Applications of Hecke Algebra in the Representation Theory of Reductive Groups

Ma, Jia-Jun

School of Mathematical Sciences
Xiamen University
Department of Mathematics

Xiamen University Malaysia

BICMR
4 Sept, 2023

Two questions

- Counting special unipotent representations of real reductive groups.
- Determining the theta correspondence over finite fields.
- Why discuss them in a single talk?

Barbasch-Vogan's definition of special unipotent representation

G: a real reductive group $\rightsquigarrow$ Langlands dual $\mathbf{G}^\vee$.

Nilpotent orbit $\check{\mathcal{O}}$ of $\mathbf{G}^\vee$.

$\rightsquigarrow$ an infinitesimal character $\lambda_{\mathcal{O}^\vee}$

$\rightsquigarrow$ the maximal primitive ideal $\mathcal{I}_{\check{\mathcal{O}}}$ with inf. char. $\lambda_{\check{\mathcal{O}}}$

- *Definition* (Barbasch-Vogan):

  An irreducible G-repn. is called *special unipotent* if

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \mathcal{I}_{\check{\mathcal{O}}}.$$

- $\mathrm{Unip}_{\check{\mathcal{O}}}(G) :=$ { special unipotent repn. attached to $\check{\mathcal{O}}$}.
- $\#\mathrm{Unip}_{\check{\mathcal{O}}}(G) = ??$

Examples

$G = \mathrm{SL}_2(\mathbb{R})$.

- $\check{\mathcal{O}} =$ principal orbit:

  $\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{$ trivial repn. $\}$

- $\check{\mathcal{O}} =$ zero orbit:

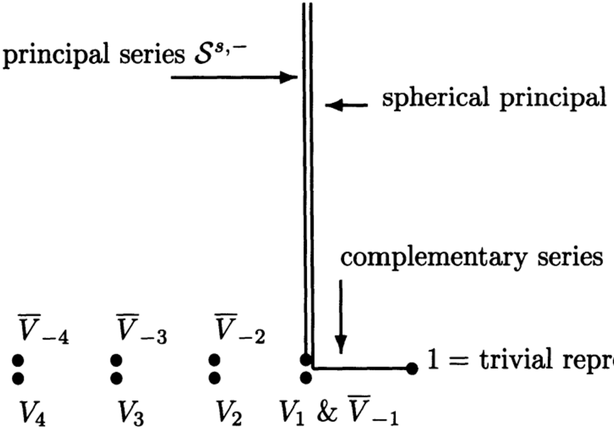
  $\mathrm{Unip}_{\check{\mathcal{O}}}(G) =$
  $\{$ 2 limit of discrete series, a spherical principle series $\}$

```
In [17]:  print(f"#Unip_(3)(SL_2(R) = {countC((3,))}")
          print(f"#Unip_(1,1,1)(SL_2(R) = {countC((1,1,1))}")

          #Unip_(3)(SL_2(R) = 1
          #Unip_(1,1,1)(SL_2(R) = 3
```

- https://www.kaggle.com/hoxidema/counting-special-unipotent-repn

Unitary dual



Elias M. Stein

Complex associated variety

- $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$
  $\Longleftrightarrow$ π has inf. char. $\lambda_{\check{\mathcal{O}}}$ and $\mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}$
- $\mathrm{Nil}(G_{\mathbb{C}}) \ni \mathcal{O}$
  $:=$ the Lusztig-Spaltenstein-Barbasch-Vogan dual of $\check{\mathcal{O}}$.
- $\mathcal{O}$ is a *special nilpotent orbit.*
- *Question:* For $\mathcal{O} \in \mathrm{Nil}(G_{\mathbb{C}})$, inf. char. λ,
  $\#\{\,\pi \in \mathrm{Irr}(G) :$ inf. char. $\pi = \lambda$ and $\mathrm{AV}_{\mathbb{C}}(G) = \overline{\mathcal{O}}\,\} = ??$.
- This is question also relevent if one consider non-special unipotent representations (defined by Losev, Mason-Brown, and Matvieievskyi).

Counting irr. repn. with a fixed asso. variety (integral case)

- *Assume:* inf. char. λ is integral.
  Fact (Joseph):
  AV(prim. ideal w. inf. char. λ) $=$ $\overline{\text{a special nilpotent orbit}}$
  Double cell $\mathcal{D}$ in $\mathrm{Irr}(W)$ $\longleftrightarrow$ the special nilpotent orbit $\mathcal{O}$.
- $\mathrm{Coh}_{[\lambda]}(G)$: the coherent continuation repn. based on $\lambda + X^*$.
- $W_\lambda := \{\, w \in W \mid w\lambda = \lambda \,\}$

Theorem

If E_8 is not a simple factor of G, then

$$\#\{\, \pi \in \mathrm{Irr}(G) \mid \text{inf. char. } = \lambda, \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} = \sum_{\tau \in \mathcal{D}} \dim \tau^{W_\lambda} \cdot [\tau : \mathrm{Coh}_{[\lambda]}(G)]$$

Counting unipotent representations

- Complex reductive groups, $\check{\mathcal{O}}$ integral, (Barbasch-Vogan)
  $\#\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \#$ Lusztig's canonical quotient of $\check{\mathcal{O}}$.
  Assume: $\check{\mathcal{O}}$ has good parity
- $\mathrm{U}(p,q)$, (Barbasch-Vogan)
  $\#\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \#$ real forms of its BV-dual $\mathcal{O}$.
- $\mathrm{SU}(p,q)$, restriction from that of $\mathrm{U}(p,q)$ or a double cover of $\mathrm{U}(p,q)$
- Real classical groups
  $\#\mathrm{Unip}_{\check{\mathcal{O}}}(G) =$ painted bi-partitions (BMSZ).
  Construction: theta correspondence
- Spin group
  very few genuine special unipotent representations.
- Exceptional group
  Atlas of Lie group

Dual pairs over finite fields

- $F := \mathbb{F}_q$ a finite field, s.t. $|F| = q$.
- (V, V'): a dual pair of Hermitian spaces

	$G = \mathrm{U}(V)$	$G' = \mathrm{U}(V')$	
(A)	unitary gp.	unitary gp.	
(B)	odd orthogonal gp.	"metaplectic" gp.	
(D)	even orthogonal gp.	symplectic gp.	$p \neq 2^r$
(C)	symplectic gp.	even orthogonal gp.	
$(\widetilde{C})$	"metaplectic" gp.	odd orthogonal gp.	

- We focus on case (C) today.

Theta lifting/Howe correspondence

- V symplectic space, V' quadratic space.
- (modified) Weil representation

$$\omega_{V,V'} := \left(\mathbf{1} \boxtimes (\xi \circ \det_{V'})^{\frac{1}{2}\dim_F V}\right) \otimes \omega_{\psi, V \otimes_F V'}$$

($\omega_{\psi, V\otimes_F V'}$: Weil representation of $\mathrm{U}(V \otimes_F V')$ a la Gérardin,
ξ the quadratic character of $F^\times$)

- Orthogonal gp. acts *geometrically* on the Schrödinger model.
- Compatible with the *conservation relation*.

Theta lift functor

Theta lift functor

$$\begin{array}{rcl}\Theta_{V,V'}\colon \mathrm{Rep}(G) & \longrightarrow & \mathrm{Rep}(G') \\ \sigma & \mapsto & \left(\omega_{V,V'} \otimes \sigma^\vee\right)_G\end{array}$$

Srinivasan

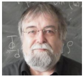
Howe

Adams

Moy

Aubert

Michel



Rouquier

Pan

Liu

Wang

Gurevich

...

Srinivasan, Weil representations of finite classical groups (1979)

Case (i), $m \leqq n$.

$$(4.3) \quad \omega_{\mathrm{unif}} = \sum_{k=0}^{m-1} \sum_{(T)\subset Sp_{2k}} \frac{1}{|W(T)|} \sum_{\theta\in\hat{T}} \varepsilon\,\varepsilon'\, R^{Sp_{2n}}_{T\times Sp_{2n-2k}}(\theta\times 1) \times R^{SO^{\varepsilon}_{2m}}_{T\times SO^{\varepsilon'}_{2m-2k}}(\theta\times 1)$$

$$+ \sum_{\substack{(T)\subset Sp_{2m}\\ T\subset SO^{\varepsilon}_{2m}}} \varepsilon(-1)^{n+m}\cdot\frac{2}{|W(T)|}\sum_{\theta\in\hat{T}} R^{Sp_{2n}}_{T}(\theta)\times R^{SO^{\varepsilon}_{2m}}_{T}(\theta).$$

Conservation relation

- $\mathcal{V}', \widetilde{\mathcal{V}}'$: Witt towers of even dim. quadratic spaces
$$\mathrm{disc}(\mathcal{V}') \neq \mathrm{disc}(\widetilde{\mathcal{V}}')$$
- *First occurrence index*
$$n_{V,\mathcal{V}'}(\sigma) := \min \{ \dim V' \mid \Theta_{V,V'}(\sigma) \neq 0, V' \in \mathcal{V}' \}$$

Theorem (Conservation relation I)

If trivial repn. of $\mathrm{GL}_1(\mathbb{F}_q)$ is not in the cuspidal support of σ, then
$$n_{V,\mathcal{V}'}(\sigma) + n_{V,\widetilde{\mathcal{V}}'}(\sigma) = 2 \dim V + \delta, \quad \text{with } \delta = 2$$

- Sun-Zhu (2014):
$\delta = \max \{$ dim. of an anisotropic quadratic space $\}$
- Pan (2002):reduction to p-adic unip. repn. (cuspidal)
- Pan (2022):reduction to the unipotent case. (general)

Parabolic inductions relevant to θ-correspondence

Definition: σ is *theta-cuspidal* $\Leftrightarrow \mathbf{1}$ of $\mathrm{GL}_1(\mathbb{F}_q) \notin$ cusp. supp. of σ.

- $V_l = V \oplus \mathbb{H}^l$ ($\mathbb{H}$ the hyperbolic space)
- A parabolic subgp. P_l of $G_l := \mathrm{U}(V_l)$ with Levi

$$L_l := \underbrace{\mathrm{GL}_1(F) \times \cdots \times \mathrm{GL}_1(F)}_{l\text{-terms}} \times \mathrm{U}(V)$$

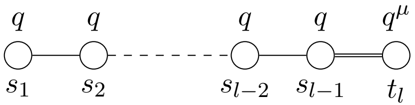
- Harish-Chandra series:
  $\mathcal{E}(l,\sigma) = \left\{ \text{irr. constituents in } \mathrm{Ind}_{P_l}^{G_l} \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{l\text{-terms}} \otimes \sigma \right\}$.

$$\sigma_l := \begin{pmatrix} \underbrace{\begin{matrix} \mathbf{1} & & \\ & \ddots & \\ & & \mathbf{1} \end{matrix}}_{l \times l} & & \\ & \sigma & \\ & & \ddots \end{pmatrix}$$

Hekce algebra $\mathcal{H}_{l,\sigma} := \mathrm{End}_{G_l}(\mathrm{Ind}_{P_l}^{G_l}\sigma_l^{\vee})$

- Howlett-Lehrer + Lusztig:

  $\mathcal{H}_{l,\sigma} \cong$ the Hecke algebra of W_l with *unequal* parameters
- $\mathrm{Norm}_{\mathrm{U}(V_l)}(L_l)/L_l \cong \mathsf{W}_l := \mathsf{S}_l \ltimes \{\pm 1\}^l$.



- $\mathcal{H}_{l,\sigma} = \langle T_s | s = s_1, \cdots, s_{l-1}, t_l \rangle$ with *Quadratic Relations*

$$(T_{s_i} + 1)(T_{s_i} - q) = 0 \quad \forall 1 \le i \le l-1$$
$$(T_{t_l} - C_1)(T_{t_l} - C_2) = 0 \quad \text{with} \quad q^{\mu} = -\frac{C_1}{C_2}$$

The operator T_{t_l}

$$(T_{t_l} - C_1)(T_{t_l} - C_2) = 0 \quad \text{with} \quad q^{\mu} = -\frac{C_1}{C_2}$$

- Let V' and $\widetilde{V}'$ be the first occurnce spaces in $\mathcal{V}'$ and $\widetilde{\mathcal{V}}'$.
- Compute the T_l-action on $\mathrm{Hom}_{G_l}(\mathrm{Ind}_{P_l}^{G_l}\sigma_l, \omega_{V_l,V'})$

  $C_1 = \gamma_{V'} q^{\dim V + \frac{1}{2}\delta - \frac{1}{2}\dim V'}$

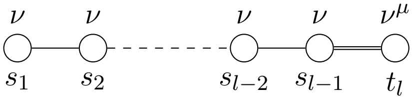
  $C_2 = \gamma_{\widetilde{V}'} q^{\dim V + \frac{1}{2}\delta - \frac{1}{2}\dim \widetilde{V}'}$

  $C_1 C_2 = -T_{t_l}^2(1) = -q^{\dim V + \frac{1}{2}\delta}$
- Compare the *powers* $\Rightarrow$ Conservation relation.

generic Hekce algebra

- $\mathsf{H}_{l,\mu} = \langle T_s | s = s_1, \cdots, s_{l-1}, t_l \rangle$ free over $\mathbb{Z}[\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}]$. with *Quadratic Relations*

$$(T_{s_i} + 1)(T_{s_i} - \nu) = 0 \quad \forall 1 \leq i \leq l-1$$
$$(T_{t_l} + 1)(T_{t_l} - \nu^{\mu}) = 0$$



Hecke bimodule and its deformation

Assume: theta-cuspidal $\sigma \overset{\Theta}{\longleftrightarrow}$ theta-cuspidal σ',

- Consider the $\mathcal{H}_{l,\sigma} \times \mathcal{H}_{l',\sigma'}$-module:

  $\mathcal{M} := \mathrm{Hom}_{G_l \times G'_{l'}}(\mathrm{Ind}_{P_l}^{G_l} \sigma_l \otimes \mathrm{Ind}_{P'_{l'}}^{G'_{l'}} \sigma'_{l'}, \omega_{V_l, V'_{l'}})$

Tits deformation

$$
\begin{array}{ccccc}
\mathcal{H}_{l,\sigma} \times \mathcal{H}_{l',\sigma'} & \xleftarrow{\nu=q} & \mathsf{H}_{l,\mu} \times \mathsf{H}_{l',\mu'} & \xrightarrow{\nu=1} & \mathbb{C}[\mathsf{W}_l \times \mathsf{W}_{l'}] \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathrm{Rep}_{\mathbb{C}}(\mathcal{H}_{l,\sigma} \times \mathcal{H}_{l',\sigma'}) & \longleftarrow & \mathrm{Rep}_R(\mathsf{H}_{l,\mu} \times \mathsf{H}_{l',\mu'}) & \longrightarrow & \mathrm{Rep}_{\mathbb{C}}(\mathsf{W}_l \times \mathsf{W}_{l'})
\end{array}
$$

Main Theorem (assume $\sigma \overset{\Theta}{\longleftrightarrow} \sigma'$ and theta-cuspidal)

Theorem (M.-Qiu-Zou)

There is an $\mathsf{H}_{l,\mu} \times \mathsf{H}_{l',\mu'}$-module M (constructed explicitly) such that

- $\mathsf{M} \otimes_R \mathbb{C}_q \cong \mathcal{M} := \mathrm{Hom}_{G_l \times G'_{l'}}(\mathrm{Ind}_{P_l}^{G_l} \sigma_l \otimes \mathrm{Ind}_{P'_{l'}}^{G'_{l'}} \sigma'_{l'}, \omega_{V_l, V'_{l'}})$
- $\mathsf{M} \otimes_R \mathbb{C}_1 \cong \displaystyle\sum_{k=0}^{\min\{l,l'\}} \mathrm{Ind}_{\mathsf{W}_{l-k} \times \triangle \mathsf{W}_k \times \mathsf{W}_{l'-k}}^{\mathsf{W}_l \times \mathsf{W}_{l'}} \mathbf{1}_{l-k} \boxtimes \varepsilon_k \boxtimes \mathbf{1}_{l'-k}.$

- Theorem + Adams-Moy $\Rightarrow$ Aubert-Michel-Rouquier + Pan
- Theorem $\Rightarrow$ General form of the conservation relation.

- When $\mu = 1$, M has a geometric realization.

Determine the correspondence between cuspidal repns.

- Lusztig's map $\mathcal{E}(G,s) \longrightarrow \mathcal{E}(G^*_s,1)$.
- Unipotent cuspidal repn are rare.
  Assume: $\mathcal{E}(G,s) \ni \sigma \leftrightarrow \sigma' \in \mathcal{E}(G,s')$ are cuspidal.
- τ_t is cuspdial GL_k repn ($k \neq 1$ or $t \neq 1$).
- Consider $\mathcal{H}_{\tau,\sigma} := \mathrm{End}(\mathrm{Ind}_{(\mathrm{GL}_k \times G)U}^{G_k} \tau \otimes \sigma)$.

Lemma

$$\mathcal{H}_{\tau,\sigma} \cong \mathcal{H}_{\tau,\sigma'}.$$

- The lemma+conservation relation
  $\rightsquigarrow$ description of theta corr. between cuspidal repns
  $\rightsquigarrow$ complete description of theta over finite field.

- Similar lemma holds in p-adic case.

Thank you for listening!