

- algebraic representation case of reductive algebraic groups / $\mathbb{C}$ (F. Sato '93, K- '14),
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The 'if' part of the theorem (+ some modification) is also true for cohomologically parabolic inductions.

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Summary

Can

- Boundedness of lengths of modules
- Boundedness of the number of irreducible modules
- Boundedness of multiplicities in restrictions and inductions

Can not

- Represent the constants C explicitly.
- Multiplicity-free?
- Irreducible?
- Unique irreducible quotient?
- Applicable only to algebraic groups (modulo connected components and covering).

Thank you for listening!

White board