Beautiful pairs of valued fields and adic spaces

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This is joint work with Pablo Cubides-Kovacsics and Martin Hils.

Berkovich analytification

Let K be an algebraically closed field and complete with respect to a non-Archimedean valuation $|\cdot| : K \to \mathbb{R}_\infty$. Valuations and non-Archimedean norms are more or less the same by $\alpha \mapsto e^{-\alpha}$.
One wish to develop a theory of analytic functions on K as in $\mathbb{C}$.
Unlike $\mathbb{C}$, the non-Archimedean axiom implies that the topology on K is totally disconnected; Hence, the locally analytic functions behave quite arbitrarily.
Historically, Tate approached this by working with a Grothendieck topology. The resulting analytic spaces have a nice function theory, but lack topological intutions.

Berkovich analytification

Berkovich approached the question by considering space of valuations.

Definition

Let K be as before and V an affine variety over K. Let V^{an} denote the set of multiplicative semivaluations $\mathcal{O}_V(V) \to \mathbb{R}_\infty$ extending that of K, and equipped with the weakest topology such for each $f \in \mathcal{O}_V(V)$, the map $p \mapsto p(f)$ is continuous.

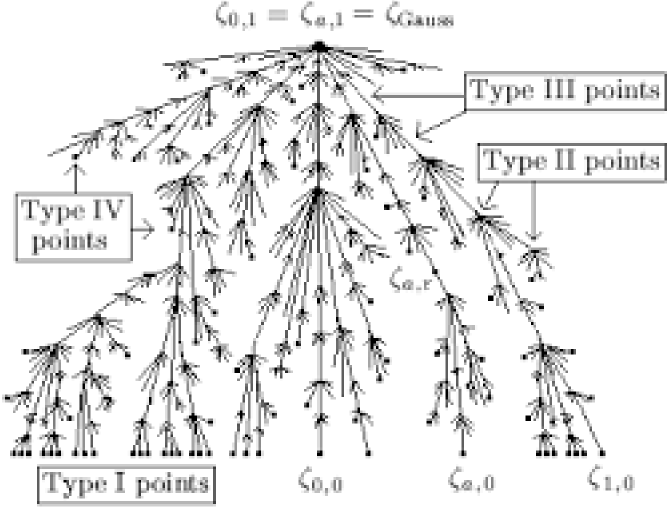
V^{an} has nice topological properties: Hausdorff(if V is seperated), locally compact, locally contractible.

Type I Points $c \in K$.

Type II Closed balls with center $c \in K$ and radius $r \in |K|$.

Type III Closed balls with center $c \in K$ and radius $r \notin |K|$.

Type IV A nested family of balls with trivial intersection (in K).



One can easily see the Berkovich affine line is contractible via the map that sends (c, t) to the ball with radius t around c.
How do we make sense of such spaces model theoretically, and in such a meaningful way so that the above map becomes a “definable” morphism in this category?

Crash course on model theory

Model theorists study definable sets in some structure. A *language* L is a collection of symbols f_i called function symbols and R_j called relation symbols. An *L-structure* is a set (M, f_i, R_j), where M is the underlying set and f_i and R_j's are interpreted as functions and relations with appropriate arities.

A *formula* $\varphi(x)$ in L is a mathematical expression one forms using the symbols above and quantifiers (over elements of the underlying set). A formula without a free variable is called a *sentence*. A *theory* T is a collection of sentences. And we say M is a model of T ($M \models T$) if the sentences in T are true in M. And we say a set $D \subseteq M$ is *definable* if there is a formula $\varphi(x)$ such that $D = \{a \in M : M \models \varphi(a)\}$. And we say a theory T is complete if there is a model M such that $T = \{\varphi : M \models \varphi\}$.

Example

The language of groups has a binary function symbol $\cdot$ and a constant symbol 1. And the theory of groups T consists of following (1) $\forall x, y, z \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (2) $\forall x \exists y \;\; x \cdot y = y \cdot x = 1$ (3) $\forall x \; 1 \cdot x = x \cdot 1 = x$. Models of T are exactly all groups. It is clear that T is not complete.

We say that M is an elementary substructure of N ($M \preceq N$) if for any formula $\varphi(x)$ with parameters in N. If $N \models \exists x \varphi(x)$ then $M \models \exists x \varphi(x)$. For example, $\mathbb{Q}^{alg} \preceq \mathbb{C}$ as fields.
Let M be a model of T, we say p is a type over M ($p \in M$) if there is an elementary extension N and $a \in N$ such that $p = \{\varphi(x) \in L_M : N \models \varphi(a)\}$. In the above example, types over $\mathbb{Q}^{alg}$ are exactly the data of generic points of varieties over $\mathbb{Q}^{alg}$.

Example (Important)

The (2-sorted) language of valued fields has two sorts VF (the valued field) and Γ_∞ (the value group) and a function symbol $v : \mathrm{VF} \to \Gamma_\infty$. The valued field sort is equipped with the usual field structure and the value group is an ordered abelian group with a distinct element ∞ and the map v is a valuation. The theory of algebraically closed valued fields asserts that VF is an algebraically closed field and v is a nontrivial surjective valuation with value group Γ.
The theory ACVF is complete modulo the characteristic of VF and residue characteristic. Moreover, the sort Γ is o-minimal. And a type in the valued field sort over some $K \models \mathrm{ACVF}$ correponds exactly the data of valuation on functions field of V/K.

The functor $V \mapsto V^{an}$ is a subfunctor $V \mapsto S_V(K)$ for $K \models \mathrm{ACVF}$. But one rarely talks about definability on type spaces, and any attempt to develop a geometric theory on it is hard. Instead, working in ACVF, Hrushovski and Loeser identified a special subset (set of generically stable types) $\widehat{V}$ of the type space as model theoretic analogues of V^{an}.
The first step of their work is to show that the set $\widehat{V}$ is strict pro-definable(a small projective limit of definable sets in ACVF). This grants them the ability to discuss definable maps and particularly, the ability to use Γ as an replacement of $\mathbb{R}$ in talking about the topology of $\widehat{V}$ definably.

Model theoretic adic space

There are other attempts on the theory of analytic spaces over non-Archimedean fields. A notable one was by Huber via the adic spaces, and has proven to be useful in recent developments in mathematics.
The goal of the talk is to establish a model theoretic analogue of adic spaces following Hrushovski and Loeser. And talk about various liftings of the results by Hrushovski and Loeser.

Definable types and pro-definability

We fix a model $K \models \mathrm{ACVF}$ and a definable set X. By *definable types* on X, we mean that that for each formula $\varphi(x; y)$, the set $\{c \in K : \varphi(x; c) \in p\}$ is definable and $X \in p$.
And we say a type $p \in S_x(K)$ is *generically stable* if p is definable and finitely satisfiable over K.
Let p be a definable type, call its canonical parameters $c_{\varphi,p}$'s. The type p is determined by the sequence $(c_{\varphi,p})_\varphi$.
In general, if we can show the φ-definition is uniform over p, meaning for each $\varphi(x, y)$ there is $\psi(y, z)$ such that for each p, the φ-definition of p is given by $\psi(y, c_{\varphi,p})$ for each p. Then we have established pro-definability.

Strict pro-definability

In general, pro-definable sets are still not well-behaved enough to study topology/geometry (E.g. Cantor set). We say a pro-definable set is *strict pro-definable* if the transition maps between the sets in the inverse limit are all surjective.
When we identify a subset C of definable types as a pro-definable set, strict pro-definabilty is equivalent to $\{c_{\varphi,p} : p \in C\}$ is definable for each φ.
In the case of $\widehat{V}$, it is well know that NIP theories have uniform definition for generically stable types, and Hrushovski and Loeser relied on heavy machinery in stable domination to show the strictness.

Strictness and beautiful pairs

Poizat first initiated the study of (proper) pairs of stable structures, and the sufficiently saturated ones are the so-called "belles paires" (beautiful pairs). They are, roughly speaking, pairs of models $M \preceq N \models T$ where M is sufficiently saturated, and N is $|M|^+$-saturated.

Poizat showed that all such pairs are elementarily equivalent, and assuming some further technical properties (nfcp), one gets that M is embedded as a pure substructure. In this case, for each L-formula $\varphi(x, y)$, let $\psi(y, z)$ be the uniform definition, the set

$$\{c \in M : \exists b \in N \ \forall y \in M \, \varphi(b, y) \Leftrightarrow \psi(y, c)\}$$

is L-definable in M, which is exactly the strictness we wanted.

Strictness and beautiful pairs

Along a similar direction, van den Dries and Lewenberg established that for any pairs of $K \preceq L \models \mathrm{RCF}$ such that K is Dedekind complete in L, they are all elementarily equivalent. Moreover, K is embedded as a pure substruture. This gives us the strictness of the set of definable types in real closed fields.

One can work by hand and show that similar property holds for pairs of divisible ordered abelian groups $M \preceq N$ such that M is Dedekind complete in N or end extensions of models of Presburger arithmetic $M \preceq N$.

Beautiful pairs of valued fields

It is well known that for pairs of (nice) valued fields, $K \preceq L$ with the property that $tp(a/K)$ is definable over K implies that L/K is *separated*, i.e. any n-dimensional K-vector space V in L admits a basis $\{v_1, ..., v_n\}$ such that $\mathrm{val}(c_1 v_1 + ... + c_n v_n) = \min \mathrm{val}(c_i v_i)$. We obtained results along the philosophy of Ax-Kochen-Ershov.

Theorem

For separated pairs of "nice" valued fields, the theory of the beautiful pairs can be axiomtized by the following: (1) (K, L) *is a separated proper pair of valued fields. (2) The residue field and value group sorts are models of the theory of the coorespoinding beautiful pairs. And for* $K \preceq L$ *a model of such theories,* K *is embedded as a pure substructure.*

Beautiful pairs of valued fields

Specific examples of such nice valued fields includes :

- Algebraically closed valued fields
- Real closed valued fields
- $\mathbb{C}((t))$
- $\mathbb{Q}_p$

Many of the corresponding type spaces can be viewed as model theoretic analogue of certain analytifications. For example, in ACVF, $S_V^{def}(K)$ corresponds to the Zariski-Riemann space of V, $\widehat{V}$ corresponds to the Berkovich space of V. And in RCVF, $\widehat{V}$ corresponds to the real analytification of V.

From now on, we will focus on ACVF. This part is joint work with Pablo Cubides-Kovacsics.

Definition

Let p be a definable type, we say that p is *bounded* if for some model K such that p is defined over, there is $K \preceq L$ with a realization of $p|K$ in L such that $\Gamma(K)$ is cofinal in $\Gamma(L)$.

We use $\widetilde{V}$ to denote the set of bounded definable types on V, and $\widetilde{V}(K)$ to denote those definable over K. We use $\widehat{V}$ to denote the set of generically stable types on V.

Theorem

$\widetilde{V}, \widehat{V}$ *are strict pro-definable.*

Like the adic space, one can topologize $\widetilde{V}$ with two different topologies.

Definition

Let U be a Zariski open and $f, g \in \mathcal{O}_V(U)$. Topologize $\widetilde{V}$ by the weakest topology such that $\{p : \infty \neq v \circ f_*(p) \leq v \circ g_*(p)\}$ is open. We topologize $\widehat{V}$ by the sets of the form $\{p \in \widehat{V} : v \circ f_*(p) < v \circ g_*(p)\}$. We say an open subset of $\widetilde{V}$ is *partially proper* if it is closed under specialization and use $\widetilde{V}_{p.p}$ to denote $\widetilde{V}$ with the partially proper topology.

Example

The valuation ring $\widetilde{\mathcal{O}}$ is an open subset of $\widetilde{\mathbb{A}^1}$ but not partially proper since the ball with valuative radius 0^- is the specialization of the generic type of $\mathcal{O}$. However, let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}$. Consider $U = \widetilde{\mathfrak{m}} \backslash \{p\}$ where p is the generic type of $\mathfrak{m}$, one can check that U is partially proper open.

Note that in the above example. Consider $\widehat{V} \subseteq \widetilde{V}$, $U \cap \widehat{\mathbb{A}^1} = \widehat{\mathfrak{m}}$, which is an open subset of $\widehat{\mathbb{A}^1}$.

Canonical extensions

The above follows from a general fact, as in the comparision of Berkovich and Huber's analytification.

Theorem

$\widehat{V} \subseteq \widetilde{V}$ and the topology on $\widehat{V}$ is the induced topology of $\widetilde{V}_{p.p}$.

An important feature in the Hrushovski-Loeser theory is the canonical extension. Namely, to define a map $h : \widehat{V} \to \widehat{W}$, it suffices to define a map $h' : V \to \widehat{W}$, there is a canonical extension to $\widehat{V}$. Similar feature exists in the category of $\widetilde{V}$, and HL's canonical extension is the restriction of the canonical extension in the category of $\widetilde{V}$.

Canonical extensions

Theorem (Hrushovski,Loeser)

Let V be a quasi-projective variety, there is a pro-definable deformation retraction $h : I \times \widehat{V} \to \widehat{V}$ with a Γ-internal and iso-definable image. Here I is a generalized interval in Γ_∞.

Theorem

The above deformation retraction lifts to $H : I \times \widetilde{V}_{p.p} \to \widetilde{V}_{p.p}$.

And similar tricks as in Hrushovski and Loeser's work descend this deformation retraction to the adic space of V.

Connection to adic/Berkovich spaces

Model theoretically, given $K \models$ ACVF and K complete with respect to the valuation, with value group $\mathbb{R}$. Let V be an variety over K, we have

$$V^{an} = \{p \in S_V(K) : \text{p is weakly orthogonal to } \Gamma\}$$
$$V^{ad} = \{p \in S_V(K) : \text{p is bounded in } \Gamma(K)\}$$

For any K, one can find a speherically complete ACVF with value group $\mathbb{R}$, call it K^{max}. One can define the restriction map

$$\pi : \widehat{V}(K^{max}) \to V^{an}$$
$$\pi : \widetilde{V}(K^{max}) \to V^{ad}$$

And the above restriction map desecends the deformation retraction.

Fibration of $\widetilde{V}$ over $\widehat{V}$

There is a canonical map $r : \widetilde{V} \to \widehat{V}$.
Let $p \in \widetilde{V}$, since p is a definable type on V, one can consider p as a definable type on $\widehat{V}$. By Hrushovski-Loeser, p has a unique limit in $\widehat{V}$, call it $r(p)$. The fiber of $r(p)$ is in canonical bijection of the set of valuations on $\mathrm{res}(r(p))$.
The fibration gives us some hands-on tools to study the structure of $\widetilde{V}$.

Recall that a set is iso-definable if it is in pro-definable bijection with a definable set. For example, $\widehat{\mathbb{A}^1}$ is iso-definable because it is canonically identified as the set of closed balls.
Using Riemann-Roch, it is not hard to see that for any curve C, $\widehat{C}$ is iso-definable.
Fibrating $\widetilde{C}$ over $\widehat{C}$, for some points $p \in \widehat{C}$, the genus of the residue curve is 0, so the fiber looks like $\mathbb{P}^1_k$. But there might be bad points.

Iso-definability of curves

Theorem

Let $p_1, ..., p_n \in \widehat{C}$, let g_{p_i} denote the genus of the residue curve at p_i. Then $\sum g_{p_i} \leq$ Genus of C.

In particular, this implies that there are only finitely many points whose residue curve is bad. Essentially, $\widetilde{C}$ looks like $\widehat{C} \times \mathbb{P}^1_k$.

Thank you for your attention!