Some Remarks on Homological Algebra

King Fai Lai

May 1, 2023

I like to ask if it is worthwhile to study homological algebra over non-commutative rings?
I believe there are lots of such results scattered everywhere and I wonder if it is convenient at least for students to collect them together.
May be you will tell me that this is a waste of time!
Or may be you have a different opinion.
Please tell me afterwards, I am sure this will help me to understand.

I begin by saying what is homological algebra for this talk. Everybody's example of a homology ring is the singular cohomology ring $\oplus H^q(X, \mathbb{Q})$ of a topological manifold with $\mathbb{Q}$-coefficients.

For us we begin with the notion that homological algebra is the study of derived functors of the derived categories of an abelian category say $\mathfrak{A}$. Standard examples of $\mathfrak{A}$:

1. The category of modules over a commutative ring R.
2. The category of $\mathscr{O}_X$-modules on a commutative ringed space $(X, \mathscr{O}_X)$.
3. The category of modules on a ringed topos.

The functors that we derive are : Hom, $\otimes$, $\boxtimes$, $\varprojlim$, $\varinjlim$, f_*, f^*, $f_!$, $f^!$.

What are their properties? For example when do they commute?

What I have just described does not include the case: a group G acting on another group A; for example A is a module over a non-commutative ring $\mathbb{Z}[G]$. Example: Galois cohomology $H^q(G,A)$, $0 \leq q \leq 2$.
But there is a situation in between : R is a commutative, A is a non-commutative R- algebra, $\mathfrak{A}$ is a category whose objects are A-modules. Can we make a lists of the properties of the derived functors on the derived category $D(\mathfrak{A})$?
There are at least two nice examples :
(i) R is a discrete valuation ring of characteristics $(0,p)$, A is a quaternion R-algebra, $\mathfrak{A}$ is the category of finite rank representations of D.
(ii) X is a compact complex manifold, R is the sheaf of holomorphic functions $\mathscr{O}_X$, A is the sheaf of differential operators on X, $\mathfrak{A}$ is the category of finite rank A-modules.

Let R be a commutative ring, A, B, C be a non-commutative R-algebras and R lies in the center of A , B, C. Suppose E is a right A-module, F is a left A-module. Then the tensor product $E \otimes_A F$ is defined to be an abelian group with certain universal property.

We say E is a (B, A)-bimodule if E is a left B-module and a right A-module and the structures of R-module induced on E by the structure of left B-module and by that of right A-module coincide. This means that the structure of (B, A)-bimodule is the same as a $B \otimes_R A^{op}$-module.

A (B, A)-bimodule morphism $f : E \to F$ means f is both a a left B-module morphism and a right A-module morphism. Let $D(B, A)$ denote the derived category of the category of complexes of (B, A)-bimodules.

Then we can define

$$- \otimes^{\mathbb{L}}_{A} - \; : \; D^{-}(B,A) \times D^{-}(A,C) \to D(B,C),$$

and

$$\mathbf{R}\operatorname{Hom}_A(-,-) : D(A,B) \times D^{+}(A,C) \to D(B,C).$$

Moreover for $E \in D^{-}(A,B)$, $F \in D^{-}(B,C)$, $G \in D^{+}(A,C)$ we have

$$\mathbf{R}\operatorname{Hom}_{(A,C)}(E \otimes^{\mathbb{L}}_{B} F, G) \xrightarrow{\approx} \mathbf{R}\operatorname{Hom}_{(B,C)}(F, \mathbf{R}\operatorname{Hom}_A(E,G))$$

But if we want to stay within a category, say we only want to work with left A-modules ?

R is a commutative ring. Let $A^{(\bullet)} = (\cdots \to A^{(m)} \to A^{(m+1)} \to \cdots)$ be an inductive system of non-commutative R-algebras. A $A^{(\bullet)}$-module is an inductive system $M^{(\bullet)} = (\cdots \to M^{(m)} \to M^{(m+1)} \to \cdots)$ where $M^{(m)}$ is a $A^{(m)}$-module. A complex of $A^{(\bullet)}$-modules is

$$
\begin{array}{ccccc}
 & \uparrow & & \uparrow & \\
\longrightarrow & M^{n,(m+1)} & \longrightarrow & M^{n+1,(m+1)} & \longrightarrow \\
 & \uparrow & & \uparrow & \\
\longrightarrow & M^{n,(m)} & \longrightarrow & M^{n+1,(m)} & \longrightarrow \\
 & \uparrow & & \uparrow &
\end{array}
$$

and we think of this as an inductive system of complexes $M^{(\bullet)} = (M^{(m)})$, where $M^{(m)} = (\cdots \to M^{n,(m)} \to M^{n+1,(m)} \to \cdots)$ is a complex of $A^{(m)}$-modules.

We denote the abelian category of $A^{(\bullet)}$-modules by $\mathbf{Mod}(A^{(\bullet)})$. We think of $M^{(\bullet)}$ as a representative of an element in the derived category $D(\mathbf{Mod}(A^{(\bullet)}))$.
Let $S^{\bullet}$ be a multiplicative set of morphisms $f^{\bullet} = (f^{(m)} : M^{(m)} \to N^{(m)})$; let $\mathbf{Mod}(A^{(\bullet)})_S$ denote the localization at S.
Assume that $S^{\bullet}$ induces a multiplicative set $S^{\natural}$ of morphisms in the derived category $D(\mathbf{Mod}(A^{(\bullet)}))$ and we localize to get $(D(\mathbf{Mod}(A^{(\bullet)})))_{S^{\natural}}$.
We ask when do we have

$$(D(\mathbf{Mod}(A^{(\bullet)})))_{S^{\natural}} \equiv D(\mathbf{Mod}(A^{(\bullet)})_{S^{\bullet}}) \ ?$$

Put $A = \varinjlim_m A^{(m)}$. We get a functor

$$\underset{\to}{l} : D(\mathbf{Mod}(A^{(\bullet)})) \to D(\mathbf{Mod}(A)).$$

Suppose that $S^\bullet$ induces a multiplicative set S of morphisms in the category $\mathbf{Mod}(A)$. Can we get

$$(D(\mathbf{Mod}(A^{(\bullet)})))_{S^\natural} \equiv D(\mathbf{Mod}(A)_S).$$

What are the analogues of these constructions for the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-objects of an ∞-category $\mathcal{C}$

Non-commutative completion

Let D be a (not necessarily commutative) ring. Say a two-sided ideal I of D is a central if it is generated by a family of elements from the center of D.
Suppose D is left noetherian, I is a central ideal, M is left D-module of finite type then we have isomorphism

$$\varprojlim_{n} D/I^n \otimes_D M \cong \varprojlim_{n} M/I^n M.$$

Inverse limits

Let $\mathcal{D}^\bullet$ be an inverse system of non-commutative rings and a $\mathcal{D}^\bullet$-module is an inverse system $\mathcal{E}^\bullet = (\mathcal{E}^{(m)}, \rho^{(m'm)})$ where $\mathcal{E}^{(m)}$ is a $\mathcal{D}^{(m)}$-module and for $m \leq m'$, $\rho^{(m'm)} : \mathcal{E}^{(m)} \overset{\rho^{(m'm)}}{\longleftarrow} \mathcal{E}^{(m')}$ is $\mathcal{D}^{(m')}$-linear. We denote the abelian category of $\mathcal{D}^\bullet$-modules by $\mathbf{Mod}(\mathcal{D}^\bullet)$. We have a left exact functor

$$\varprojlim : \mathbf{Mod}(\mathcal{D}^\bullet) \to \mathbf{Mod}(\varprojlim \mathcal{D}^{(m)}).$$

We ask if we can pass to the derived category

$$\mathbf{R}\varprojlim : D(\mathbf{Mod}(\mathcal{D}^\bullet)) \to D(\mathbf{Mod}(\varprojlim \mathcal{D}^{(m)}))$$

Such theories have been applied to study the derived functors on modules over non-commutative rings of the form

$$\mathcal{D}^{\dagger} := \varinjlim_{m} \varprojlim_{i} \mathcal{D}_i^m.$$

Let us write $\widehat{\mathcal{D}}^{(m)} = \varprojlim_i \mathcal{D}_i^m$. We ask can we get $D(\mathbf{Mod}(\mathcal{D}^{\dagger}))$ from $D(\mathbf{Mod}(\widehat{\mathcal{D}}^{(\bullet)}))$ by localizations of derived categories ?

Topos

Let $\mathscr{T}$ be a topos. A morphism of topos $f : \mathscr{T} \to \mathscr{T}'$ is $(f^* \dashv f_*, \phi)$ where $f^* : \mathscr{T}' \to \mathscr{T}$ is left adjoint to $f_* : \mathscr{T} \to \mathscr{T}'$, $\phi : \mathrm{Hom}_{\mathscr{T}}(f^* \cdot, ?) \to \mathrm{Hom}_{\mathscr{T}'}(\cdot, f_* ?)$ is the adjunction isomorphism.

Assume that we have in $\mathscr{T}$ a commutative ring object $\mathcal{R}$ and a non-commutative $\mathcal{R}$-algebra object $\mathcal{D}$. We can consider the category $Mod(\mathcal{D})$ of $\mathcal{D}$-module in $\mathscr{T}$.

Topos : Inverse limits

(1) Let I be a partially ordered set. Let $\mathscr{T}$ be a topos. We introduce the topos $\mathscr{T}_I$ of projective systems of objects of $\mathscr{T}$ indexed by I. The objects are projective systems $E_\bullet = (E_i)_{i\in I}$ and an arrow is a morphisms of projective systems, i.e. a family of morphisms $E_i \to F_i$ commuting with the transition maps of $E_\bullet, F_\bullet$.

(2) A a ring (resp. a module) of $\mathscr{T}_I$ (resp. , ...) is an projective system of rings (resp. modules) of $\mathscr{T}$.

So to say $A_\bullet$ is a ring in $\mathscr{T}_I$ means each A_i is a ring object in the category $\mathscr{T}$ and $A_\bullet$ is a projective system of rings.
And $M_\bullet$ is a $A_\bullet$-module means $M_i \in \mathscr{T}$ is an A_i-module and $M_\bullet$ is a projective system of modules.
For a projective system of rings $A_\bullet$, the *derived category* $D(A_\bullet)$ is the derived category of left $A_\bullet$-modules seen as objects of the topos $\mathscr{T}_I$.

(3) An increasing map $\varphi : I' \to I$ defines a morphism of topos

$$\varphi_{\mathscr{T}} = (\varphi_{\mathscr{T}}^{-1} \dashv \varphi_{\mathscr{T}*}) : \mathscr{T}_{I'} \to \mathscr{T}_I$$

given by
the inverse image functor $(\varphi_{\mathscr{T}}^{-1} E_\bullet)_{i'} = E_{\varphi(i')}$ and
the direct image functor
$(\varphi_{\mathscr{T}*} E'_\bullet)_i = \varprojlim_{\varphi(i') \leq i} E'_{i'}$.
For $i \in I$ denote by I'^{o}_{i} the subset of I' consisting of i' such that $\varphi(i') \geq i$ and equipped with the ordering opposite to that of I'.
Put $(\varphi_{\mathscr{T}!} E'_\bullet)_i = \varinjlim_{i' \in I'^{o}_{i}} E'_{i'}$. Then

$$\mathscr{T}_{I'} \; \begin{array}{c} \xrightarrow{\varphi_{\mathscr{T}!}} \\ \xleftarrow{\varphi_{\mathscr{T}}^{-1}} \\ \xrightarrow[\varphi_{\mathscr{T}*}]{} \end{array} \; \mathscr{T}_I$$

where each functor is a left adjoint of the functor below.

(4) Consider the case $\varphi : I \to \{*\}$. Then we get a topos morphism

$$\varphi_{\mathscr{T}} = (\varphi_{\mathscr{T}}^{-1} \dashv \varphi_{\mathscr{T}*}) : \mathscr{T}_I \to \mathscr{T}.$$

In this case for $F \in \mathscr{T}$, we have $(\varphi_{\mathscr{T}}^{-1} F)_i = F$, and

$$\varphi_{\mathscr{T}*}(E_\bullet) = \varprojlim_{i \in I} E_i.$$

Topos : Direct limits

Let I be a filtered set. Let $\mathscr{T}$ be a topos. We introduce the topos $\mathscr{T}^I$ of inductive systems of objects of $\mathscr{T}$ indexed by I. The objects are injective systems $E = (E^i)_{i\in I}$ and an arrow is a morphisms of inductive systems, i.e. a family of morphisms $E^i \to F^i$ commuting with the transition maps of $E^\bullet, F^\bullet$. This defines a topos.

For $E = (E^i)$ in $\mathscr{T}^I$ put

$$\underset{\rightarrow}{l}(E^\bullet) = \varinjlim_{i \in I} E^i.$$

If $A^\bullet$ is a ring of $\mathscr{T}^I$, then $A = \varinjlim_{i \in I} A^i$ is a ring of $\mathscr{T}$.

We get a functor taking a $A^\bullet$-module to an A-module and this passes to a functor on derived categories

$$\underset{\rightarrow}{l} : D(A^\bullet) \rightarrow D(A).$$

Perhaps you observe that I am only describing a portion of the theory of Grothendieck's 6 operations (Ayoub, *Les six opérations de Grothendieck*, Astérisque, 314, 315, 2007).
Let B be a Noetherian and finite-dimensional base scheme.
Denote by $\mathfrak{S}$ the category of separated finite-type smooth schems over B with smooth morphisms.
Write $\mathfrak{C}$ for the category of symmetric monoidal stable ∞-categories ($\otimes$ denotes the monoidal structure).
We assume that for each $\mathcal{C}$ in $\mathfrak{C}$ we fix a right adjoint $\underline{\mathrm{Hom}}(\star, \bullet) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ of $\otimes$, so we have adjunction $\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)) \cong \mathrm{Hom}(A \otimes B, C)$.
Let me give an incomplete description of a "*data of coefficients*" just to show the common features that we usually want.

A **data of coefficients** is a pair of functors $D^* : \mathfrak{S}^{op} \to \mathfrak{C}$, $D_! : \mathfrak{S} \to \mathfrak{C}$. For every smooth map $f : X \to Y$ in $\mathfrak{S}$, write $f^* := D^*(f) : D^*(Y) \to D^*(X)$ and $f_! := D_!(f) : D_!(X) \to D_!(Y)$

We assume that f^* is compatible with composition of maps $((g \circ f)^* \cong f^* \circ g^*)$,
compatible with the symmetric monoidal structure $(f^*(A \otimes B) \cong f^*A \otimes f^*B$, for $A, B \in D^*(Y))$,
and f^* has a right adjoint $f_* : D^*(X) \to D^*(Y)$ such that when f is proper we have
the projection formula $f_*A \otimes B \cong f_*(A \otimes f^*B)$ for $A \in D^*(X)$, $B \in D^*(Y)$;
and base change $g^*f_* \cong {f'}_*{g'}^*$ for each cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} .$$

We assume that $f_!$ is compatible with composition of maps $((g \circ f)_! \cong g_! \circ f_!)$, and has a right adjoint $f^! : D_!(Y) \to D_!(X)$ such that when D^*, $D_!$ agree on X, Y we have
the projection formula $f_! A \otimes B \cong f_!(A \otimes f^* B)$
and base change $g^* f_! \cong f'_! g'^*$ for each cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} .$$

Let us quote from Peter **Scholze**, *6 Functor Formalisms*, incomplete lecture notes for a course in Winter 2022/23 Bonn. "In the context where $D(X)$ is an ∞-category, there have been at least two formalizations of the datum of a 6-functor formalism:

(1) By **Liu Yifeng** and **Zheng Weizhe** (*Enhanced six operations and base change theorems for artin stacks*) in the context of étale cohomology of schemes. Their formalization is firmly rooted in Luries foundational works.

(2) By D. **Gaitsgory** and N. **Rozenblyum** (*A study in derived algebraic geometry*,Vol. I, 2017) in the context of coherent cohomology of schemes. Their formalization makes use of the formalism of $(\infty, 2)$-categories..

Recently, **Mann** (6-*Functor Formalism in Rigid-Analytic Geometry*, 318pp, 2022) has found a definition that combines the best of both worlds, it is firmly rooted in Luries formalism."

To connect us with the familiar world : let $\mathcal{A}$ be an abelian category with enough projective objects. Lurie uses differential graded nerves to construct an ∞-category $D^{-}(\mathcal{A})$ and we quote

"the homotopy category of $D^{-}(\mathcal{A})$ can be identified with the derived category of $\mathcal{A}$ studied in classical homological algebra" (**Lurie**, *Higher Algebra*, Remark 1.3.2.9).

The locally convex topological vector spaces analogue of the previous sections with quasi-abelian categories replacing abelian categories.

Derived functors on categories of representations of Lie groups.

To study the stable ∞-category - see the p-adic local Langlands correspondence of Fargues, Scholze, Emerton, Toby,...

I hope you agree that whether it is modules over non-commutative algebras, topos or stable ∞-category, I am talking about different bits of the same thing that needs to be organized better.

I would like to end by quoting from a paper of Emerton - Gee :

"*If we were writing these in the twentieth century, we would then proceed to explain that this obliges us to work with various derived or triangulated categories, and we would go on to explain which ones. But we will take advantage of contemporary advances in homological algebra by working with stable ∞-categories, which are especially useful to work with when one wants to apply various gluing or limiting processes.* "

THAT'S ALL FOLKS !

THANK YOU