Gan-Gross-Prasad conjectures for general linear groups

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Classical examples of restriction problems

1. Symmetric group S_n: irr. repn. π_λ parametrized by Young tableaux λ

$$\pi_\lambda|_{S_{n-1}} = \bigoplus_{\lambda'} \pi_{\lambda'},$$

where λ' runs through all tableaux by removing one box in λ

2. Unitary groups $U(n)$: irr. repn. π_μ parametrized by highest weights in $\mathbb{Z}^n$

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$$\mu_1 \geq \mu'_1 \geq \mu_2 \geq \ldots \geq \mu'_{n-1} \geq \mu_n$$

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1. Other real forms: $\mathrm{GL}(n, \mathbb{R})$, $U(p, q)$
2. Other fields $\mathrm{GL}(n, F)$ for local and global fields F

Another viewpoint of branching laws

Notation: $G_n = \mathrm{GL}_n(F)$, where F is a local field or a global field. Representations are over $\mathbb{C}$.

For $\pi_1 \in \mathrm{Irr}(G_{n+1})$, $\pi_2 \in \mathrm{Irr}(G_n)$,

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \cong \mathrm{Hom}_{\Delta G_n}(\pi_1 \boxtimes \pi_2^\vee, \mathbb{C})$$

Latter space $G_{n+1} \times G_n / \Delta G_n$ is spherical (i.e. expectation on finite multiplicity on the Hom-spaces) and lies in the framework of relative Langlands program of Sakellaridis-Venkatesh:

1. (Unitary version) Spectral decomposition for $L^2(G_{n+1} \times G_n / \Delta G_n)$
2. (Smooth version) Study the space $C^\infty(G_{n+1} \times G_n / \Delta G_n)$

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1. Mirabolic subgroup:

$$M_{n+1} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset G_{n+1}$$

2. Two steps restriction:

$$G_n \hookrightarrow M_{n+1} \subset G_{n+1}$$

via embedding

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

Restriction to mirabolic

Restriction to M_{n+1}:

1. Kirillov conjecture (Unitary version) (Bernstein (non-Archimedean), Sahi, Sahi-Stein, Baruch (Archimedean)): Any unitary irreducible G_n representation restricted to M_n is topologically irreducible.
2. Duflo geometric interpretation of Archimdean Kirillov conjecture in terms of orbit method (J. Yu)
3. Indecomposability (Smooth version) (Bernstein-Zelevinsky): For non-Archimedean F, any smooth irreducible G_n representation restricted to M_n is indecomposable.

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Restriction to GL

1. Multiplicity One (AGRS, SZ): $\pi_1 \in \mathrm{Irr}(G_{n+1})$ and $\pi_2 \in \mathrm{Irr}(G_n)$

$$\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) \leq 1$$

2. Generic branching law (Local GGP, JPSS 1983): $\pi_1 \in \mathrm{Irr}(G_{n+1})$, $\pi_2 \in \mathrm{Irr}(G_n)$ both generic. Then

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \neq 0$$

3. Prasad conjecture on Generic Ext-branching law (C.-Savin 2018): $\pi_1 \in \mathrm{Irr}(G_{n+1})$, $\pi_2 \in \mathrm{Irr}(G_n)$ both generic. Then

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0$$

for $i \geq 1$.

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Restriction to GL

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2. Projectivity criteria (C.-Savin 18, C. 19): Let $\pi \in \mathrm{Irr}(G_{n+1})$. Then $\pi|_{G_n}$ is projective if and only if π is generic and any irreducible G_n-quotient of π is generic.
3. Non-tempered branching law (non-temepred GGP conj, proved in C. 2020 preprint)
4. Generalizations to Bessel and Fourier-Jacobi models (C. 2020)

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Local Langlands correspondence

Let W_F be the Weil group of F. Let WD_F be the Weil-Deligne group. Let

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-Archimedean} \\ W_F & \text{if } F \text{ is Archimedean} \end{cases}$$

$\Phi(G)$= set of L-parameters i.e. set of admissible maps:

$$\psi : WD \to {}^L G.$$

Local Langlands correspondence asserts a natural finite surjection:

$$\mathrm{Irr}(G) \to \Phi(G)$$

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Arthur parameters

An Arthur parameter is the set of ${}^LG$-orbits of maps

$$\psi : WD_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG = \mathrm{GL}_n(\mathbb{C})$$

such that $\psi|_{WD_F}$ has bounded image i.e. has tempered Langlands parameter, and the restriction to $\mathrm{SL}_2(\mathbb{C})$-factor is algebraic.

1. The notion of Arthur parameter (and packet) is to remedy some properties failed in the tempered L-packet e.g. endoscopy theory.
2. Example: $\mathrm{triv} \boxtimes \mathrm{Sym}^k(\mathbb{C})$ gives the trivial representation

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From A-parameter to L-parameter

- $\mathrm{Sym}^k(\mathbb{C}^2)$: $(k+1)$-diml irr. rep. of $\mathrm{SL}_2(\mathbb{C})$
- Arthur parameter, as a finite $WD_F \times \mathrm{SL}_2(\mathbb{C})$-representation ψ, takes the form

$$M_A = \sum_d M_d \otimes \mathrm{Sym}^d(\mathbb{C}^2), \quad (1)$$

  where each M_d is a tempered representation of WD_F
- Given a Arthur parameter ψ, one obtains a L-parameter:

$$\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}), \quad w \in \mathrm{WD}$$

Relevant Arthur parameters

Definition (Gan-Gross-Prasad)

Let M_A and N_A be Arthur parameters. Then M_A and N_A are **relevant** if there exist tempered *WD*-representations $M_0^+, \ldots, M_r^+$ and $M_0^-, \ldots, M_s^-$ such that

$$M_A = \sum_{d=0}^{r} M_d^+ \otimes \mathrm{Sym}^d(\mathbb{C}^2) \oplus \sum_{e=1}^{s} M_e^- \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2),$$

$$N_A = \sum_{d=1}^{r} M_d^+ \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) \oplus \sum_{e=0}^{s} M_e^- \otimes \mathrm{Sym}^{e}(\mathbb{C}^2).$$

Remark: The notion of relevant is symmetric.

Gan-Gross-Prasad conjecture

Conjecture (Gan-Gross-Prasad $\sim$ 2019)

Let F be a local field. Let π_M and π_N be Arthur type representations of G_{n+1} and G_n respectively. Then

$$\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0 \Leftrightarrow M_A \text{ and } N_A \text{ are relevant.}$$

Theorem (C. 2020)

If F is non-Archimedean, then the conjecture is true.

Previous results: GGP, Gurevich, Gourevitch-Sayag (Archimedean)

Global Gan-Gross-Prasad conjecture: Generic

Period: Let π and π' be irreducible cuspidal automorphic repns of G_{n+1} and G_n respectively. For $\psi \in \pi$ and $\psi' \in \pi'$, period is defined as:

$$F(\psi \otimes \psi') = \int_{H(F)\backslash H(\mathbb{A})} \psi(g)\bar{\psi}'(g)dg$$

1. cuspidal $\Rightarrow$ integral absolutely convergent
2. (JPSS)

$$F(\psi \otimes \psi') \neq 0$$

if and only if the L-function

$$L(s, \mathrm{std}_{n+1} \otimes \mathrm{std}_n, \psi \otimes \psi') \neq 0$$

at $s = \frac{1}{2}$.

Global Gan-Gross-Prasad conjecture: General

In general, the period may not be absolutely convergent and need regularization..

Ichino-Yamana: Over a number field, define a regularized period via certain mixed truncation (based on Jacquet-Lapid-Rogawski)

$$\int_{H(F)\backslash H(\mathbb{A})} \Lambda^T(\phi)(g)\bar{\phi}'(g)dg$$

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A result of Ichino-Yamana

$\mathcal{A}(G)$: the space of automorphic forms of G, with subspaces:

$$\mathcal{A}_{\mathrm{cusp}}(G) \subset \mathcal{A}_{\mathrm{disc}}(G) \subset \mathcal{A}(G)$$

Classification of discrete automorphic representations of $G_n(\mathbb{A})$ (Mœglin-Waldspurger):

1. a pair (τ, k)
2. τ: a cuspidal automorphic repn of $G_{n/k}$
3. k divides n, and $n = kl$
4. $\nu(g) = |\det(g)|_F$
5. π is the unique irreducible quotient of

$$\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})} \nu^{(l-1)/2}\tau \boxtimes \ldots \boxtimes \nu^{-(l-1)/2}\tau$$

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A result of Ichino-Yamana

1. (Ichino-Yamana) Suppose π, π' are discrete automorphic representations which are not 1-dimensional. Then for $\phi \in \pi$ and $\phi' \in \pi'$, $F(\phi, \phi')$ is absolutely convergent and is equal to zero unless both π and π' are cuspidal.
2. Localizing π and π' to a non-Archimedean place is a Speh representation. The Arthur parameter of π_ν takes the form:

$$N \otimes \mathrm{Sym}^k(\mathbb{C}),$$

where N is an indecomposable WD-representation (i.e. discrete series).

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Tools to prove the local conjecture

Restricting to mirabolic subgroup (i.e. Bernstein-Zelevinsky theory):

$$M_{n+1} = G_n \ltimes F^n$$

and G_n acts on F^n by two orbits: zero orbit and open orbit. This gives a filtration:

$$0 \to \mathrm{Ind}_{M_n N}^{G_n} \pi_{N,\psi} \otimes \psi \to \pi|_{M_{n+1}} \to \pi_N \to 0,$$

where

$$N = \left\{ \begin{pmatrix} I_n & * \\ & 1 \end{pmatrix} \right\} \cong F^n.$$

Repeating the process gives a Bernstein-Zelevinsky filtration. To explicate the filtration, one needs a notion of derivatives.

Tools: Derivatives

- Derivative: Let $R_i = \left\{ \begin{pmatrix} I_{n-i} & x \\ & u \end{pmatrix} : u \in U_i \right\}$. The i-th derivative, as G_{n-i}-repn

$$\pi^{(i)} = \text{ (normalized) } \psi\text{-twisted Jacquet functor of } R_i,$$

  where ψ is generic character on the part U_i.
- Level of π: largest integer such that $\pi^{(i)} \neq 0$.
- Shifted derivative: $\pi^{[i]} = \nu^{1/2} \cdot \pi^{(i)}$ e.g.

$$\mathrm{triv}_n^{[1]} = \mathrm{triv}_{n-1}$$

- For the level k^* of π, define $\pi^- = \pi^{[k^*]}$.

Tools: Derivatives

Imposing the Gelfand-Kazhdan involution

$$\theta(g) = g^{-t},$$

we have the left derivative of π:

$$^{(i)}\pi = \theta(\theta(\pi)^{(i)})$$

and the shifted derivatives

$$^{[i]}\pi = \nu^{-1/2} \cdot {}^{(i)}\pi$$

A consequence of Zelevinsky theory: When $k^* =$ level of π,

$$\pi^- = \pi^{[k^*]} \cong {}^{[k^*]}\pi$$

is irreducible.

Theorem (C. 2019)

Let $\pi \in \mathrm{Irr}(G_n)$. If i is not the level of π, then $\pi^{[i]}$ and $^{[i]}\pi$ do not have isomorphic irreducible quotients and do not have isomorphic irreducible submodules.

Another tool: Dual restriction

Duality for restriction:

Proposition

Let $\pi_1 \in \mathrm{Alg}(G_{n+1})$ and $\pi_2 \in \mathrm{Alg}(G_n)$. For all i,

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2^\vee) \cong \mathrm{Ext}^i_{G_{n+1}}(\pi_2 \times \sigma, \pi_1^\vee)$$

for a suitable choice of cuspidal representation $\sigma \in \mathrm{Alg}(\mathrm{GL}_2)$.

Proof of 'if direction' (Sketch)

Suppose (M_A, N_A) are relevant. Write

$$\pi_M = u \times \pi'_M, \quad \pi_N = u^- \times \pi'_N,$$

where $u = u_{\rho_1}(m_1, d_1)$ (chosen specially from dual restriction)

$$(\pi'_M, \pi'_N) \text{ relevant}$$

$$\Updownarrow \text{ Induction}$$

$$\sigma \times \pi'_M \text{ has a quotient } \pi'_N$$

$$\Updownarrow \text{ GGP type reduction}$$

$$\mathcal{RS}(\pi'_M) \text{ has a quotient of } \pi'_N$$

$$\Downarrow \text{ exactness of product}$$

$$u^- \times \mathcal{RS}(\pi'_M) \text{ has a quotient of } u^- \times \pi'_N$$

$$\Updownarrow \text{ Filtration on product}$$

$$u \times \pi'_M \text{ has a quotient of } u^- \times \pi'_N$$

Bessel and Fourier-Jacobi models

Bessel model (for odd corank) and Fourier-Jacobi model (for even corank):

- Bessel and Fourier-Jacobi subgroups:

$$H = \left\{ \begin{pmatrix} u_1 & * & * \\ & g & * \\ & & u_2 \end{pmatrix} : u_1 \in U_{m_1}, u_2 \in U_{m_2}, g \in G' \right\},$$

  where $G' = \{\mathrm{diag}(1, g') \in G_{r+1} : g' \in G_r\}$,
  $r = n - m_1 - m_2 - 1$.
- U_H= unipotent radical of H
- Let $\psi : U_H \to \mathbb{C}$ be generic i.e. dense orbit by $T \rtimes U_H$
  Corank=$m_1 + m_2 + 1$ (above not include corank 0 case)

Bessel and Fourier-Jacobi models

Restriction problem: For $\pi_1 \in \mathrm{Irr}(G_n)$, $\pi_2 \in \mathrm{Irr}(G_r)$,

$$m_H(\pi_1, \pi_2) = \dim \mathrm{Hom}_H(\pi_1 \otimes \psi\delta_H^{-1/2}, \pi_2)$$

Theorem (C. 2020)

Suppose $\pi_1 \in \mathrm{Irr}(G_n)$ and $\pi_2 \in \mathrm{Irr}(G_r)$ with Arthur parameters M_A and N_A. Then $m_H(\pi_1, \pi_2) \neq 0$ if and only if M_A and N_A are relevant.

A missing case: Equal rank

Let $S(F^n)$ be the Bruhat-Schwartz space. Transfer to the problem of

$$\mathrm{Hom}_{G_n}(\pi_1 \otimes \nu^{-1/2} S(F^n), \pi_2) \cong \mathrm{Hom}_{G_n}((\chi \times \pi_1)|_{G_n}, \pi_2)$$

for a suitable character χ of $F^\times$.

Theorem (C. 2020)

Let $\pi_1, \pi_2 \in \mathrm{Irr}(G_n)$ with Arthur parameters. Then $\mathrm{Hom}_{G_n}(\pi_1 \otimes \nu^{-1/2} S(F^n), \pi_2) \neq 0$ if and only if π_1, π_2 are relevant.

Summary

1. Prasad conjecture on Generic Ext-branching law (C.-Savin 2018): $\pi_1 \in \mathrm{Irr}(G_{n+1})$, $\pi_2 \in \mathrm{Irr}(G_n)$ both generic. Then
$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0$$
for $i \geq 1$.
2. Analogue of Kirillov conjecture (C. 2019 Preprint): Let $\pi \in \mathrm{Irr}(G_{n+1})$. Then each Bernstein-component of $\pi|_{G_n}$ is indecomposable.
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Thank you!