Dilatations and Néron blowups

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Bosch-Lütkebohmer-Raynaud(-...) dilatations

Let X be a scheme of finite type over a DVR R.
Note $\tilde{R} = R/\pi R$ and $\tilde{X} = X \times_R \tilde{R}$. Let $Y \underset{closed}{\subset} \tilde{X}$.

Definition/Proposition - Bosch-Lütkebohmert-Raynaud dilatations

There exists a canonical pair $(X', u), u : X' \to X$ such that if Z is a flat R-scheme and if $v : Z \to X$ is an R-morphism such that $\tilde{v} : \tilde{Z} \to \tilde{X}$ factors through Y, then v factors through u.
The pair (X', u) is called the dilatation with center Y.

Example (First congruence group)

G flat group scheme over R, $e : Spec(R) = Y \subset \tilde{G}$ neutral section. Then $G' =: G_1$ is the first congruence group. We have
$G_1(R) = \ker(G(R) \to G(R/\pi R))$.

Bosch-Lütkebohmer-Raynaud dilatations

Some properties of BLR dilatations

BLR dilatations preserve group scheme structures.
BLR dilatations commute with unramified base change.

Fact

BLR dilatations do not commute with ramified base change. This comes from the fact that BLR dilatations is built on the concept of special fibres and special fibres do not commute with base change in the ramified case.

Fact

Conceptually, the hypothesis that π is a uniformizer is nowhere used in BLR dilatations.

Goal of this talk

• Introduce a general notion of dilatation for schemes, generalizing BLR dilatations.
• As long as to do, let us introduce the most general notion as possible.
• Give some properties, look at the group scheme case, and indicate some applications of this theory of dilatations.

For proofs and references, see arXiv 2001.03597 *Néron blowups and low-degree cohomological applications*.

Most proofs use EGA and Stackproject.

Definition of dilatations through affine blowup algebras

Fix $Z \underset{closed}{\subset} D \underset{closed}{\subset} X$ schemes (D loc. principal, $\mathcal{O}_X$ structure sheaf of X)
Let $\mathcal{I} \supset \mathcal{J}$ the ideal of $\mathcal{O}_X$ such that $Z = V(\mathcal{I})$ and $D = V(\mathcal{J})$.
Let $Bl_{\mathcal{I}}\mathcal{O}_X := \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \dots$ (Rees $\mathcal{O}_X$-algebra). If $\mathcal{J} = (b)$ is principal $b \in \Gamma(X, \mathcal{J})$, then we put $Bl_{\mathcal{I}}\mathcal{O}_X[\mathcal{J}^{-1}] := Bl_{\mathcal{I}}\mathcal{O}_X[b^{-1}]$. If $\mathcal{J}$ is only locally principal, we define $Bl_{\mathcal{I}}\mathcal{O}_X[\mathcal{J}^{-1}]$ by glueing, it is naturally a graded algebra, locally $deg(\frac{i}{b^k}) = n - k$ ($i \in \mathcal{I}^n$).

Definition

The affine blowup algebra $Bl[\frac{\mathcal{I}}{\mathcal{J}}]$ is the degree zero part of $Bl_{\mathcal{I}}\mathcal{O}_X[\mathcal{J}^{-1}]$.
The dilatation of X with center (Z, D) is $Spec(Bl[\frac{\mathcal{I}}{\mathcal{J}}]) := Bl_Z^D X$.

We have a morphism $Bl_Z^D X \to X$ of schemes corresponding to $\mathcal{O}_X \to Bl[\frac{\mathcal{I}}{\mathcal{J}}]$.

Characterizations of dilatations

Proposition

Let $Z \subset D \subset X$ as before. Then:

1. $Bl_Z^D X$ is the open subscheme of the blowup $Bl_Z X = Proj(Bl_{\mathcal{I}} \mathcal{O}_X)$ defined by the complement of $V_+(\mathcal{J})$.
2. If D is a Cartier divisor, $Bl_Z^D X$ is the closed subscheme of the affine projecting cone $C_Z X = Spec(\oplus_{n \geq 0} \mathcal{I}^n \otimes \mathcal{J}^{-n})$ defined by the equation $\varrho - 1$, where $\varrho \in \mathcal{I} \otimes \mathcal{J}^{-1}$ is the image of 1 under the inclusion $\mathcal{O}_X = \mathcal{J} \otimes \mathcal{J}^{-1} \subset \mathcal{I} \otimes \mathcal{J}^{-1}$.
3. If $v : T \to X$ is a morphism in $Sch_X^{D\text{-}reg}$ such that $v\,|_D \colon T\,|_D \to X\,|_D$ factors through Z, then v factors through $Bl_Z^D X \to X$. Moreover $Bl_Z^D X \to X$ is the only canonical scheme satisfying this universal property.

Here $Sch_X^{D\text{-}reg}$ is the category of schemes $T \to X$ such that $T \times_X D \subset T$ is an effective Cartier divisor on T. If $T' \to T$ is flat and $T \to X$ is an object in this category, so is the composition $T' \to T \to X$.

Dilatations in the affine case

If $X = Spec(B)$, $D = Spec(B/J)$ and $Z = Spec(B/I)$ $(J = (b) \subset I \subset B)$.

Then $Bl_Z^D(X) = Spec(Bl[\frac{I}{J}])$ where $Bl[\frac{I}{J}] = \{\frac{x}{b^n} \mid n \in \mathbb{N}, x \in I^n\}/\sim$

$$\frac{x}{b^n} \sim \frac{y}{b^m} \Leftrightarrow \exists k \geq 0 \ b^k(b^m x - b^n y) = 0.$$

Examples

• If X is an R-scheme and Y is a closed subscheme of the special fibre $\tilde{X}$. Then BLR dilatation $X' \to X$ is equal to $Bl_Y^{\tilde{X}} X \to X$.

• $Bl_D^D X = X$

• Let G be a flat group scheme over R. Let $D = G \times_R R/\pi^n R$ where $n \in \mathbb{N}$. Let Z be the neutral section of the $R/\pi^n R$-group scheme D. Then $Bl_Z^D G$ is the n-th congruence subgroup, it is denoted G_n. We have

$$G_n(R) = \ker(G(R) \to G(R/\pi^n R)).$$

• We will see other examples in Timo's talk.

Properties of dilatations - Functoriality

Let $Z' \subset D' \subset X'$ and $Z \subset D \subset X$ as usual.

Proposition - Functoriality

A morphism $X' \to X$ such that its restriction to D' (resp. Z') factors through D (resp. Z) induces a unique morphism $Bl_{Z'}^{D'} X' \to Bl_Z^D X$ such that the following diagram of schemes

$$\begin{array}{ccc} Bl_{Z'}^{D'} X' & \longrightarrow & Bl_Z^D X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

commutes.

In particular we get a canonical map $Bl_{Z'}^{D'} X' \to Bl_Z^D X \times_X X'$.

Properties of dilatations - Base change

$Z \subset D \subset X$ and let $X' \to X$ a morphism. Let $\underset{=Z\times_X X'}{Z'} \subset \underset{=D\times_X X'}{D'} \subset X'$ the preimage of $Z \subset D \subset X$, $Bl_{Z'}^{D'} X'$ is well-defined.

Proposition - Base change

If $Bl_Z^D X \times_X X' \to X'$ is an object of $Sch_{X'}^{D'-reg}$, then

$$Bl_{Z'}^{D'} X' \overset{\simeq}{=} Bl_Z^D X \times_X X' \quad \text{(canonical isomorphism).}$$

Example

Let k'/k with ramification e and ring of integers R'/R (e.g. $k = \mathbb{Q}_p$). Let G be a flat group scheme over R. Then

$$(G_r) \times_R R' \overset{\simeq}{=} (G \times_R R')_{er}.$$

$$(G_r) \times_R R' \simeq (Bl_e^{G\times_R R/\pi^r} G) \times_R R' \overset{Prop}{\simeq} Bl_{e'}^{\overset{=(G\times_R R')\times_{R'} R'/\pi'^{er}}{(G\times_R R/\pi^r)\times_R R'}} (G \times_R R') \simeq (G \times_R R')_{er}$$

Properties of dilatations - Exceptional divisor

Proposition

Let $Z \subset D \subset X$ as usual. Let $Bl_Z^D X \to X$ the dilatation. We have

$$\underset{preimage\ \ of\ \ D}{Bl_Z^D X \times_X D} = \underset{preimage\ \ of\ \ Z}{Bl_Z^D X \times_X Z}.$$

This is an effective Cartier divisor on $Bl_Z^D X$, called the exceptional divisor.

In order to describe the exceptional divisor, let $\mathcal{C}_{Z/D}$ and $\mathcal{N}_{Z/D} = \mathcal{C}_{Z/D}^{\vee}$ be the conormal and normal sheaves of Z in D.

Theorem - Description of the exceptional divisor

Assume that $D \subset X$ is an effective Cartier divisor, and $Z \subset D$ is a regular immersion. Write $\mathcal{J}_Z := \mathcal{J}|_Z$.

1. The exceptional divisor $Bl_Z^D X \times_X Z \to Z$ is Zariski locally over Z isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.
2. If $H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z) = 0$ (for example if Z is affine), then $Bl_Z^D X \times_X Z \to Z$ is globally isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.
3. If Z is a transversal intersection then $Bl_Z^D X \times_X Z \to Z$ is globally and canonically isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \to Z$.

Here, Z is a transversal intersection means that there is a cartesian square

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D \end{array}$$

of closed subschemes whose vertical maps are regular immersions.

Properties of dilatations - Relative case

Let us assume that we have a diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & D = X \times_S S_0 & \hookrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S_0 & \underset{\text{Eff. Cart. Div.}}{\hookrightarrow} & S \end{array}$$

Theorem - Properties of dilatations - Relative Case

1. If $Z \subset D$ is regular, then $Bl_Z^D X \to X$ is of finite presentation.
2. If $Z \subset D$ is regular, the fibers of $Bl_Z^D X \times_S S_0 \to S_0$ are connected (resp. irreducible, geometrically connected, geometrically irreducible) if and only if the fibers of $Z \to S_0$ are.
3. If $X \to S$ is flat and if moreover one of the following holds:
   - (i) $Z \subset D$ is regular, $Z \to S_0$ is flat and S, X are locally noetherian,
   - (ii) $Z \subset D$ is regular, $Z \to S_0$ is flat and $X \to S$ is locally of finite presentation,
   - (iii) the local rings of S are valuation rings,

   then $Bl_Z^D X \to S$ is flat.
4. If both $X \to S$, $Z \to S_0$ are smooth, then $Bl_Z^D X \to S$ is smooth.
5. Assume $BL_Z^D X \to S$ is flat. Let $S' \to S$ flat then

$$(Bl_Z^D X) \times_S S' \simeq Bl_{Z'}^{D'}(X') \quad \text{(canonical isomorphism)} \ \ (\bullet' = \bullet \times_S S').$$

Iterated dilatations

$Z \subset D_0 \subset X_0$ as before, and assume they sit in a cartesian diagram of closed subschemes

$$\begin{array}{ccc} W & \hookrightarrow & X_0 \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D_0 \end{array}$$

such that the vertical maps are Cartier divisor inclusions. In this situation we can construct a sequence of dilatations $\cdots \to X_r \to X_{r-1} \to \cdots \to X_1 \to X_0$ and closed immersions $i_r : W \hookrightarrow X_r$, as follows. We let $i_0 : W \hookrightarrow X_0$. Let $u_1 : X_1 \to X_0$ be $Bl_Z^{D_0} X_0$, and D_1 the preimage of D_0 in X_1.

Proposition

There is a canonical closed immersion $i_1 : W \hookrightarrow X_1$ lifting i_0. Moreover, we again have a cartesian diagram $\begin{array}{ccc} W & \hookrightarrow & X_1 \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D_1 \end{array}$ where the vertical maps are Cartier divisor inclusions.

The sequence $\cdots \to X_r \to X_{r-1} \to \cdots \to X_1 \to X_0$ is obtained by iterating this construction.

Proposition

Let $\cdots \to X_r \to X_{r-1} \to \cdots \to X_1 \to X_0$ be the sequence of dilatations constructed previously. Let rD be the r-th multiple of D_0 as a Cartier divisor, and $rZ := W \cap rD$. Then the composition $X_r \to X$ is the dilatation of (rZ, rD) inside X.

Example

Let R be a DVR and π a uniformizer. Let $W = Spec(R)$ and $Z = Spec(R/\pi)$. Let $G = G_0$ be a group scheme over R and $D_0 = G \times_S Z$. Let $e : W \hookrightarrow G$ be the unit section. The obtained sequence $\cdots \to G_r \to G_{r-1} \to \cdots \to G_1 \to G_0$ is the usual sequence of congruence group schemes.

Dilatations of group schemes : Néron blowups

Let S be a scheme and $S_0 \subset S$ be a locally principal closed subscheme. Let $G \to S$ be a group scheme over S, then $G_0 = G \times_S S_0$ is an S_0-group scheme. Let $H \subset G_0$ be a closed subgroup scheme over S_0.

Dilatation of group schemes : Néron blowups

The dilatation $Bl_H^{G_0} G$ is called the Néron blowup of G with center the S_0-group scheme H. Let us denote the Néron blowup by $\mathcal{G}$.

The map $\mathcal{G} \to G$ is affine. Its restriction over $S \backslash S_0$ induces an isomorphism $\mathcal{G}|_{S \backslash S_0} \simeq G|_{S \backslash S_0}$. Its restriction over S_0 factors as $\mathcal{G}_0 \to H \subset G_0$.

Néron blowups

$H \subset G_0 \subset G$ as before

Proposition - some facts

• The scheme $\mathcal{G} \to S$ represents the contravariant functor $Sch_S^{S_0\text{-}reg} \to Groups$ given for $T \to S$ by the set of all S-morphisms $T \to G$ such that the induced morphism $T_0 \to G_0$ factors through $H \subset G_0$.
• $\mathcal{G}$ is a group object in the category $Sch_S^{S_0\text{-}reg}$.
• Product in $Sch_S^{S_0\text{-}reg}$ are given by $X_1 \times_{Sch_S^{S_0\text{-}reg}} X_2 = Bl_{(X_1 \times_S X_2) \times_S S_0}(X_1 \times_S X_2)$.
• If $\mathcal{G} \to S$ is flat, then $\mathcal{G} \times_S \mathcal{G} = \mathcal{G} \times_{Sch_S^{S_0\text{-}reg}} \mathcal{G}$.

Corollary

If $\mathcal{G} \to S$ is flat, then $\mathcal{G} \to S$ is an S-group scheme.

Theorem - Properties of Néron blowups

Let $\mathcal{G} \to G$ be the Néron blowup of G in H along S_0.

1. If $G \to S$ is (quasi-)affine, then $\mathcal{G} \to S$ is (quasi-)affine.
2. If $G \to S$ is (locally) of finite presentation and $H \subset G_0$ is regular, then $\mathcal{G} \to S$ is (locally) of finite presentation.
3. If $H \to S_0$ has connected fibres and $H \subset G_0$ is regular, then $\mathcal{G} \times_S S_0 \to S_0$ has connected fibres.
4. Assume that $G \to S$ is flat and one of the following holds:
   (i) $H \subset G_0$ is regular, $H \to S_0$ is flat and S, G are locally noetherian,
   (ii) $H \subset G_0$ is regular, $H \to S_0$ is flat and $G \to S$ is locally of finite presentation,
   (iii) the local rings of S are valuation rings,

   then $\mathcal{G} \to S$ is flat.
5. If both $G \to S$, $H \to S_0$ are smooth, then $\mathcal{G} \to S$ is smooth.
6. Assume that $\mathcal{G} \to S$ is flat. If $S' \to S$ is a scheme such that $S'_0 := S' \times_S S_0$ is an effective Cartier divisor on S', then the base change $\mathcal{G} \times_S S' \to S'$ is the Néron blowup of $G \times_S S'$ in $H \times_{S_0} S'_0$ along S'_0.

In cases (4) and (5), the map $\mathcal{G} \to S$ is a group scheme.

Exceptional divisor of Néron blowups

Assume that $G \to S$ is flat, locally finitely presented and $H \to S_0$ is flat, regularly immersed in G_0. Let $\mathcal{G} \to G$ be the dilatation of G in H with exceptional divisor $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Let $\mathcal{J}$ be the ideal sheaf of G_0 in G and $\mathcal{J}_H := \mathcal{J}|_H$. Let V be the restriction of the normal bundle $\mathbb{V}(\mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}) \to H$ along the unit section $e_0 : S_0 \to H$.

Proposition - Group structures on the exceptional divisor

1. Locally over S_0, there is an exact sequence of S_0-group schemes $1 \to V \to \mathcal{G}_0 \to H \to 1$.
2. If H lifts to a flat S-subgroup scheme of G, there is globally an exact, canonically split sequence $1 \to V \to \mathcal{G}_0 \to H \to 1$.
3. If $G \to S$ is smooth, separated and $\mathcal{G} \to G$ is the dilatation of the unit section of G, there is a canonical isomorphism of smooth S_0-group schemes $\mathcal{G}_0 \simeq Lie(G_0/S_0) \otimes N_{S_0/S}^{-1}$ where $N_{S_0/S}$ is the normal bundle of S_0 in S.

Some applications

- "Moy-Prasad" isomorphism for group schemes

$$G_s(R)/G_r(R) \simeq \mathfrak{g}_s(R)/\mathfrak{g}_r(R) \quad 0 \leq \frac{r}{2} \leq s \leq r.$$

- Identifications of moduli stacks

$$Bun_{Bl_H^N G} \simeq Bun_{(G,H,N)}.$$

Thank you very much for your attention