Analytic continuation of branching laws for unitary representations

Jan Frahm (Aarhus University)

Peking University – POINTS
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Branching laws for unitary representations – General

Let G be a group, $H \subseteq G$ a subgroup and π an irreducible representation of G.

Branching problem

Understand the restriction $\pi|_H$ of π to H.

- If G is compact and π continuous, then $\dim \pi < \infty$, and if additionally $H \subseteq G$ is closed:

$$\pi|_H \simeq \bigoplus_{\tau \in \widehat{H}} m(\pi, \tau) \cdot \tau \qquad \text{with } m(\pi,\tau) = \dim \operatorname{Hom}_H(\pi|_H, \tau) \in \mathbb{N}.$$

  Examples: Classical branching laws by Weyl and Murnaghan for (G, H) a pair of successive orthogonal/unitary/symplectic groups in terms of highest weights $\rightsquigarrow$ interlacing conditions

- For non-compact real reductive groups $H \subseteq G$ and unitary representations π:

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m(\pi, \tau) \cdot \tau \, d\mu_\pi(\tau)$$

  where $m(\pi, \cdot) : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ is a multiplicity function (unique up to measure zero) and μ_π a measure on the unitary dual $\widehat{H}$.

Branching laws for unitary representations – Special cases

Branching problem – refined

Given π, determine the multiplicity function $m(\pi,\cdot):\widehat{H}\to\mathbb{N}\cup\{\infty\}$ and the measure μ_π.

Examples

1. $H=K$ a maximal compact subgroup $\rightsquigarrow$ the measure μ_π is discrete, $m(\pi,\tau)<\infty$ and the branching law is the K-type decomposition (e.g. Blattner's Formula for discrete series)
2. $G=G'\times G'$ and $H=\operatorname{diag}G'$ corresponds to the decomposition of tensor products $\pi_1\otimes\pi_2$
3. $H=H_1\times H_2\subseteq G=\mathrm{Sp}(V,\omega)$: Howe's dual pair correspondence

Some previous work

- $(G,H)=(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}),\operatorname{diag}\mathrm{SL}(2,\mathbb{R}))$: Repka '78
- $(G,H)=(\mathrm{Spin}(1,n),P)$: Liu–Oshima–Yu '20
- Kobayashi (90's): criterion for discrete decomposability (μ_π is discrete)
  $\rightsquigarrow$ algebraic and geometric methods available to determine $m(\pi,\tau)=\dim\mathrm{Hom}_H(\pi|_H,\tau)$
  (see e.g. Kobayashi, Loke, Duflo–Vargas, Y. Oshima, ...) $\rightsquigarrow$ happens rarely

Unitary vs. smooth multiplicities

In general: μ_π has both a continuous and a discrete part
$\rightsquigarrow$ $m(\pi,\tau) = \dim \mathrm{Hom}_H(\pi|_H, \tau)$ only holds for τ in the discrete spectrum

Reason: $A_\pi : \pi|_H \xrightarrow{\sim} \int_{\widehat{H}}^{\oplus} m(\pi,\tau)\cdot\tau \, d\mu_\pi(\tau)$ not pointwise defined in general, analogy:

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{ix\cdot\xi}\, d\xi, \quad f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \qquad \textbf{but } f \mapsto \widehat{f}(\xi) \text{ not defined on } L^2(\mathbb{R})$$

$\rightsquigarrow$ pass to the space π^∞ of *smooth vectors*

Unitary $\rightarrow$ smooth

The composition

$$\pi^\infty \hookrightarrow \pi \xrightarrow{A_\pi} \int_{\widehat{H}}^{\oplus} m(\pi,\tau)\cdot\tau\, d\mu_\pi(\tau)$$

is *pointwise defined*, i.e. $A_\pi v = (A_{\pi,\tau}v)_\tau$ $(v \in \pi^\infty)$ for some intertwining operators $A_{\pi,\tau} \in \mathrm{Hom}_H(\pi^\infty|_H, \tau) = \mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty)$. In particular,

$$m(\pi,\tau) \leq m_\infty(\pi^\infty, \tau^\infty) = \dim \mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty).$$

$\rightsquigarrow$ All unitary branching laws can be constructed from smooth intertwining operators.

Multiplicities for smooth representations

For a real reductive group G let $\widehat{G}_\infty$ denote the equivalence classes of irreducible smooth Fréchet representations of moderate growth (also referred to as Casselman–Wallach representations).

Theorem

1. (Kobayashi–Oshima '13) $m_\infty(\pi,\tau)<\infty$ for all $\pi\in\widehat{G}_\infty$, $\tau\in\widehat{H}_\infty$
   $\Leftrightarrow$ the pair (G,H) is *strongly spherical*
   $\Leftrightarrow$ the homogeneous space $(G\times H)/\operatorname{diag}(H)$ is real spherical
   $\Leftrightarrow$ a minimal parabolic $P_G\times P_H\subseteq G\times H$ has an open orbit on $(G\times H)/\operatorname{diag}(H)$
   $\Leftrightarrow$ $\#(P_H\backslash G/P_G)<\infty$
2. (Sun–Zhu '12) $m_\infty(\pi,\tau)\leq 1$ for all $\pi\in\widehat{G}_\infty$, $\tau\in\widehat{H}_\infty$ if (G,H) is one of the pairs

$$(\mathrm{GL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C})),\quad (\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R})),\quad (\mathrm{U}(p,q+1),\mathrm{U}(p,q)),$$
$$(\mathrm{SO}(n+1,\mathbb{C}),\mathrm{SO}(n,\mathbb{C})),\quad (\mathrm{SO}(p,q+1),\mathrm{SO}(p,q)),$$

Classification: Kobayashi–Matsuki ('14) and Knop–Krötz–Pecher–Schlichtkrull ('19)

Examples: $(G,H)=(\mathrm{O}(1,n)\times\mathrm{O}(1,n),\mathrm{O}(1,n))$, $(\mathrm{O}(2,2n),\mathrm{U}(1,n))$, $(E_{6(-26)},\mathrm{Spin}(9,1)\times\mathbb{R})$

Branching problem for smooth representations

Problem

For a given strongly spherical pair (G, H), determine $m_\infty(\pi, \tau) \in \mathbb{N}$ for (all) $\pi \in \widehat{G}_\infty$, $\tau \in \widehat{H}_\infty$.

$\widehat{G}_\infty$ is known by the Langlands classification:

1. In terms of L-packets: Gan–Gross–Prasad conjectures for classical groups
   $\rightsquigarrow$ both global and local conjectures, relations to L-functions, period integrals, etc.
2. In terms of principal series: more complicated structure, but advantage of *analytic families of representations* (principal series representations)

Strategy for unitary branching problems

Let (G, H) be a strongly spherical reductive pair and π an irreducible unitary representation of G.

Strategy to decompose $\pi|_H$

1. Embed π^∞ into a (possibly non-unitary) principal series representation $\pi_{\xi,\lambda}$ ($\lambda \in \mathfrak{a}_{\mathbb{C}}^*$)
2. Decompose $\pi_{\xi,\lambda}$ for $\lambda \in i\mathfrak{a}^*$ using Mackey theory and Plancherel theory
3. Analytically extend the decomposition to all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$
4. Restrict the decomposition to $\pi^\infty \subseteq \pi_{\xi,\lambda}$

Key idea: The unitary structure of the representation π might be complicated, whereas the unitary structure of the unitary principal series $\pi_{\xi,\lambda}$, $\lambda \in i\mathfrak{a}^*$, is easy ($L^2$-space)

Difficulty: Analytic continuation from $i\mathfrak{a}^*$ to $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

Step I: Principal series embedding

Casselman Embedding Theorem

Every irreducible unitary representation π is a subrepresentation (or, alternatively, a quotient) of a principal series representation

$$\pi^\infty \hookrightarrow \pi_{\xi,\lambda} = C^\infty\!-\mathrm{Ind}_{P_G}^G(\xi \otimes e^\lambda \otimes 1)$$

induced from a minimal parabolic subgroup $P_G = M_G A_G N_G$ and $\xi \in \widehat{M_G}$, $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$.

Similar notation for principal series of H: $\tau_{\eta,\nu} = C^\infty\!-\mathrm{Ind}_{P_H}^H(\eta \otimes e^\nu \otimes 1)$ for $\eta \in \widehat{M_H}$, $\nu \in \mathfrak{a}_{H,\mathbb{C}}^*$

$\rightsquigarrow$ Every intertwining operator $\pi^\infty|_H \to \tau^\infty$ can be obtained from an intertwining operator $\pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu}$ between principal series by composition with embedding and quotient map.

$\rightsquigarrow$ construct (and classify) symmetry breaking operators between principal series representations:

$$\pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu}$$

Construction of SBOs

Theorem (F. '17)

Let (G,H) be one of the multiplicity-one pairs. Then, for generic $(\lambda,\nu)\in\mathfrak{a}_{G,\mathbb{C}}^*\times\mathfrak{a}_{H,\mathbb{C}}^*$:

$$\dim\operatorname{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})=\dim\operatorname{Hom}_M(\xi|_M,\eta|_M)\in\{0,1\},$$

and in case $=1$ there exists a non-trivial meromorphic family $A_{(\xi,\lambda),(\eta,\nu)}\in\operatorname{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})$.

(+ similar results for *spherical* principal series of general strongly spherical pairs, where the multiplicities additionally depend on $\#(P_H\backslash G/P_G)_{\text{open}}$)

Proof.
- Identify $A\in\operatorname{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})$ with its distribution kernel $K\in\mathcal{D}'(G)$
- Upper bounds: Bruhat's theory of invariant distributions
- Lower bounds: explicit construction of meromorphic families of distribution kernels from complex powers of matrix coefficients of finite-dimensional representations □

$\rightsquigarrow$ explicit meromorphic families $A_{(\xi,\lambda),(\eta,\nu)}$ of integral operators depending on complex parameters $(\lambda,\nu)\in\mathfrak{a}_{G,\mathbb{C}}^*\times\mathfrak{a}_{H,\mathbb{C}}^*$ $\rightsquigarrow$ poles and residues?

Classification of SBOs

Classification results

1. (Kobayashi–Speh '15 & '18) $(G,H)=(\mathrm{O}(1,n+1),\mathrm{O}(1,n))$: $\xi=\Lambda^p\mathbb{C}^n$, $\eta=\Lambda^q\mathbb{C}^{n-1}$
2. (Clerc 16' & '17) $(G,H)=(\mathrm{O}(1,n)\times\mathrm{O}(1,n),\operatorname{diag}\mathrm{O}(1,n))$: $\xi=1$, $\eta=1$
3. (F.–Weiske '19) $(G,H)=(\mathrm{U}(1,n;\mathbb{F}),\mathrm{U}(1,m;\mathbb{F})\times\mathrm{U}(n-m;\mathbb{F}))$, $\mathbb{F}=\mathbb{C},\mathbb{H},\mathbb{O}$: $\xi=1$, $\eta=1$
   $\mathbb{F}=\mathbb{C}$: unitary groups, $\mathbb{F}=\mathbb{H}$: symplectic groups, $\mathbb{F}=\mathbb{O}$: $(G,H)=(F_{4(-16)},\mathrm{Spin}(1,8))$
4. (Ditlevsen–F. '21) $(G,H)=(\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$: holomorphic normalization of the meromorphic families

Strategy: Classify distribution kernels $K\in\mathcal{D}'(G)$ that define intertwining operators in the non-compact picture $\rightsquigarrow$ P_H-invariant distributions on $\overline{N}_G\simeq\overline{\mathfrak{n}}_G$

General observation: The meromorphic families together with their residues yield most SBOs, but there can occur *sporadic* SBOs that are not related to the meromorphic families (should not contribute to the decomposition of unitary representations)

Step II: Decompose unitary principal series (Mackey)

- $\pi_{\xi,\lambda}$ ($\lambda \in i\mathfrak{a}^*$) is unitary on $L^2(G/P_G, \mathcal{V}_{\xi,\lambda})$, where $\mathcal{V}_{\xi,\lambda} = G \times_{P_G} (\xi \otimes e^{\lambda+\rho} \otimes 1) \to G/P_G$
- (G, H) strongly spherical $\Rightarrow$ H acts on G/P_G with finitely many orbits
- Let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be the open H-orbits $\Rightarrow$ $\mathcal{O}_1 \cup \ldots \cup \mathcal{O}_n \subseteq G/P_G$ open dense
- The restriction to $\mathcal{O}_1 \cup \ldots \cup \mathcal{O}_n$ is an H-equivariant isometric isomorphism

$$L^2(G/P_G, \mathcal{V}_{\xi,\lambda}) \xrightarrow{\sim} L^2(\mathcal{O}_1, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}_1}) \oplus \cdots \oplus L^2(\mathcal{O}_n, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}_n}).$$

- Note that the condition that (G, H) is strongly spherical implies that each open H-orbit $\mathcal{O}_j$ is real spherical, i.e. P_H has an open orbit on $\mathcal{O}_j$ $\Rightarrow$ $L^2(\mathcal{O}_j, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}_j})$ has finite multiplicities

Assumption (Plancherel formula)

The explicit decomposition of $L^2(\mathcal{O}_j, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}_j})$ is known for all j.

$\rightsquigarrow$ explicit decomposition of $\pi_{\xi,\lambda}|_H$ for $\lambda \in i\mathfrak{a}^*$

Decomposing unitary principal series

Examples

- $(G,H) = (\mathrm{U}(1,n+1;\mathbb{F}), \mathrm{U}(1,n;\mathbb{F})) \Rightarrow \mathcal{O} \simeq \mathrm{U}(1,n;\mathbb{F})/\mathrm{U}(n;\mathbb{F})$
  $\rightsquigarrow$ $\mathcal{O}$ is essentially a Riemannian symmetric space
  $\rightsquigarrow$ Plancherel formula is very explicit (Harish-Chandra, Helgason, Camporesi)
- $(G,H) = (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \operatorname{diag} \mathrm{SL}(2,\mathbb{R})) \Rightarrow \mathcal{O} \simeq \mathrm{SL}(2,\mathbb{R})/MA$
  $\rightsquigarrow$ $\mathcal{O}$ is a one-sheeted pseudo-Riemannian hyperboloid
  $\rightsquigarrow$ Plancherel formula known in principle (Faraut, Rossmann, Strichartz), but not explicit enough
- $(G,H) = (\mathrm{GL}(3,\mathbb{R}), \mathrm{GL}(2,\mathbb{R})) \Rightarrow \mathbb{R}^\times \to \mathcal{O} \to \mathrm{GL}(2,\mathbb{R})/MA$ fibration
  $\rightsquigarrow$ similar as above

Step III: Analytically extend the decomposition

Given an *explicit* decomposition $\pi_{\xi,\lambda}|_H \simeq \bigoplus_\eta \int^\oplus \tau_{\eta,\nu}\, d\mu_{\xi,\lambda,\eta}(\nu)$ for $\lambda \in i\mathfrak{a}^*$:

$$\|f\|_{\xi,\lambda}^2 = \sum_\eta \int \|A_{(\xi,\lambda),(\eta,\nu)} f\|_{\eta,\nu}^2\, d\mu_{\xi,\lambda,\eta}(\nu) \qquad \text{with} \qquad A_{(\xi,\lambda),(\eta,\nu)} \in \mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$$

Using standard intertwining operators, the left hand side can be made holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$\|f\|_{\xi,\lambda}^2 = \langle f, T_{\xi,\lambda}^w \overline{f} \rangle$$

$\rightsquigarrow$ extend the right hand side analytically

Tools

- *Functional equations* for the composition of a symmetry breaking operator with a standard intertwining operator: $A_{(w\xi,w\lambda),(\eta,\nu)} \circ T_{\xi,\lambda}^w = \text{const} \times A_{(\xi,\lambda),(\eta,\nu)}$
- Meromorphic properties of $T_{\xi,\lambda}^w$, $A_{(\xi,\lambda),(\eta,\nu)}$ and $d\mu_{\xi,\lambda,\eta}(\nu)$: poles and residues

$\rightsquigarrow$ additional (more singular) spectrum might occur as residue when shifting the contour of integration in order to extend the right hand side analytically

Step IV: Restrict the decomposition to $\pi \hookrightarrow \pi_{\xi,\lambda}$

Given an explicit decomposition for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$\|f\|_{\xi,\lambda}^2 = \sum_{\eta} \int \|A_{(\xi,\lambda),(\eta,\nu)} f\|_{\eta,\nu}^2 \, d\mu_{\xi,\lambda,\eta}(\nu)$$

$\rightsquigarrow$ restrict to $f \in \pi^\infty \subseteq \pi_{\xi,\lambda}$

Tools

Requires a detailed analysis of the mapping properties of $A_{(\xi,\lambda),(\eta,\nu)} \in \mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$ to determine the image of $A_{(\xi,\lambda),(\eta,\nu)}|_{\pi^\infty}$:

- K-type analysis
- Functional equations

$\rightsquigarrow$ in general hard (many composition factors, high K-multiplicities)

Results – rank one unitary groups

$(G,H) = (\mathrm{U}(1,n+1), \mathrm{U}(1,n))$ (F.–Weiske '20)

We consider all unitary subrepresentations $\pi \subseteq \pi_{\xi,\lambda}$ for $\dim\xi = 1$

$\mathcal{O} = \mathrm{U}(1,n)/\mathrm{U}(n) \rightsquigarrow$ explicit Plancherel formula by Shimeno '94

$$\|f\|^2_{\xi,\lambda} = \bigoplus_{\dim\eta=1} \left[\int_{i\mathfrak{a}_H^*} \|A_{(\xi,\lambda),(\eta,\nu)} f\|^2_{\eta,\nu} \frac{d\nu}{|c(\xi,\lambda,\eta,\nu)|^2} \right.$$
$$\left. + \sum_{\nu} \|A_{(\xi,\lambda),(\eta,\nu)} f\|^2_{\eta,\nu} \operatorname{Res}_{\mu=\nu} \frac{1}{c(\xi,\lambda,\eta,\nu)c(\xi,\lambda,\eta,-\nu)} \right]$$

- $\pi =$ *unitary principal series:* direct integral + highest/lowest weight representations
- $\pi =$ *complementary series:* finitely many complementary series appear as residues
- $\pi =$ *relative discrete series:* finitely many highest/lowest weight and relative discrete series representations appear as residues
- $\pi =$ *unitary highest/lowest weight representation:* the continuous spectrum vanishes and only highest/lowest weight representations survive in the discrete spectrum

Results – rank one orthogonal groups

$(G, H) = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ (Weiske '20)

Consider all unitary subrepresentations $\pi \subseteq \pi_{\xi,\lambda}$ for $\xi = \Lambda^p\mathbb{C}^n$
$\rightsquigarrow$ comprises in particular all irreducible representations with non-trivial $(\mathfrak{g}, K)$-cohomology

$\mathcal{O} = \mathrm{O}(1, n)/\mathrm{O}(n) \rightsquigarrow$ explicit Plancherel formula by Camporesi '94

$$\|f\|^2_{\xi,\lambda} = \bigoplus_{\substack{\eta=\Lambda^q\mathbb{C}^{n-1}\\ q=p,p-1}} \Bigg[\int_{i\mathfrak{a}_H^*} \|A_{(\xi,\lambda),(\eta,\nu)}f\|^2_{\eta,\nu} \frac{d\nu}{|c(\xi,\lambda,\eta,\nu)|^2} + \sum_{\nu} \|A_{(\xi,\lambda),(\eta,\nu)}f\|^2_{\eta,\nu} \operatorname{Res}_{\mu=\nu} \frac{1}{c(\xi,\lambda,\eta,\nu)c(\xi,\lambda,\eta,-\nu)}\Bigg]$$

- $\pi =$ *unitary principal series:* direct integral + possibly two discrete series (for $p = \frac{n}{2}$)
- $\pi =$ *complementary series:* finitely many complementary series appear as residues
- $\pi =$ *representation with cohomology:* continuous spectrum + finitely many complementary series + one representation with cohomology

In progress

Current projects

- $(\mathrm{GL}(3,\mathbb{R}),\mathrm{GL}(2,\mathbb{R}))$: goal to decompose all $\pi\in\widehat{G}$
  $\rightsquigarrow$ first unitary branching laws for $\mathrm{GL}(3,\mathbb{R})$
- $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}),\operatorname{diag}\mathrm{SL}(2,\mathbb{R}))$: reprove Repka's tensor product decompositions
  $\rightsquigarrow$ uniform proof for all types of tensor products
- $(\mathrm{O}(p,q),\mathrm{O}(p,q-1))$: branching laws for degenerate principal series and its quotients
  $\rightsquigarrow$ rep's associated to elliptic coadjoint orbits + minimal representation
- $(\mathrm{PSL}(2,E)/\mathrm{PSL}(2,F))$ for E/F a quadratic extension of p-adic fields: branching laws for unitary principal series, complementary series and Steinberg representation
  $\rightsquigarrow$ analogues of Prasad's results in the smooth category

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