Lecture Notes on Homological Algebra

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Contents

Introduction

These are lecture notes for a one-semester graduate course on **homological algebra**, given at Peking University in the Fall 2026 semester. The goal is to develop the core machinery (abelian categories, derived functors, spectral sequences, derived categories, and simplicial methods) at a level suitable for ambitious undergraduates and beginning graduate students.

Most of the content is distilled from the published textbooks [9] and [10], with variations and omissions: the categorical language of Chapter 1 and the module appendix Appendix B are drawn largely from Vol. 1, the rest from Vol. 2. For the latest PDF versions of these books, errata and links to source repositories, see `https://wwli.asia`.

Prerequisites

We assume familiarity with undergraduate algebra, including the basics of module theory (Appendix B collects what is needed), together with the bare basics of category theory (functors, natural transformations, adjunctions, limits; to be reviewed in Chapter 1). A passing acquaintance with algebraic topology (singular homology, homotopy groups) is helpful but not required; the relevant notions will be recalled as needed. No commutative algebra is assumed.

Chapter map

- Chapter 1 reviews the categorical language: categories, functors, natural transformations, the Yoneda lemma, adjunctions, limits, localization, and Kan extensions.
- Chapter 2 develops additive and abelian categories, together with the vocabulary of complexes and exactness, the diagram lemmas (snake and five), and projective and injective objects; $R\text{-}\mathsf{Mod}$ is the running example throughout.
- Chapter 3 develops the homotopy theory of complexes: the homotopy category, mapping cones, double complexes, and truncation.
- Chapter 4 constructs the classical derived functors and treats Ext, Tor, $\lim^1$, and their applications.
- Chapter 5 treats Serre subcategories, quotient categories, and the Grothendieck group.
- Chapter 6 surveys spectral sequences, from filtered complexes and exact couples to the Grothendieck spectral sequence.
- Chapter 7 recasts the theory on triangulated and derived categories: localization, the Verdier quotient, and Ext as morphisms in the derived category.
- Chapter 8 revisits the derived functors at the level of derived categories: total derived functors as Kan extensions, resolutions, the Grothendieck spectral sequence, and the derived bifunctors RHom, $\mathrm{R}\lim$, and $\overset{\mathrm{L}}{\otimes}$.

- Chapter 9 covers simplicial methods: simplicial sets, nerves, geometric realization, and the Dold–Kan correspondence.
- Two appendices provide background: Appendix A (the Freyd–Mitchell embedding theorem, ind- and pro-objects, and the adjoint functor theorem) and Appendix B (modules, tensor products, and flatness).

Conventions

A few global conventions, to be laid down once and for all:

- All rings are associative and unital, with $1 \neq 0$ unless stated otherwise.
- "Module" means left module unless qualified.
- We write $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}_{\geq 1} = \{1, 2, 3, \cdots\}$ for the non-negative and the positive integers, respectively.
- We use $A \subset B$ to mean that A is a subset of B; if A is a proper subset, we write $A \subsetneq B$. The symbols $\subset$ and $\supset$ are also used for inclusions of subobjects, without implying properness.
- Complexes are cohomologically graded by default: $d^n : C^n \to C^{n+1}$.
- The objects of a category $\mathcal{C}$ form a class $\mathrm{Ob}(\mathcal{C})$, but for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are required to form a set.

Throughout, we state definitions and basic results for abelian categories, but the reader should feel free to chase diagrams in the category of modules; the Freyd–Mitchell embedding theorem (Appendix A) justifies this. We will say so explicitly each time.

How to read these notes

The main references are [9] and [10], with the texts [17], [5], [4], [6] as further reading. Proofs that become too technical are sketched, with a pointer to the literature.

There are exercises scattered in various sections, many of which are important properties or examples of the theory. The reader is encouraged to try them out.

Use of AI

The book was built from the syllabus and LaTeX sources of the Chinese textbooks [9], [10], by the **GLM-5.*** series of large language models running inside the **Pi Coding Agent**. The models are open-weight and the harness is open-source software, in accordance with mathematicians' moral standards.

More concretely, the AI's tasks were

- selecting and organizing the relevant material into the present form,
- converting the LaTeX sources into Typst,
- tweaking the Typst templates,
- reviewing the subsequent revisions and expansions.

All content has been audited line-by-line by a human; the revisions and expansions were either performed or steered by the human author.

Chapter 1

Categories, Functors, and the Yoneda Lemma

This chapter is a rapid review of the categorical language that pervades homological algebra. The reader has probably met categories, functors, and natural transformations before; our aim is to fix notation, to record the few results we shall quote constantly: the Yoneda lemma, the characterization of equivalences, and the fact that adjoints preserve limits, as well as to develop two topics that will be indispensable for (Chapter 7), namely ***localization*** and ***Kan extensions*** . Proofs of routine assertions are left to the reader; those of the structural results are given in full or sketched, with references for the longer arguments.

1.1 Categories, objects, and morphisms

We follow the founders' own story: Eilenberg and Mac Lane introduced functors to make precise the notion of a *natural transformation*, and introduced categories only to say what a functor is. We must, however, tell the story in reverse.

Definition 1.1.1 A ***category*** $\mathcal{C}$ consists of

- a class of ***objects*** $\mathrm{Ob}(\mathcal{C})$;
- a class $\mathrm{Mor}(\mathcal{C})$ of ***morphisms*** , equipped with source and target maps to $\mathrm{Ob}(\mathcal{C})$; for $X, Y \in \mathrm{Ob}(\mathcal{C})$ we assume

  $$\mathrm{Hom}_{\mathcal{C}}(X,Y) := \{f \in \mathrm{Mor}(\mathcal{C}) : f \text{ has source } X \text{ and target } Y\} \tag{1.1.1}$$

  is a set, and call its elements **morphisms** from X to Y;
- for each object X an ***identity morphism*** $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$;
- a ***composition*** law

  $$\begin{aligned} \circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (f,g) &\mapsto f \circ g, \end{aligned} \tag{1.1.2}$$

  abbreviated fg, satisfying associativity and the unit laws

  $$\begin{aligned} f(gh) &= (fg)h, \\ f \circ \mathrm{id}_X = f &= \mathrm{id}_Y \circ f \quad (\text{for } f : X \to Y). \end{aligned} \tag{1.1.3}$$

We write $f : X \to Y$ or $X \xrightarrow{f} Y$ for a morphism. A ***commutative diagram*** is one in which any two directed paths with the same start and end compose to the same morphism; for instance

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "h"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd} \quad \text{commutes} \quad \text{iff} \quad gf = h. \tag{1.1.4}$$

The phrase "the diagram commutes" is used so often that we shall usually leave it implicit. A morphism $f : X \to Y$ is an ***isomorphism*** if it admits a two-sided inverse $g : Y \to X$ (necessarily unique); we then write $f : X \xrightarrow{\sim} Y$ and $g = f^{-1}$. The set of isomorphisms $X \to Y$ is $\mathrm{Isom}_{\mathcal{C}}(X, Y)$, and

$$\mathrm{End}_{\mathcal{C}}(X) := \mathrm{Hom}_{\mathcal{C}}(X, X), \quad \mathrm{Aut}_{\mathcal{C}}(X) := \mathrm{Isom}_{\mathcal{C}}(X, X) \tag{1.1.5}$$

are the ***endomorphisms*** and ***automorphisms*** of X: $\mathrm{End}(X)$ is a monoid under composition and $\mathrm{Aut}(X)$ a group.

Definition 1.1.2 A ***subcategory*** $\mathcal{C}' \subset \mathcal{C}$ means a category with $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all objects $X, Y \in \mathcal{C}'$, the inclusions preserving identities and composition. It is a ***full subcategory*** if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all X, Y, in which case $\mathcal{C}'$ is determined by its objects.

The intersection $\bigcap_{i \in I} \mathcal{C}_i$ of a family of subcategories $\mathcal{C}_i \subset \mathcal{C}$ indexed by a set I is defined in the evident manner.

Example 1.1.3 The basic examples, to which we return constantly:

- Set (***sets*** and functions), Grp (groups), Ab (***abelian groups*** , written additively), Ring (rings), Top (topological spaces).
- For a ring R, the categories $R\text{-}\mathsf{Mod}$ (***left R-modules***) and $\mathsf{Mod}\text{-}R$ (right R-modules). The category Ab is $\mathbb{Z}\text{-}\mathsf{Mod}$.
- Over a field $\Bbbk$, the category $\mathsf{Vect}(\Bbbk)$ of vector spaces and its full subcategory $\mathsf{Vect}_f(\Bbbk)$ of finite-dimensional spaces.
- A ***preordered set*** $(P, \leq)$, as defined in [9, Definition 1.2.1], is the same thing as a category in which each Hom-set has at most one element: there is a unique morphism $p \to p'$ iff $p \leq p'$. In particular each $n \in \mathbb{Z}_{\geq 0}$ is the linearly ordered category $\mathbf{0} \to \mathbf{1} \to \cdots \to \boldsymbol{n-1}$ (identities suppressed); $\mathbf{0}$ is the ***empty category*** and $\mathbf{1}$ the category with one object and one morphism.

  Perhaps more familiar is the notion of a ***partially ordered set*** : a preordered set $(P, \leq)$ such that $x \leq y$ and $y \leq x$ imply $x = y$.
- The ***discrete category*** $\mathsf{Disc}(S)$ on a set S has objects S and only identity morphisms.

The reader will notice a subtlety: Set has "all" sets as objects, but the collection of all sets is not itself a set. This is why we ask $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ to be merely classes, not necessarily sets. If $\mathrm{Ob}(\mathcal{C})$ is a set, we say $\mathcal{C}$ is a ***small category*** in contrast, a category in the sense of Definition 1.1.1 is also called ***locally small*** .

Another way to handle this set-theoretic nuisance once and for all is to fix a ***Grothendieck universe*** $\mathcal{U}$ (see also Appendix A) and control the sizes of "all" sets, groups, rings, etc. by declaring them to be "$\mathcal{U}$-small sets".[1] In this framework, $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ can be taken to be sets, and we say $\mathcal{C}$ is locally $\mathcal{U}$-small (or a $\mathcal{U}$-category) if every Hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{U}$-

[1] Yet another approach is to bound the size of structures under consideration by a chosen cut-off cardinal κ.

small; it is a $\mathcal{U}$-small category if moreover $\mathrm{Mor}(\mathcal{C})$ is $\mathcal{U}$-small. We henceforth drop the universe from the notation and assume, unless stated otherwise, that every category is locally small; the resulting theory is then essentially that of Definition 1.1.1.

Definition 1.1.4 A morphism $f : X \to Y$ is a ***monomorphism*** (mono) if $fg = fh \Rightarrow g = h$ for all objects W and $g, h : W \to X$ (written $f : X \hookrightarrow Y$); it is an ***epimorphism*** (epi) if $gf = hf \Rightarrow g = h$ for all objects W and $g, h : Y \to W$ (written $f : X \twoheadrightarrow Y$). Equivalently, f is mono iff post-composition f_* is injective on Hom-sets, and epi iff pre-composition f^* is (the notation f_*, f^* is introduced in Example 1.3.2).

A left-invertible morphism is mono and a right-invertible one is epi; a morphism is an isomorphism iff it is both left- and right-invertible. In Set, Grp, R-Mod the categorical notions of mono/epi agree with the familiar injective/surjective ones, but this fails in general (see Exercise 1.1.10). We shall be especially careful with this distinction in Chapter 2.

Definition 1.1.5 The ***opposite category*** $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, with composition $f \circ^{\mathrm{op}} g = g \circ f$.

The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ reverses all arrows and satisfies $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. This is the source of the ***duality principle*** : every notion and every theorem in category theory come with a *dual*, obtained by reversing all arrows (equivalently, by working in $\mathcal{C}^{\mathrm{op}}$). The dual of "monomorphism" is "epimorphism", the dual of "product" is "coproduct", the dual of "limit" is "colimit" (to be discussed in Section 1.5), and so on.

A ***groupoid*** is a category in which every morphism is an isomorphism; the basic example is the ***fundamental groupoid*** $\Pi_1(X)$ of a topological space.

Definition 1.1.6 An object X in a category $\mathcal{C}$ is said to be ***initial*** (resp. ***terminal***) if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(Y, X)$) is a singleton for every $Y \in \mathrm{Ob}(\mathcal{C})$. If X is both initial and terminal, we say X is a ***zero*** object.

Initial and terminal objects are dual concepts: X is initial in $\mathcal{C}$ if and only if it is terminal in $\mathcal{C}^{\mathrm{op}}$. They are useful in formulating universal properties, by virtue of the following uniqueness up to unique isomorphism.

Proposition 1.1.7 Let X and X' be initial (resp. terminal) objects in a category $\mathcal{C}$. Then there exists a unique morphism $f : X \to X'$, which is necessarily an isomorphism.

Proof Since X is initial (resp. X' is terminal), there exists a unique $f : X \to X'$. By symmetry, there exists a unique $g : X' \to X$. Then $gf : X \to X$ must be id_X since $\mathrm{Hom}_{\mathcal{C}}(X, X)$ is also a singleton; similarly $fg = \mathrm{id}_{X'}$. □

For example, in R-Mod the zero module $\{0\}$ is a zero object.

Any zero object of a category, if it exists, will be written as 0.

We record one last condition on index categories, needed later for filtered colimits (Chapter 2) and the Ind-completion (Appendix A).

Definition 1.1.8 A nonempty category I is called ***filtered*** if

- for all $i, j \in \mathrm{Ob}(I)$, there exist an object k and morphisms $i \to k$ and $j \to k$;
- for all pairs of parallel morphisms $f, g : i \to j$, there exist an object k and a morphism $h : j \to k$ such that $hf = hg$.

A partially ordered set $(P, \leq)$ (Example 1.1.3) is said to be filtered if the corresponding category is; in this case, the second condition above is vacuous.

Exercise 1.1.9 Consider morphisms $X \xrightarrow{a} Y \xrightarrow{b} Z$ in any category $\mathcal{C}$. Suppose that either b is mono or a is epi. Prove that $ba : X \to Z$ is an isomorphism if and only if a and b are both isomorphisms. *Hint*. For the "only if" direction, the candidate for b^{-1} is $a(ba)^{-1}$.

Exercise 1.1.10 Consider the category Top of all topological spaces and continuous maps between them. Show that the monomorphisms (resp. epimorphisms) are exactly the injective (resp. surjective) continuous maps. If we only allow Hausdorff spaces, as assumed in [9, §2.1], then the epimorphisms are exactly the continuous maps with dense image.

1.2 Functors and natural transformations

With categories available, the story now unfolds in its natural order: functors first, then the natural transformations they were invented to serve.

Definition 1.2.1 A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ consists of a map on objects $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and a map on morphisms, compatible with source, target, identities, and composition:

$$\begin{gathered} F(f) : F(X) \to F(Y) \quad (\text{for } f : X \to Y), \\ F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \,. \end{gathered} \tag{1.2.1}$$

Definition 1.2.2 A functor $F : \mathcal{C} \to \mathcal{D}$ is ***faithful*** if $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is injective for all X, Y; ***full*** if it is surjective; and ***essentially surjective*** if every object of $\mathcal{D}$ is isomorphic to some $F(X)$.

The ***essential image*** of F is the full subcategory of $\mathcal{D}$ consisting of the objects isomorphic to some $F(X)$; thus F is essentially surjective iff its essential image equals $\mathcal{D}$.

Thus a subcategory $\mathcal{C}' \subset \mathcal{C}$ gives an inclusion functor, which is always faithful; it is full iff $\mathcal{C}'$ is a full subcategory. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ (and its many cousins) forgets structure and is faithful but not full.

Definition 1.2.3 A ***natural transformation*** $\alpha : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ (sometimes written as $\alpha : F \Rightarrow G$) is a family of morphisms $\alpha_X : F(X) \to G(X)$, one for each $X \in \mathcal{C}$, such that for every $f : X \to Y$ in $\mathcal{C}$ the square

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd} \tag{1.2.2}$$

commutes. If every α_X is an isomorphism, α is a ***natural isomorphism*** and we write $\alpha : F \xrightarrow{\sim} G$ (in practice one often simply writes $F = G$).

It is customary to depict a natural transformation $\alpha : F \Rightarrow G$ using **2-cells**:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \tag{1.2.3}$$

Natural transformations may be composed in two ways. The ***vertical composition*** of $\alpha : F \to G$ and $\beta : G \to H$ is $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$. The ***horizontal composition*** of $\alpha : F_1 \to F_2$ (between functors $\mathcal{C}' \to \mathcal{C}$) and $\beta : G_1 \to G_2$ (between functors $\mathcal{C} \to \mathcal{D}$) is the natural transformation $\beta \circ \alpha : G_1F_1 \to G_2F_2$ whose component at X is the common diagonal of the square

$$\begin{tikzcd} G_1F_1X \arrow[r, "\beta_{F_1X}"] \arrow[d, "G_1(\alpha_X)"'] & G_2F_1X \arrow[d, "G_2(\alpha_X)"] \\ G_1F_2X \arrow[r, "\beta_{F_2X}"'] & G_2F_2X \end{tikzcd} \tag{1.2.4}$$

Both compositions are associative and compatible; we use the same symbol $\circ$ for both, disambiguating in words when necessary.

Definition 1.2.4 A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an ***equivalence of categories*** if it admits a ***quasi-inverse*** $G : \mathcal{C}_2 \to \mathcal{C}_1$, i.e. natural isomorphisms $GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_1}$ and $FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_2}$. If moreover $GF = \mathrm{id}$ and $FG = \mathrm{id}$, then F is an ***isomorphism of categories*** with inverse G.

In practice equivalence is the useful notion and isomorphism of categories is too strict. The following characterization is the criterion we shall always use.

Theorem 1.2.5 For a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, the following are equivalent.

- F is an equivalence of categories.
- F is fully faithful and essentially surjective.

Proof ($\Rightarrow$) Let G be a quasi-inverse and $\psi : GF \xrightarrow{\sim} \mathrm{id}$, $\varphi : FG \xrightarrow{\sim} \mathrm{id}$. Then $\varphi_Z : F(GZ) \xrightarrow{\sim} Z$ exhibits F as essentially surjective. To see F is faithful, note that the composite

$$\mathrm{Hom}(X,Y) \xrightarrow{F} \mathrm{Hom}(F(X),F(Y)) \xrightarrow{G} \mathrm{Hom}(GF(X),GF(Y)) \to \mathrm{Hom}(X,Y) \tag{1.2.5}$$

is the identity, where the last map is $\alpha \mapsto \psi_Y \circ \alpha \circ \psi_X^{-1}$; hence F is injective on Hom-sets. Dually, the composite obtained from $\varphi : FG \xrightarrow{\sim} \mathrm{id}$ shows that G is faithful. For fullness, given $u : FX \to FY$, put $g := \psi_Y \circ G(u) \circ \psi_X^{-1} : X \to Y$; naturality of ψ gives $GF(g) = G(u)$, whence $F(g) = u$ since G is faithful.

($\Leftarrow$) This direction is slightly subtler and uses the *axiom of choice* in the form of skeleta. A ***skeleton*** of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ meeting every isomorphism class exactly once; every category has a skeleton, and the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence. If F is fully faithful and essentially surjective, choose for each $Z \in \mathcal{C}_2$ an object $X_Z \in \mathcal{C}_1$ and an isomorphism $\varphi_Z : F(X_Z) \xrightarrow{\sim} Z$; then define G on objects by $G(Z) = X_Z$ and on morphisms by transport along the φ_Z. One checks $GF \xrightarrow{\sim} \mathrm{id}$ and $FG \xrightarrow{\sim} \mathrm{id}$. Full details are in [9, Theorem 2.2.13]. □

Example 1.2.6 Let Mat be the category whose objects are $n \in \mathbb{Z}_{\geq 0}$ and whose morphisms $n \to m$ are $m \times n$ matrices over a field $\Bbbk$, composed by matrix multiplication. The functor $\mathsf{Mat} \to \mathsf{Vect}_f(\Bbbk)$ sending $n \mapsto \Bbbk^{\oplus n}$ and a matrix A to the linear map $v \mapsto Av$ is fully faithful (this is the matrix representation of a linear map) and essentially surjective (every finite-dimensional space is isomorphic to $\Bbbk^{\oplus \dim V}$). By Theorem 1.2.5 it is an equivalence. Likewise, the dual-space functor gives an equivalence $\mathsf{Vect}_f(\Bbbk)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Vect}_f(\Bbbk)$.

1.3 Functor categories

We first record how to multiply categories in a naive way, then package functors themselves into a category.

Definition 1.3.1 Given categories $\mathcal{C}_i$ indexed by a set I, the ***product*** $\prod_{i \in I} \mathcal{C}_i$ has objects $(X_i)_i$ and morphisms $(f_i)_i$ componentwise; the ***coproduct*** $\bigsqcup_{i \in I} \mathcal{C}_i$ (a.k.a. disjoint union) has objects $\bigcup_{i \in I} \mathrm{Ob}(\mathcal{C}_i)$ and $\mathrm{Hom}(X_j, X'_k) = \varnothing$ for $j \neq k$. A functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ is a ***bifunctor*** .

The above involves Cartesian products and disjoint unions of classes indexed by a set I, which can be safely defined. We will only need the case $|I| \leq 3$.

Example 1.3.2 The assignment $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a bifunctor, the ***Hom-functor***

$$\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}. \tag{1.3.1}$$

For morphisms $f : X' \to X$ and $g : Y \to Y'$, a morphism $\varphi : X \to Y$ is sent to $g\varphi f : X' \to Y'$. We write $f^*\varphi = \varphi f$ (***pullback*** along f, i.e. pre-composition) and $g_*\varphi = g\varphi$ (***pushout*** along g, i.e. post-composition); then $(f_1 f_2)^* = f_2^* f_1^*$ and $(g_1 g_2)_* = (g_1)_* (g_2)_*$.

Definition 1.3.3 For categories $\mathcal{C}, \mathcal{D}$, the ***functor category*** $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ has functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations as morphisms, composed vertically. We also write it $\mathcal{D}^{\mathcal{C}}$.

A natural transformation $F \Rightarrow G$ is given by an element of the product $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(FX, GX)$, subject to the naturality condition. The product is a set when $\mathcal{C}$ is small; in practice we often assume $\mathcal{C}$ small, since otherwise the natural transformations $F \to G$ may form a proper class, and $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ fails to be locally small (as its Hom-sets may be proper classes).

There is a natural identification

$$\mathrm{Fct}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \tag{1.3.2}$$

sending $\alpha : F \to G$ to $\alpha^{\mathrm{op}} : G^{\mathrm{op}} \to F^{\mathrm{op}}$. Independently of smallness, a natural transformation is an isomorphism in $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ iff each of its components is.

Definition 1.3.4 The ***center*** of $\mathcal{C}$ is $Z(\mathcal{C}) := \mathrm{End}(\mathrm{id}_{\mathcal{C}})$, the monoid of natural endomorphisms of the identity functor.

For any $\alpha, \beta \in Z(\mathcal{C})$, naturality of α applied to the morphism $\beta_X : X \to X$ gives $\alpha_X \beta_X = \beta_X \alpha_X$, so $Z(\mathcal{C})$ is always a commutative monoid; it is a useful invariant, and an equivalence $\mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$ induces an isomorphism $Z(\mathcal{C}_1) \xrightarrow{\sim} Z(\mathcal{C}_2)$.

Remark 1.3.5 In the literature, functors $\mathcal{C}_1^{\mathrm{op}} \to \mathcal{C}_2$, or equivalently $\mathcal{C}_1 \to \mathcal{C}_2^{\mathrm{op}}$, are often called ***contravariant*** functors from $\mathcal{C}_1$ to $\mathcal{C}_2$, whereas the bona fide functors $\mathcal{C}_1 \to \mathcal{C}_2$ are called ***covariant*** ones.

1.4 Representable functors and the Yoneda lemma

We come to the first deep result of the subject, which formalizes the idea that an object is determined by the morphisms into it. Fix a category $\mathcal{C}$. Let us assume $\mathcal{C}$ is small to avoid set-theoretic issues. Define the ***presheaf*** category $\mathcal{C}^\wedge$ and its dual version $\mathcal{C}^\vee$, the ***copresheaf*** category,

$$\mathcal{C}^\wedge := \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \quad \mathcal{C}^\vee := \mathrm{Fct}(\mathcal{C}, \mathsf{Set})^{\mathrm{op}}; \tag{1.4.1}$$

the terminology is borrowed from sheaf theory. The two ***Yoneda embeddings***

$$\begin{aligned} h_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^\wedge, \quad S \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, S), \\ k_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^\vee, \quad S \mapsto \mathrm{Hom}_{\mathcal{C}}(S, \cdot), \end{aligned} \tag{1.4.2}$$

record, respectively, the contravariant and covariant Hom-functors represented by S.

One readily sees that $(\mathcal{C}^\vee)^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^\wedge$, compatibly with the two Yoneda embeddings. Therefore, we will focus on $h_{\mathcal{C}}$.

Theorem 1.4.1 (*Yoneda's lemma*) For every object $S \in \mathrm{Ob}(\mathcal{C})$ and every $A \in \mathrm{Ob}(\mathcal{C}^\wedge)$, the evaluation map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), A) &\to A(S) \\ \varphi &\mapsto \varphi_S(\mathrm{id}_S) \end{aligned} \tag{1.4.3}$$

is a bijection, natural in both S and A (i.e., an isomorphism between the bifunctors). In particular $h_{\mathcal{C}}$ is fully faithful (and dually so is $k_{\mathcal{C}}$).

Proof The map is natural in S and A by inspection. To invert it, given $u \in A(S)$, define a natural transformation $\tilde{u} : h_{\mathcal{C}}(S) \to A$ by

$$\tilde{u}_T(f : T \to S) := A(f)(u) \in A(T). \tag{1.4.4}$$

Naturality in T is exactly the commutative square expressing A's functoriality, and $\tilde{u}_S(\mathrm{id}_S) = A(\mathrm{id}_S)(u) = u$, so the two constructions are inverse to one another. Taking $A = h_{\mathcal{C}}(T)$ gives a bijection $\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), h_{\mathcal{C}}(T)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(S, T)$, i.e. full faithfulness of $h_{\mathcal{C}}$, once it is shown to be inverse to the map induced by $h_{\mathcal{C}}$, which we leave as an exercise. $\square$

Remark 1.4.2 The smallness of $\mathcal{C}$ only serves to ensure that $\mathcal{C}^\wedge$ is a (locally small) category. In general, the recipe above still gives a bijection between the natural transformations $\mathrm{Hom}_{\mathcal{C}}(\cdot, S) \to A$ (forming a class, a priori) and the elements of $A(S)$, and the naturality in (S, A) holds. This will suffice for many applications.

The proof highlights a technique we shall use repeatedly: to give a natural transformation $h_{\mathcal{C}}(S) \to A$ is the same as to give a single element $u \in A(S)$.

Definition 1.4.3 A functor $A : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is ***representable*** if $A \simeq h_{\mathcal{C}}(X)$ for some object X, called a representing object or representative of A.

By the Yoneda lemma, representatives are unique up to unique isomorphism: if $h_{\mathcal{C}}(X) \xrightarrow{\sim} A \xrightarrow{\sim} h_{\mathcal{C}}(Y)$, then $X \xrightarrow{\sim} Y$ uniquely; smallness of $\mathcal{C}$ is not required here (Remark 1.4.2). The element $u \in A(X)$ corresponding to id_X via a chosen isomorphism $A \simeq h_{\mathcal{C}}(X)$ is called the ***universal element*** : to specify a representative X of A, together with an isomorphism $A \simeq h_{\mathcal{C}}(X)$, is the same as to specify a pair (X, u) with $u \in A(X)$ for which the induced map $\tilde{u}_T : \mathrm{Hom}_{\mathcal{C}}(T, X) \to A(T)$ is a bijection for every T, and then $\varphi_T(f : T \to X) = A(f)(u)$. In this guise representability is exactly the language of ***universal properties*** : a universal property asserts that a certain presheaf is representable. We shall see this concretely with limits in Section 1.5.

Remark 1.4.4 The Yoneda lemma also implies that $\mathcal{C}$ embeds fully faithfully into the presheaf category $\mathcal{C}^\wedge$, which admits all small limits and colimits in the sense of Section 1.5 (*pointwise* construction, see Proposition 1.5.8 below). This is the foundation of the *Yoneda philosophy*: to study an object X, study the functor $\mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ it represents.

1.5 Limits and colimits

Many constructions in mathematics are characterized by ***universal properties*** : products, kernels, free objects, completions. We encode all of them at once via the diagonal functor.

Definition 1.5.1 For categories I and $\mathcal{C}$, assuming I small, the ***diagonal functor***

$$\Delta : \mathcal{C} \to \mathcal{C}^I = \mathrm{Fct}(I, \mathcal{C}) \tag{1.5.1}$$

sends an object X to the constant functor $i \mapsto X$ and a morphism f to the constant natural transformation.

Definition 1.5.2 Given functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the ***comma category*** (F/G) has

- as objects: triples (A, f, B) where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is a morphism of $\mathcal{C}$;
- as morphisms $(A, f, B) \to (A', f', B')$: pairs $a : A \to A'$ in $\mathcal{A}$ and $b : B \to B'$ in $\mathcal{B}$ for which the square

$$\begin{tikzcd} F(A') \arrow[r, "f'"] & G(B') \\ F(A) \arrow[u, "F(a)"] \arrow[r, "f"'] & G(B) \arrow[u, "G(b)"'] \end{tikzcd} \tag{1.5.2}$$

commutes. Identities and composition are componentwise, so the evident projections define functors $\mathcal{A} \leftarrow (F/G) \to \mathcal{B}$.

Remark 1.5.3 (*cones and cocones*) Let $\mathbf{1}$ denote the category with one object and one morphism. When one argument is a single object $c \in \mathrm{Ob}(\mathcal{C})$ (viewed as a functor $\mathbf{1} \to \mathcal{C}$), the comma category (c/G) is the ***coslice category*** (category of objects under c) and (F/c) the ***slice category*** (category of objects over c). In the definition below, the diagram $\alpha : I \to \mathcal{C}$ is an object of $\mathcal{C}^I$, so (α/Δ) is the category of *cocones* under α: its objects are pairs $(X \in \mathrm{Ob}(\mathcal{C}), \alpha \Rightarrow \Delta(X))$. Dually (Δ/β) is the category of *cones* over β.

Definition 1.5.4 Let I be a category, assumed to be small. For a functor $\alpha : I \to \mathcal{C}$ (a ***diagram*** of shape I):

- the ***colimit*** (notation colim, a.k.a. ***inductive limit***) is the initial object (Definition 1.1.6) of the comma category (α/Δ);
- the ***limit*** $\lim \beta$ (a.k.a. ***projective limit***) of $\beta : I^{\mathrm{op}} \to \mathcal{C}$ is the terminal object (Definition 1.1.6) of (Δ/β) (here Δ is the diagonal of shape I^{op}; the opposite in the domain of β is just a convention).

When they exist, limits and colimits are unique up to unique isomorphism by Proposition 1.1.7.

The colimit of $\alpha : I \to \mathcal{C}$ is said to be filtered if I is filtered, and the limit of $\beta : I^{\mathrm{op}} \to \mathcal{C}$ is said to be cofiltered if the indexing category I^{op} is cofiltered (i.e. I is filtered). Since the indexing category I is assumed small in this section, the corresponding colimits and limits are also said to be small.

Unwinding the universal property, a ***cone*** over $\beta : I^{\mathrm{op}} \to \mathcal{C}$ with apex L is a family of maps $p_i : L \to \beta(i)$ making the triangles

$$\begin{tikzcd} L \arrow[d, "p_i"'] \arrow[dr, "p_j"] & \\ \beta(i) & \beta(j) \arrow[l, "\beta(\varphi)"] \end{tikzcd} \qquad (\varphi : i \to j \text{ in } I) \tag{1.5.3}$$

commute, and $\lim \beta$ is the *terminal* cone: every cone factors uniquely through it. Dually, a ***cocone*** under $\alpha : I \to \mathcal{C}$ is a family $\iota_i : \alpha(i) \to L$ with $\iota_j \circ \alpha(\varphi) = \iota_i$ for all $\varphi : i \to j$, and colim α is the *initial* cocone. The duality $\lim(\beta^{\mathrm{op}}) = \operatorname{colim} \beta$ (in $\mathcal{C}^{\mathrm{op}}$) records that limits and colimits are dual notions.

The morphisms $\iota_i : \alpha(i) \to \operatorname{colim} \alpha$ and $p_j : \lim \beta \to \beta(j)$ appearing in cones and cocones are called **structure morphisms**.

Example 1.5.5 (*limits as familiar constructions*) Discrete, empty, and simple finite shapes give the classical notions.

▷ **Products and coproducts** Taking I discrete yields the ***product*** $\prod_i X_i$ (terminal cone) and ***coproduct*** $\bigsqcup_i X_i$ (initial cocone), characterized by

$$\begin{aligned} \mathrm{Hom}\left(Z, \prod_i Y_i\right) &\simeq \prod_i \mathrm{Hom}(Z, Y_i), \\ \mathrm{Hom}\left(\bigsqcup_i X_i, Z\right) &\simeq \prod_i \mathrm{Hom}(X_i, Z); \end{aligned} \tag{1.5.4}$$

here the structure morphisms p_i and ι_j are left implicit.

In Set these are Cartesian product and disjoint union; in Ab the coproduct is the direct sum $\oplus_i X_i$. Binary products and coproducts are written as $X \times Y$ and $X \sqcup Y$, respectively, and analogously $X_1 \times \cdots \times X_n$ and $X_1 \sqcup \cdots \sqcup X_n$ in the finite case.

▷ **Terminal and initial objects** Taking I empty gives the terminal object $*$ (every object maps to it uniquely) and the initial object (maps uniquely to every object); see Definition 1.1.6. Note that terminal (resp. initial) objects can also be interpreted as empty products (resp. coproducts).

▷ **Equalizers and coequalizers** For parallel morphisms $f, g : X \rightrightarrows Y$, the ***equalizer*** $\ker(f, g) \hookrightarrow X$ is the universal object on which f and g agree, and the ***coequalizer*** $Y \twoheadrightarrow \operatorname{coker}(f, g)$ is its dual:

$$\begin{tikzcd} \ker(f,g) \arrow[r, hook] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd} \quad \text{(and dually)} \tag{1.5.5}$$

They can be interpreted as limits and colimits. Consider the category I depicted as $\bullet \rightrightarrows \bullet$ (no other morphisms besides the identities); to give a functor $\alpha : I \to \mathcal{C}$ is the same as giving parallel arrows $f, g : X \rightrightarrows Y$ in $\mathcal{C}$, and (α/Δ) is identified with the category of morphisms $\varphi : Y \to Z$ such that $\varphi f = \varphi g$. Therefore $\operatorname{colim} \alpha$, being the initial object in (α/Δ), is the coequalizer of f, g. Dually for equalizers.

▷ **Pullbacks and pushouts** The shape $\bullet \to \bullet \leftarrow \bullet$ gives the ***pullback*** (or ***fibered product***) $X \times_Z Y$ and its dual shape $\bullet \leftarrow \bullet \to \bullet$ the ***pushout*** $X \sqcup_Z Y$:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\boxplus"] & X \arrow[d, "f'"] \\ Y \arrow[r, "g'"'] & X \sqcup_Z Y \end{tikzcd} \tag{1.5.6}$$

A commutative square with a $\square$ in the middle denotes a pullback, and $\boxplus$ a pushout.

(Co)limits can be constructed by hand in concrete categories. In Set,

$$\lim \beta = \left\{ (x_i)_i \in \prod_i \beta(i) : \beta(\sigma)(x_j) = x_i \text{ for all } \sigma : i \to j \right\}, \tag{1.5.7}$$

a subobject of the product; and

$$\begin{gathered} \operatorname{colim} \alpha = \left(\bigsqcup_i \alpha(i) \right) / \sim, \\ x \sim \alpha(\sigma)(x) \text{ for } \sigma : i \to j. \end{gathered} \tag{1.5.8}$$

Proposition 1.5.6 Suppose $\mathcal{C}$ has all (small) products and equalizers. Then $\mathcal{C}$ has all (small) limits. Dually, coproducts and coequalizers yield all colimits.

Proof Given $\beta : I^{\text{op}} \to \mathcal{C}$, form the products
- $P = \prod_{i \in \operatorname{Ob}(I)} \beta(i)$ and
- $Q = \prod_{\sigma \in \operatorname{Mor}(I)} \beta(s\sigma)$, writing $s(\sigma), t(\sigma)$ for the source and target of a morphism $\sigma : i \to j$ in I.

Using the universal property of products, define $f, g : P \rightrightarrows Q$ to be the morphisms with components $f_\sigma = p_{s\sigma}$ and $g_\sigma = \beta(\sigma)p_{t\sigma}$.

We claim that $\ker(f, g)$ together with the structure morphisms to the $\beta(i)$ (inherited from $\ker(f, g) \to P$) gives $\lim \beta$. Indeed, giving morphisms φ_i to $\beta(i)$ for all $i \in \mathrm{Ob}(I)$ amounts to giving a morphism φ to P; the compatibility conditions for $(\varphi_i)_i$ (to get an object of (Δ/β)) amount to $f\varphi = g\varphi$, i.e. that φ factors through $\ker(f, g)$.

The case of colimits is proved in the dual manner. □

A category with all small limits (resp. colimits) is said to be ***complete*** (resp. ***cocomplete***). By Proposition 1.5.6, completeness is equivalent to having products and equalizers, and cocompleteness to coproducts and coequalizers. The categories Set, Top, Grp, Ab, and R-Mod are all complete and cocomplete.

Functors act on diagrams and hence on (co)limits. A functor $F : \mathcal{C} \to \mathcal{D}$ ***preserves limits*** if $F(\lim \beta) \simeq \lim(F\beta)$ canonically, and dually for colimits. The two facts below are used constantly.

Proposition 1.5.7 (*Hom preserves limits*) For every object X,
- $\mathrm{Hom}_{\mathcal{C}}(X, \cdot) : \mathcal{C} \to \mathsf{Set}$ preserves all limits that exist;
- $\mathrm{Hom}_{\mathcal{C}}(\cdot, X) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ preserves all limits that exist (i.e. takes colimits in $\mathcal{C}$ to limits of sets).

Proof This amounts to the canonical isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, \lim_i \beta(i)) \simeq \lim_i \mathrm{Hom}_{\mathcal{C}}(X, \beta(i))$, which simply restates the definition of $\lim \beta$. The second assertion follows by applying the first in $\mathcal{C}^{\mathrm{op}}$. □

(Co)limits in a presheaf category are computed pointwise:

Proposition 1.5.8 (*pointwise (co)limits in presheaves*) For small I and any $\alpha : I \to \mathcal{C}^\wedge$ (assume also $\mathcal{C}$ small, so that $\mathcal{C}^\wedge$ is a category in the sense of Definition 1.1.1), the presheaf

$$(\operatorname{colim} \alpha)(S) := \operatorname{colim}(\alpha(\cdot)(S)) \quad (S \in \mathcal{C}) \tag{1.5.9}$$

is the colimit of α in $\mathcal{C}^\wedge$; limits are formed dually. In particular $\mathcal{C}^\wedge$ is complete and cocomplete, and a limit $\lim \beta$ in $\mathcal{C}$ exists iff the presheaf $S \mapsto \lim \mathrm{Hom}(S, \beta(i))$ is representable.

We record one more useful consequence: limits may be computed iteratively.

Lemma 1.5.9 (*Fubini for (co)limits*) For $\alpha : I \times J \to \mathcal{C}$, assuming the (co)limits exist,

$$\operatorname*{colim}_j \operatorname*{colim}_i \alpha(i, j) \simeq \operatorname{colim} \alpha \simeq \operatorname*{colim}_i \operatorname*{colim}_j \alpha(i, j), \tag{1.5.10}$$

and dually for limits.

Proof This is routine once you have mastered the definitions; see [9, Lemma 2.7.10]. □

Exercise 1.5.10 Show that products in a category $\mathcal{C}$ satisfy the following canonical isomorphisms

$$\begin{gathered}(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z),\\ X \times * \simeq X \simeq * \times X,\\ X \times Y \simeq Y \times X\end{gathered} \tag{1.5.11}$$

where the products are assumed to exist, and $*$ is any terminal object of $\mathcal{C}$. Generalize these to arbitrary products (possibly infinite), and state dual versions for coproducts and initial objects.

Hint. For products, use the Yoneda embedding $h_{\mathcal{C}}$ to reduce to set-theoretic products.

Exercise 1.5.11 Show that the equalizer $\ker(f, g) \to X$ is a monomorphism and the coequalizer $Y \to \operatorname{coker}(f, g)$ is an epimorphism.

Exercise 1.5.12 Show that the pullback of a monomorphism is a monomorphism, and dually the pushout of an epimorphism is an epimorphism. *Hint*. Describe pullbacks via Proposition 1.5.7 and use the characterization of monomorphisms in Definition 1.1.4.

1.6 Adjoint functors

Adjoint functors, introduced by Daniel Kan in 1958, pervade mathematics. The informal slogan is: a pair of adjoint functors expresses a *universal mapping property* relating two categories.

Definition 1.6.1 An ***adjunction*** (or ***adjoint pair***) is a triple (F, G, φ) where

$$F : \mathcal{C}_1 \to \mathcal{C}_2 \quad \text{and} \quad G : \mathcal{C}_2 \to \mathcal{C}_1 \tag{1.6.1}$$

are functors and φ is a natural isomorphism

$$\varphi : \operatorname{Hom}_{\mathcal{C}_2}(F(\cdot), \cdot) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}_1}(\cdot, G(\cdot)). \tag{1.6.2}$$

We say F is ***left adjoint*** to G (and G ***right adjoint*** to F), and express the adjunction either by the diagram

$$F : \mathcal{C}_1 \rightleftarrows \mathcal{C}_2 : G \tag{1.6.3}$$

or the shorthand $F \dashv G$.

Spelled out, φ gives a bijection $\varphi_{X,Y} : \operatorname{Hom}_{\mathcal{C}_2}(FX, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}_1}(X, GY)$, natural in $X \in \operatorname{Ob}(\mathcal{C}_1)$ and $Y \in \operatorname{Ob}(\mathcal{C}_2)$. The data of an adjunction can be packaged without mentioning Hom-sets, via two natural transformations:

Definition 1.6.2 The ***unit*** $\eta : \mathrm{id}_{\mathcal{C}_1} \to GF$ and ***counit*** $\varepsilon : FG \to \mathrm{id}_{\mathcal{C}_2}$ of an adjunction (F, G, φ) are defined by

$$\eta_X := \varphi_{X,FX}(\mathrm{id}_{FX}), \quad \varepsilon_Y := \varphi^{-1}_{GY,Y}(\mathrm{id}_{GY}). \tag{1.6.4}$$

The unit and counit determine φ by the formulas

$$\begin{aligned} \varphi(f) &= Gf \circ \eta_X \quad (\text{for } f : F(X) \to Y), \\ \varphi^{-1}(g) &= \varepsilon_Y \circ Fg \quad (\text{for } g : X \to G(Y)), \end{aligned} \tag{1.6.5}$$

and they satisfy the ***triangle identities***

$$\begin{aligned} G &\overset{\eta G}{\Rightarrow} GFG \overset{G\varepsilon}{\Rightarrow} G = \mathrm{id}_G, \\ F &\overset{F\eta}{\Rightarrow} FGF \overset{\varepsilon F}{\Rightarrow} F = \mathrm{id}_F \, . \end{aligned} \tag{1.6.6}$$

Conversely, any pair of natural transformations $\eta : \mathrm{id} \to GF$ and $\varepsilon : FG \to \mathrm{id}$ satisfying Equation (1.6.6) determines a unique adjunction (an upcoming exercise). This offers another characterization of adjunctions.

Example 1.6.3 (*free-forgetful adjunctions*) The free vector space functor $V : \mathsf{Set} \to \mathsf{Vect}(\Bbbk)$, $X \mapsto \bigoplus_{x \in X} \Bbbk\, x$, is left adjoint to the forgetful functor $U : \mathsf{Vect}(\Bbbk) \to \mathsf{Set}$: a function $X \to U(W)$ is the same thing as a linear map $V(X) \to W$, naturally in both variables. The same pattern gives free group $\dashv$ forgetful, tensor algebra $\dashv$ forgetful, etc.

Example 1.6.4 (*Hom–tensor adjunction*) The central example for these notes. If R, S are rings and M is an (S, R)-bimodule, the functor $M \otimes_R (\cdot) : R\text{-}\mathsf{Mod} \to S\text{-}\mathsf{Mod}$ is left adjoint to $\mathrm{Hom}_S(M, \cdot) : S\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$:

$$\mathrm{Hom}_S(M \otimes_R N, P) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, P)) \tag{1.6.7}$$

naturally in $N \in R\text{-}\mathsf{Mod}$ and $P \in S\text{-}\mathsf{Mod}$. This is proved in Theorem B.4.3; it will be our running example in Chapter 2 and the foundation of the derived functors Tor and Ext in Chapter 4.

Remark 1.6.5 Given functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ which are mutually quasi-inverse, i.e. there exist natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}_1} \xrightarrow{\sim} GF$ and $\varepsilon : FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_2}$, we can keep η and adjust ε (or conversely) so that they form the unit and counit of an adjunction; then $G \dashv F$ as well, with unit ε^{-1} and counit η^{-1}. Such data $(F, G, \eta, \varepsilon)$ are called ***adjoint equivalences*** , and the above says that all equivalences can be promoted to adjoint equivalences. The proof is instructive; see [9, Theorem 2.6.12].

The theorem below is the single most useful preservation result in the subject.

Theorem 1.6.6 (*adjoints preserve* (*co*)*limits*) Let (F, G, φ) be an adjunction. Then
- F preserves all colimits that exist in $\mathcal{C}_1$;
- G preserves all limits that exist in $\mathcal{C}_2$.

Proof We prove that G preserves limits (the other half is dual). For $\beta : I^{\mathrm{op}} \to \mathcal{C}_2$ and any $X \in \mathrm{Ob}(\mathcal{C}_1)$, there are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_1}(X, G(\lim \beta)) &\simeq \mathrm{Hom}_{\mathcal{C}_2}(F(X), \lim \beta) && \text{by the adjunction } \varphi \\ &\simeq \lim \mathrm{Hom}_{\mathcal{C}_2}(F(X), \beta(i)) && \text{since Hom preserves limits} \\ &\simeq \lim \mathrm{Hom}_{\mathcal{C}_1}(X, G\beta(i)) && \text{by the adjunction } \varphi \\ &\simeq \mathrm{Hom}_{\mathcal{C}_1}(X, \lim(G\beta)) && \text{since Hom preserves limits,} \end{aligned} \tag{1.6.8}$$

holding for every X; by the Yoneda lemma (Theorem 1.4.1) this yields $G(\lim \beta) \simeq \lim(G\beta)$, as claimed. The two appeals to "Hom preserves limits" are Proposition 1.5.7. □

By Theorem 1.6.6 applied to Example 1.6.4, the tensor product $M \otimes_R (\cdot)$ preserves colimits, while $\mathrm{Hom}_S(M, \cdot)$ preserves limits.

Exercise 1.6.7 Verify Equation (1.6.6) directly from the definitions of unit and counit. Then prove the converse: natural transformations $\eta : \mathrm{id} \to GF$, $\varepsilon : FG \to \mathrm{id}$ satisfying Equation (1.6.6) determine a unique adjunction $F \dashv G$.

Exercise 1.6.8 (*uniqueness of adjoints*) Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ have two right adjoints $G_1, G_2 : \mathcal{C}_2 \to \mathcal{C}_1$, given by adjunctions (F, G_1, φ_1) and (F, G_2, φ_2). Composing yields, for each $Y \in \mathrm{Ob}(\mathcal{C}_2)$, bijections

$$\psi_Y : \mathrm{Hom}_{\mathcal{C}_1}(X, G_1Y) \xrightarrow{\varphi_1^{-1}} \mathrm{Hom}_{\mathcal{C}_2}(FX, Y) \xrightarrow{\varphi_2} \mathrm{Hom}_{\mathcal{C}_1}(X, G_2Y), \tag{1.6.9}$$

natural in X.

(i) Show that there is a unique morphism $\theta_Y : G_1Y \to G_2Y$ with $\psi_Y(f) = \theta_Y f$ for all $f : X \to G_1Y$ (Yoneda lemma, Remark 1.4.2).

(ii) Show that the θ_Y assemble into a natural isomorphism $\theta : G_1 \xrightarrow{\sim} G_2$; thus the right adjoint of F is unique up to canonical isomorphism. State and prove the dual assertion for left adjoints.

Hint. For the naturality in (ii), fix a morphism $g : Y \to Y'$ of $\mathcal{C}_2$ and evaluate the naturality of the ψ_Y in Y at id_{G_1Y}, then apply (i) at Y'. For the isomorphism, exchange G_1 and G_2.

Exercise 1.6.9 More generally, let (F_i, G_i, φ_i), $i = 1, 2$, be adjunctions. Show that every natural transformation $u : G_1 \to G_2$ induces, via the composite bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_2}(F_1X, Y) \xrightarrow{\varphi_1} \mathrm{Hom}_{\mathcal{C}_1}(X, G_1Y) &\xrightarrow{f \mapsto u_Y f} \mathrm{Hom}_{\mathcal{C}_1}(X, G_2Y) \\ &\xrightarrow{\varphi_2^{-1}} \mathrm{Hom}_{\mathcal{C}_2}(F_2X, Y) \end{aligned} \tag{1.6.10}$$

and the Yoneda lemma, a natural transformation $v : F_2 \to F_1$ in the opposite direction; when $F_1 = F_2 = F$ and $u = \theta$, verify that $v = \mathrm{id}_F$.

1.7 Additive categories: a first glimpse of kernels

The categories of greatest interest in homological algebra (R-Mod, Ab, sheaves of modules, etc.) carry an *additive* structure: Hom-sets are abelian groups and composition is bilinear. We give the definitions here and defer the full theory to Chapter 2.

Definition 1.7.1 A ***preadditive category*** (or Ab-category) is a category in which each Hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is endowed with the structure of an abelian group, written additively, and composition is bilinear:

$$(f + g)h = fh + gh, \quad f(g + h) = fg + fh. \tag{1.7.1}$$

A functor $F : \mathcal{A} \to \mathcal{B}$ between preadditive categories is called an ***additive functor*** if it induces group homomorphisms on Hom-sets.

Remark 1.7.2 More generally, given a commutative ring $\Bbbk$, a $\Bbbk$-***linear category*** is a category $\mathcal{C}$ in which each $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is endowed with a $\Bbbk$-module structure, and the composition is $\Bbbk$-bilinear. A functor between $\Bbbk$-linear categories is said to be $\Bbbk$-***linear*** if it induces $\Bbbk$-module homomorphisms on Hom-sets. A preadditive category is nothing but a $\mathbb{Z}$-linear category.

Definition 1.7.3 A preadditive category $\mathcal{C}$ is ***additive*** if it has a zero object and for all objects X_1, X_2 there is an object Z together with morphisms

$$\begin{tikzcd} X_1 \arrow[r, shift right, "\iota_1"'] & Z \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & X_2 \arrow[l, shift left, "\iota_2"] \end{tikzcd} \tag{1.7.2}$$

satisfying the identities $p_1\iota_1 = \mathrm{id}$, $p_2\iota_2 = \mathrm{id}$, $p_2\iota_1 = 0$, $p_1\iota_2 = 0$ and $\iota_1 p_1 + \iota_2 p_2 = \mathrm{id}_Z$; we also say that $(Z, \iota_1, \iota_2, p_1, p_2)$, or simply Z, is the ***biproduct*** of X_1, X_2, denoted as $X_1 \oplus X_2$.

Lemma 1.7.4 Let X_1 and X_2 be objects of a preadditive category $\mathcal{C}$. The following are equivalent:

(i) the product $X_1 \times X_2$ exists,
(ii) the coproduct $X_1 \sqcup X_2$ exists,
(iii) a biproduct $(Z, \iota_1, \iota_2, p_1, p_2)$ exists.

If any of the above holds, then (Z, p_1, p_2) gives the product and (Z, ι_1, ι_2) gives the coproduct.

Proof This is largely formal. See [9, Theorem 3.4.9]. □

The biproduct $X_1 \oplus X_2$ is thus unique up to unique isomorphism whenever it exists. One can define the multivariate version $X_1 \oplus \cdots \oplus X_n$ in a similar manner; it can also be constructed iteratively as $(X_1 \oplus X_2) \oplus X_3$, and so forth.

In a preadditive category the ***zero morphism*** $0 : X \to Y$ is the zero element of the group $\mathrm{Hom}_{\mathcal{C}}(X, Y)$; in an additive category it is equivalently the unique map factoring through the zero object, and we may hence speak of ***kernels*** and ***cokernels*** .

The following is based on Exercise 1.5.11.

Definition 1.7.5 In an additive category, the ***kernel*** of $f : X \to Y$ is the equalizer of f and 0, i.e. the terminal $\ker f \hookrightarrow X$ with $f \circ (\ker f) = 0$; the ***cokernel*** is the coequalizer, the initial $Y \twoheadrightarrow \operatorname{coker} f$ with $(\operatorname{coker} f) \circ f = 0$:

$$\ker f \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow \operatorname{coker} f. \tag{1.7.3}$$

As a spoiler, an ***abelian category*** is an additive category in which every morphism has a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel; equivalently, every morphism is **strict**, meaning that the canonical map $\operatorname{coim} f \to \operatorname{im} f$ (Definition 2.1.1) is an isomorphism. The category R-Mod is the prototype; the Freyd–Mitchell embedding theorem (Appendix A) reduces diagram-chasing in any (small) abelian category to module categories. Chapter 2 develops this in detail; for now we note only that the categories Ab and R-Mod are abelian, and that this is the natural setting for the diagram lemmas, complexes, and derived functors to come.

Exercise 1.7.6 Let $\mathcal{C}$ be an additive category and $f, g : X \to Y$ morphisms. Write ι_i^X, p_i^X and ι_i^Y, p_i^Y for the structure morphisms of the biproducts $X \oplus X$ and $Y \oplus Y$, $f \oplus g : X \oplus X \to Y \oplus Y$ for the unique morphism with $p_1^Y(f \oplus g) = f p_1^X$ and $p_2^Y(f \oplus g) = g p_2^X$, the **diagonal** $\delta_X : X \to X \oplus X$ for the unique morphism with $p_i^X \delta_X = \mathrm{id}_X$, and the **codiagonal** $\check{\delta}_Y : Y \oplus Y \to Y$ for the unique morphism with $\check{\delta}_Y \iota_i^Y = \mathrm{id}_Y$.

(i) Prove that $f + g = \check{\delta}_Y \circ (f \oplus g) \circ \delta_X$.
(ii) Deduce that a functor between additive categories preserving finite products (the empty one included) is additive; thus the addition on the Hom-sets is determined by the zero object and the biproducts alone, and additivity is a property, not a structure.

Hint. For (i), express everything in terms of the ι_i and p_i and use the bilinearity of composition. For (ii), the right-hand side of (i) involves only finite (co)products and the zero object.

1.8 Kan extensions and localization

We conclude with two constructions that will reappear, somewhat disguised, when we build the derived category and the derived functors. Both are best motivated as *universal solutions* to extension problems.

Definition 1.8.1 Let $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{E}$ be functors. The ***left Kan extension*** $(\operatorname{Lan}_K F, \eta)$ of F along K consists of a functor $\operatorname{Lan}_K F : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\eta : F \Rightarrow (\operatorname{Lan}_K F) \circ K$, universal in the sense that any $\xi : F \Rightarrow LK$ factors uniquely as $\xi = (\chi K) \circ \eta$ for some $\chi : \operatorname{Lan}_K F \Rightarrow L$:

$$
\begin{tikzcd}
\mathcal{C} \arrow[d, "K"'] \arrow[dr, bend left, "F"] & \\
\mathcal{D} \arrow[r, "L"'] \arrow[ur, phantom, "\overset{\xi}{\Leftarrow}"] & \mathcal{E}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[d, "K"'] \arrow[dr, bend left, "F"] & \\
\mathcal{D} \arrow[r, "\operatorname{Lan}_K F"'] \arrow[r, bend right=70, "L"'] \arrow[r, phantom, bend right=40, "\Downarrow \chi"] \arrow[ur, phantom, "\overset{\eta}{\Leftarrow}"] & \mathcal{E}
\end{tikzcd}
\tag{1.8.1}
$$

Dually, the ***right Kan extension*** $(\operatorname{Ran}_K F, \varepsilon)$ is universal among functors R with a natural transformation $RK \Rightarrow F$; the structure map is $\varepsilon : (\operatorname{Ran}_K F) \circ K \Rightarrow F$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[d, "K"'] \arrow[dr, bend left, "F"] & \\
\mathcal{D} \arrow[r, "R"'] \arrow[ur, phantom, "\overset{\delta}{\Rightarrow}"] & \mathcal{E}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[d, "K"'] \arrow[dr, bend left, "F"] & \\
\mathcal{D} \arrow[r, "\operatorname{Ran}_K F"'] \arrow[r, bend right=70, "R"'] \arrow[r, phantom, bend right=40, "\Uparrow \theta"] \arrow[ur, phantom, "\overset{\varepsilon}{\Rightarrow}"] & \mathcal{E}
\end{tikzcd}
\tag{1.8.2}
$$

When they exist for all F, Lan_K and Ran_K become left and right adjoints to precomposition $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$.

Recall that $\mathbf{1}$ is the category with one object and one morphism. For any category $\mathcal{C}$, denote by $!$ the unique functor $\mathcal{C} \to \mathbf{1}$.

Kan extensions subsume both limits and adjunctions: taking $K = !$, one finds $\operatorname{Lan}_! F = \operatorname{colim} F$ and $\operatorname{Ran}_! F = \lim F$ (Example 1.8.2); and an adjunction $F \dashv G$ exhibits $G = \operatorname{Lan}_{F(\mathrm{id})}$ and $F = \operatorname{Ran}_{G(\mathrm{id})}$. Kan extensions are ubiquitous; the reason Kan extensions matter in this book is that *derived functors are Kan extensions* (Chapter 7).

Example 1.8.2 (*limits and colimits as Kan extensions*) For $F : \mathcal{C} \to \mathcal{E}$ and $! : \mathcal{C} \to \mathbf{1}$ the unique functor, $\operatorname{Lan}_! F$ exists iff $\operatorname{colim} F$ exists in $\mathcal{E}$ (and then equals it), and $\operatorname{Ran}_! F$ exists iff $\lim F$ exists.

Proposition 1.8.3 The left or right Kan extension of F along K, if it exists, is unique up to a unique isomorphism in the functor category $\mathcal{E}^{\mathcal{D}}$, compatible with the structure maps.

Proof Suppose (L,η) and (L',η') are both left Kan extensions of F along K. The universal property of (L,η) applied to the data (L',η') yields a unique $\chi : L \Rightarrow L'$ with $\eta' = (\chi K)\circ\eta$; symmetrically there is a unique $\chi' : L' \Rightarrow L$ with $\eta = (\chi' K)\circ\eta'$. Then $(\chi'\chi)K \circ \eta = (\chi' K)\circ \eta' = \eta = (\mathrm{id}_L\, K)\circ\eta$, so the uniqueness clause applied to (L,η) itself forces $\chi'\chi = \mathrm{id}_L$; likewise $\chi\chi' = \mathrm{id}_{L'}$. Hence χ is an isomorphism, and it is the unique natural transformation $\alpha : L \Rightarrow L'$ with $\eta' = (\alpha K)\circ\eta$; in particular, the unique isomorphism between the extensions (L,η) and (L',η'). The right Kan case is dual. □

The second construction adds inverses to a class of morphisms.

Definition 1.8.4 Let $S \subset \mathrm{Mor}(\mathcal{C})$ contain all identities and be closed under composition. The ***localization*** $\mathcal{C}[S^{-1}]$ (a.k.a. ***category of fractions*** , in the sense of Gabriel–Zisman), is a category, possibly a large one (see Remark 1.8.7), equipped with a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that

- $Q(s)$ is an isomorphism for every $s \in S$;
- for every category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$ sending each $s \in S$ to an isomorphism, there is a unique $F[S^{-1}] : \mathcal{C}[S^{-1}] \to \mathcal{D}$ with $F = F[S^{-1}] \circ Q$.

The localization is unique up to unique isomorphism. It can always be constructed formally: its morphisms are finite formal "zig-zags" $\cdots as^{-1}bt^{-1}\cdots$ with $a, b \in \mathrm{Mor}(\mathcal{C})$ and $s, t \in S$ such that composition makes sense, but these zig-zags are hard to compute with. The situation improves dramatically when S is a ***multiplicative system*** :

Definition 1.8.5 A class $S \subset \mathrm{Mor}(\mathcal{C})$ is said to be a left multiplicative system if

▷ **(S1)** $\mathrm{id}_X \in S$ for all $X \in \mathrm{Ob}(\mathcal{C})$;
▷ **(S2)** S is closed under composition;
▷ **(S3)** for every $X \xrightarrow{s\in S} Z \xleftarrow{f} Y$ there exists $X \xleftarrow{f'} W \xrightarrow{s'\in S} Y$ with $sf' = fs'$;
▷ **(S4)** if $sf = sg$ with $s \in S$, there is $t \in S$ with $ft = gt$.

If S is a left multiplicative system in $\mathcal{C}^{\mathrm{op}}$, we say S is a right multiplicative system.

If S is both a left and a right multiplicative system, we say S is a ***multiplicative system*** .

When S is a left (resp. right) multiplicative system, the zig-zags collapse: every morphism of $\mathcal{C}[S^{-1}]$ can be written as a single fraction as^{-1} (resp. $s^{-1}a$) with $s \in S$. This is the *calculus of fractions* of Gabriel and Zisman. We sketch the construction for a left multiplicative system, following [10, §1.9]; the right case is dual, with every arrow reversed, and the verifications that we omit are routine applications of (S1)–(S4).

In the diagrams, arrows belonging to S will be drawn like $\rightarrowtail$.

Theorem 1.8.6 (*Gabriel–Zisman*) Let S be a left multiplicative system in $\mathcal{C}$. The localization $\mathcal{C}[S^{-1}]$ can then be described concretely, as follows.

- Objects: the objects of $\mathcal{C}$;
- Morphisms $X \to Y$: equivalence classes $[Z; s, a]$ of ***left fractions*** , a left fraction being a span $X \xleftarrow{s\in S} Z \xrightarrow{a} Y$; the class $[Z; s, a]$ is also written, suggestively, as the fraction as^{-1}. Two left fractions $(Z; s, a)$ and $(Z'; s', a')$ are declared equivalent, in symbols $\sim$, when there exist

morphisms $u : W \to Z$ and $v : W \to Z'$ with $su = s'v \in S$ and $au = a'v$, in the commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\
X & W \arrow[l] \arrow[r] \arrow[u, "u"] \arrow[d, "v"] & Y \\
& Z' \arrow[ul, "s'"] \arrow[ur, "a'"'] &
\end{tikzcd} \tag{1.8.3}$$

reflexivity is immediate ($u = v = \mathrm{id}$); transitivity draws on (S3) and (S4);

- the composition of left fractions $(U; s, a) : X \to Y$ and $(V; t, b) : Y \to Z$: apply (S3) to $U \xrightarrow{a} Y \xleftarrow{t \in S} V$ so as to complete it to a commutative square

$$\begin{tikzcd}
& & W \arrow[dl, "r"'] \arrow[dr, "c"] & & \\
& U \arrow[dl, "s"'] \arrow[dr, "a"] & & V \arrow[dl, "t"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd} \tag{1.8.4}$$

the composite being $X \xleftarrow{sr} W \xrightarrow{bc} Z$, i.e. $[V; t, b] \circ [U; s, a] := [W; sr, bc]$; here $tc = ar$ by commutativity and $sr \in S$ by (S2). The result is independent of the representatives and of the chosen completion; the identity at X is $[X; \mathrm{id}_X, \mathrm{id}_X]$, and the composition is associative (complete three spans at once, applying (S3) thrice);

- define $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ to be id on objects and $Qf := [X; \mathrm{id}_X, f]$ on morphisms, so that every morphism of $\mathcal{C}[S^{-1}]$ is a fraction $Q(a)Q(s)^{-1}$.

Dually, a right multiplicative system leads to a localization via right fractions $X \xrightarrow{a} Z \xleftarrow{s \in S} Y$, the fractions now reading $s^{-1}a$; when S is a multiplicative system the two constructions agree, by the uniqueness clause of Definition 1.8.4.

Proof *The category axioms.* That $\sim$ is an equivalence relation, that the composition is well defined and associative, and that Q is a functor: each item is a diagram chase with (S1)–(S4), recorded in [10, §1.9] and [3]. As a sample, associativity is checked by completing three composable spans to a single commutative diagram, via three applications of (S3), from which both bracketings can be read off.

Q inverts S. Composing $Q(s) = [X; \mathrm{id}_X, s]$ with $[X; s, \mathrm{id}_X]$ in either order yields $[X; \mathrm{id}_X, \mathrm{id}_X]$ and $[X; s, s]$: the former is the identity at X by definition, while the latter is equivalent to the identity at Y via the refinement with $u = \mathrm{id}_X$ and $v = s$.

The universal property. A functor $F : \mathcal{C} \to \mathcal{D}$ inverting S leaves us no choice: the extension must send the fraction $Q(a)Q(s)^{-1}$ to $F(a)F(s)^{-1}$, which settles uniqueness. Conversely, this prescription is well defined, since F maps equivalent spans to equal fractions.

The conditions in Definition 1.8.4 are therefore met. □

Remark 1.8.7 (*size issues*) The definition must allow the localization to be a large category. Indeed, although $\mathrm{Hom}(X,Y)$ is a set for all $X,Y \in \mathrm{Ob}(\mathcal{C})$, the spans $X \leftarrow\!\!\prec Z \to Y$ range over a proper class in general, and quotienting a proper class by an equivalence relation need not return a set: the morphisms $Q(X) \to Q(Y)$ of $\mathcal{C}[S^{-1}]$ form, a priori, merely a class. If $\mathcal{C}$ is small, then so is $\mathcal{C}[S^{-1}]$. More generally, it suffices that for every X the spans over X contain a small *cofinal* subfamily, in the sense that every span is refined by one from the subfamily: then every morphism $X \to Y$ has a representative with apex in the subfamily, and the morphisms $X \to Y$ form a set. In Chapter 5 this issue is disposed of for Serre quotients, by a direct construction.

Proposition 1.8.8 Let S be a left (resp. right) multiplicative system in $\mathcal{C}$, with localization functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$. Morphisms $f, g : X \to Y$ of $\mathcal{C}$ satisfy $Qf = Qg$ if and only if there exists $s \in S$ with $fs = gs$ (resp. $sf = sg$).

Proof The condition is sufficient: $fs = gs$ yields $Q(f)Q(s) = Q(fs) = Q(gs) = Q(g)Q(s)$, and $Q(s)$ is invertible. Conversely, suppose S is a left multiplicative system and $Qf = Qg$, i.e. the left fractions $(X; \mathrm{id}_X, f)$ and $(X; \mathrm{id}_X, g)$ are equivalent. Since both apexes are X itself, the common refinement amounts to a single morphism $s : W \to X$ in S with $fs = gs$. The right case is dual. □

The calculus of fractions is the form of localization that will build the derived category $\mathsf{D}(\mathcal{A})$ from the homotopy category $\mathsf{K}(\mathcal{A})$ by inverting the *quasi-isomorphisms* (Definition 3.1.3); the verifications that these form a multiplicative system will occupy a good part of Chapter 7.

Chapter 2

Additive and Abelian Categories

Abelian categories are the natural home for homological algebra: they are the additive categories with kernels and cokernels in which every morphism admits an *image* and *coimage*, related by an isomorphism. The prototype is the category R-Mod of left modules over a ring R; the abstract theory is designed so that all the familiar diagram-chasing arguments (exact sequences, the snake and five lemmas, kernels and cokernels) make sense and remain true without ever speaking of elements.

We met additive categories, zero objects, biproducts, kernels, and cokernels in Section 1.7. This chapter turns to abelian categories in earnest. After the definition of abelian categories (Section 2.1) and the language of complexes, exactness, and exact functors (Section 2.2), we develop the running example of modules (Section 2.3), establish the subobject calculus and the isomorphism theorems (Section 2.4), and prove the *diagram lemmas* (snake and five, Section 2.5). Three short supplementary sections then treat direct-sum decompositions (Section 2.6), projective and injective objects (Section 2.7), and the Grothendieck categories (Section 2.8).

A warning about method. In a general abelian category objects have no elements, so the elementary diagram-chasing one learns for modules is not literally available. We replace it by a single exactness criterion (Lemma 2.5.1), due in this form to Grothendieck, that plays the role of elementwise verification. Alternatively, the *Freyd–Mitchell embedding theorem* (Theorem A.1.1) embeds every small abelian category fully faithfully and exactly into some R-Mod, so that elementwise chasing is after all legitimate. We shall use both points of view as convenient.

2.1 From additive to abelian categories

We begin by recalling the relevant definitions from Section 1.7. A ***preadditive category*** is one whose Hom-sets are abelian groups and whose composition is bilinear; it is ***additive*** if it moreover has a zero object and finite ***biproducts*** $X_1 \oplus X_2$, characterized in terms of structure morphisms $\iota_i : X_i \to X_1 \oplus X_2$ and $p_i : X_1 \oplus X_2 \to X_i$. In a preadditive category the ***zero morphism*** $0 : X \to Y$ is the zero element of the Hom-group (Section 1.7). When kernels and cokernels exist, we write them $\ker f \hookrightarrow X$ and $Y \twoheadrightarrow \operatorname{coker} f$; in an additive category with kernels and cokernels they admit the familiar characterizations: f is mono iff $\ker f = 0$, and epi iff $\operatorname{coker} f = 0$. This follows by recalling that ker and coker are defined as equalizers and coequalizers with the zero morphism, respectively.

Definition 2.1.1 Let $\mathcal{A}$ be an additive category with kernels and cokernels. For a morphism $f : X \to Y$, the ***image*** and ***coimage*** of f are

$$\mathrm{im}(f) := \ker(Y \to \operatorname{coker} f), \quad \mathrm{coim}(f) := \operatorname{coker}(\ker f \to X). \tag{2.1.1}$$

There is a *canonical* morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ induced by f, characterized as the unique morphism making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads] & Y \\ \mathrm{coim}(f) \arrow[r] & \mathrm{im}(f) \arrow[u, hook] \end{tikzcd} \tag{2.1.2}$$

Unwinding the universal properties, $\mathrm{im}(f) \hookrightarrow Y$ is the smallest subobject of Y through which f factors, and $X \twoheadrightarrow \mathrm{coim}(f)$ is the largest quotient of X from which f descends; see Section 2.4 for the exact definitions. The canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ measures the gap between *quotient of the domain* and *subobject of the codomain*.

Definition 2.1.2 A morphism f is ***strict*** if the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

Definition 2.1.3 An ***abelian category*** is an additive category $\mathcal{A}$ such that
- every morphism has a kernel and a cokernel, and
- every morphism is strict.

Remark 2.1.4 One shows that an abelian category is equivalently an additive category with kernels and cokernels in which every monomorphism is a kernel and every epimorphism is a cokernel (the formulation of Section 1.7); the strictness condition packages both requirements at once.

Remark 2.1.5 Because finite products, coproducts, equalizers, and coequalizers all exist in an abelian category (equalizers are kernels, coequalizers are cokernels, and products coincide with coproducts as biproducts), an abelian category has all finite ***limits*** and ***colimits*** . It need not have infinite ones.

Convention 2.1.6 In an abelian category $\mathcal{A}$, the coproduct $\coprod_{i \in I} X_i$ of a family of objects, if it exists, is also called the ***direct sum*** , written alternatively as $\bigoplus_{i \in I} X_i$.

Example 2.1.7 For every ring R, the category R-Mod of left R-modules is abelian, as is the category Mod-R of right modules; taking $R = \mathbb{Z}$ gives Ab. The category of Banach spaces over $\mathbb{C}$ is additive but *not* abelian: not every morphism is strict.

Proposition 2.1.8 If $\mathcal{A}$ is abelian, so is $\mathcal{A}^{\mathrm{op}}$; and for any small category I, the functor category $\mathcal{A}^I$ is abelian (kernels, cokernels, images formed object-wise).

Proof The duality $\mathcal{A} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ exchanges kernels with cokernels and images with coimages, so strictness is self-dual. For $\mathcal{A}^I$, all the constructions are made object-wise: $(\ker \alpha)_i = \ker(\alpha_i)$, etc., and a morphism α is strict iff each component α_i is. □

In an abelian category every morphism $f : X \to Y$ admits a unique ***epi-mono factorization***

$$X \overset{f}{\twoheadrightarrow} \operatorname{im}(f) \hookrightarrow Y, \tag{2.1.3}$$

so that $\ker(f) = \ker(X \to \operatorname{im}(f))$ and $\operatorname{coker}(f) = \operatorname{coker}(\operatorname{im}(f) \to Y)$, as one readily verifies. Equivalently, since $\operatorname{coim}(f) \simeq \operatorname{im}(f)$, we may write $X \twoheadrightarrow \operatorname{coim}(f) \simeq \operatorname{im}(f) \hookrightarrow Y$.

A second basic fact, used constantly, concerns the interplay of pullbacks and pushouts with epimorphisms and monomorphisms. In an abelian category, by expressing equalizers as kernel of differences, the ***fibered product*** (pullback) of $X \xrightarrow{f} Z \xleftarrow{g} Y$ is seen to be $\ker\left(X \oplus Y \xrightarrow{(f,-g)} Z\right)$; dually the ***fibered coproduct*** (pushout) of $X \leftarrow Z \to Y$ is the cokernel of $Z \to X \oplus Y$.

Proposition 2.1.9 Consider a commutative square in an abelian category

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd} \tag{2.1.4}$$

- If it is a pullback and g is epi, then it is also a pushout and f is epi.
- Dually, if it is a pushout and f is mono, then it is also a pullback and g is mono.

Proof The case of R-Mod is a pleasant exercise (Section 2.3); the general case is [10, Proposition 2.1.6]. □

In particular, pulling back an epimorphism yields an epimorphism, and pushing out a monomorphism yields a monomorphism. This is the abstract reason that the snake lemma works.

2.2 Complexes and exactness

We now introduce the language in which the results of this chapter are stated: complexes and exactness. Everything takes place in an additive category (for exactness statements: an abelian one); the systematic theory of complexes is developed in Chapter 3.

Definition 2.2.1 A ***complex*** in an additive category $\mathcal{A}$ is a sequence of morphisms

$$\cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots \tag{2.2.1}$$

which can be bounded, unbounded in either direction, or cyclic, such that $d^n d^{n-1} = 0$ for all n. We often denote the whole complex by $X^\bullet = (X^n, d^n)_n$, or simply X. The term X^n is called the degree n object of $X^\bullet$, and we sometimes write $d^n_X = d^n$.

Suppose that $\mathcal{A}$ is abelian. The complex $X^\bullet$ is said to be exact at the term X^n if $\operatorname{im} d^{n-1} = \ker d^n$ as subobjects of X^n; an everywhere exact complex is said to be an ***exact sequence*** .

Definition 2.2.2 Let $X^\bullet$ and $Y^\bullet$ be complexes in an additive category $\mathcal{A}$. A morphism (or "chain map") $f : X^\bullet \to Y^\bullet$ is a sequence of morphisms $f^n : X^n \to Y^n$ in $\mathcal{A}$, such that $d^n_Y f^n = f^{n+1} d^n_X$ for all n.

For complexes that are unbounded in both directions, the above makes them into an additive category $\mathsf{C}(\mathcal{A})$: composition and the biproducts are defined degreewise. Bounded complexes are dealt with by padding with zero objects, as in Chapter 3, where the theory is developed systematically.

Every additive functor $F : \mathcal{A} \to \mathcal{B}$ induces an additive functor $X^\bullet \mapsto F(X^\bullet) = (F(X^n), F(d^n))_n$ between the categories of complexes.

Remark 2.2.3 The complexes in Definition 2.2.1 are also called **cochain complexes**. In contrast, the **chain complexes** use decreasing lower indices

$$X_\bullet = \left[\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots\right] \tag{2.2.2}$$

such that $d_{n-1}d_n = 0$. Morphisms between chain complexes are defined in a similar manner. All chain complexes in $\mathcal{A}$ form a category $\mathsf{Ch}(\mathcal{A})$. One can switch between cochain and chain complexes by simply setting $X_n = X^{-n}$ and $d_n = d^{-n}$.

Example 2.2.4 Given a morphism $f : X \to Y$ in an abelian category $\mathcal{A}$,

- the complex $0 \to X \xrightarrow{f} Y$ is exact if and only if $\ker f = 0$, if and only if f is mono;
- dually, the complex $X \xrightarrow{f} Y \to 0$ is exact if and only if $\operatorname{coker} f = 0$, if and only if f is epi.

Example 2.2.5 Consider a complex $0 \to X' \xrightarrow{f} X \xrightarrow{g} X''$, so that f factorizes as $X' \twoheadrightarrow \operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f \hookrightarrow \ker g$ (recall Definition 2.1.1). The exactness at X amounts to $\operatorname{im} f \xrightarrow{\sim} \ker g$, and the exactness at X' amounts to $X' \xrightarrow{\sim} \operatorname{coim} f$. Hence the complex is exact if and only if the induced $X' \to \ker g$ is an isomorphism, by Exercise 1.1.9.

Dually, consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$, so that g induces $\operatorname{coker} f \to X''$. Its exactness is equivalent to g inducing $\operatorname{coker} f \xrightarrow{\sim} X''$.

Summing up, exact sequences of the types above are nothing but kernels and cokernels, respectively.

Definition 2.2.6 A ***short exact sequence*** in $\mathcal{A}$ is a sequence of the form

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \tag{2.2.3}$$

with f mono, g epi, and $\operatorname{im} f = \ker g$. By Example 2.2.5, this is equivalent to f inducing $X' \xrightarrow{\sim} \ker g$ and g inducing $\operatorname{coker} f \xrightarrow{\sim} X''$.

Example 2.2.7 Let X' and X'' be objects in an abelian category $\mathcal{A}$. Then

$$0 \to X' \xrightarrow{\iota_1} X' \oplus X'' \xrightarrow{p_2} X'' \to 0 \tag{2.2.4}$$

is a short exact sequence, where ι_1 and p_2 are the canonical structure morphisms of the biproduct.

Recall from Proposition 1.5.6 that in any category limits can be built from products and equalizers, and colimits from coproducts and coequalizers, provided the relevant (co)products, (co)equalizers, etc. exist. For the next definition we only need finite (co)limits, i.e. those for which the indexing category I has only finitely many morphisms. In an additive category the building blocks coalesce (Lemma 1.7.4), which makes the following equivalences available.

Lemma 2.2.8 Additive functors between additive categories preserve finite products and coproducts.

Proof It suffices to consider binary products and coproducts, both coincide with biproducts by Lemma 1.7.4. As biproducts are defined by equations of morphisms under addition, they are preserved by any additive functor. □

Definition 2.2.9 An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is
- ***left exact*** if it preserves kernels (equivalently, finite limits);
- ***right exact*** if it preserves cokernels (equivalently, finite colimits);
- ***exact*** if it is both left and right exact.

Indeed, the equivalences follow from the fact that finite limits (resp. colimits) are built from finite products (resp. coproducts) and equalizers (resp. coequalizers), and the latter reduce to kernels (resp. cokernels).

An exact sequence $\cdots \to X^n \to X^{n+1} \to \cdots$ in an abelian category $\mathcal{A}$ is said to be preserved by an additive functor $F : \mathcal{A} \to \mathcal{B}$ if $\cdots \to F(X^n) \to F(X^{n+1}) \to \cdots$ is exact in $\mathcal{B}$.

Proposition 2.2.10 For an additive functor $F : \mathcal{A} \to \mathcal{B}$, the following are equivalent:
(i) F is exact,
(ii) F preserves all short exact sequences,
(iii) F preserves all exact sequences.

Proof (i) $\Leftrightarrow$ (ii). F is exact if and only if it preserves both kernels and cokernels, if and only if F preserves all short exact sequences by Definition 2.2.6.

(ii) $\Leftrightarrow$ (iii). We need only show $\Rightarrow$. Every exact sequence $\cdots \to X^n \xrightarrow{d^n} X^{n+1} \to \cdots$ (bounded or not) in $\mathcal{A}$ can be canonically split into short exact sequences

$$0 \to \ker d^n \to X^n \to \ker d^{n+1} \to 0 \tag{2.2.5}$$

for every n such that X^n is not the rightmost term of the complex. Applying F gives short exact sequences in $\mathcal{B}$; they splice together to give the desired exact sequence in $\mathcal{B}$

$$\cdots \to FX^n \xrightarrow{Fd^n} FX^{n+1} \to \cdots. \tag{2.2.6}$$

We refer to [10, Proposition 2.8.2] for the formal details. □

Proposition 2.2.11 For any object T, the functors $\mathrm{Hom}(T, \cdot)$ and $\mathrm{Hom}(\cdot, T)$ are left exact.

Proof It follows from Proposition 1.5.7. □

Proposition 2.2.12 Let (F, G, φ) be an adjunction where F and G are additive functors between abelian categories. Then
- F is right exact,
- G is left exact.

Proof Combine Definition 2.2.9 with Theorem 1.6.6. □

2.3 The running example: modules

Before further developing the abstract theory, let us record concretely how everything looks in R-Mod. This is the category we shall chase diagrams in whenever the abstract arguments become opaque, justified a posteriori by Freyd–Mitchell; the module-theoretic background is consolidated in Appendix B.

A morphism $f : M \to N$ of R-modules has

$$\begin{aligned} \ker f = f^{-1}(0) \subset M, \quad \operatorname{coker} f = N/f(M), \\ \operatorname{im} f = f(M) \subset N, \quad \operatorname{coim} f = M/\ker f. \end{aligned} \tag{2.3.1}$$

The canonical morphism $\operatorname{coim} f \to \operatorname{im} f$ is the familiar isomorphism $M/\ker f \simeq \operatorname{im} f$, so R-Mod is indeed abelian.

Example 2.3.1 (*the Hom–tensor adjunction, revisited*) We met in Example 1.6.4 the adjunction, for an (S, R)-bimodule M,

$$\operatorname{Hom}_S(M \otimes_R N, P) \simeq \operatorname{Hom}_R(N, \operatorname{Hom}_S(M, P)), \tag{2.3.2}$$

natural in $N \in R$-Mod and $P \in S$-Mod. Being a left adjoint, the functor $M \otimes_R (\cdot)$ is right exact: it preserves cokernels, hence takes right-exact sequences to right-exact sequences. Dually its right adjoint $\operatorname{Hom}_S(M, \cdot)$ is left exact: it preserves kernels. Neither need be exact, and measuring the failure is the very origin of the derived functors Tor and Ext of Chapter 4.

Concretely, for a short exact sequence $0 \to N' \to N \to N'' \to 0$, applying $M \otimes_R (\cdot)$ yields an exact sequence

$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0, \tag{2.3.3}$$

but the leftmost map need not be injective; this is the failure of exactness anticipated in Example 2.3.1, to be measured by Tor in Chapter 4.

Example 2.3.2 (*base change*) For a ring homomorphism $R \to S$, the ***extension of scalars*** $S \otimes_R \cdot : R$-Mod $\to S$-Mod is left adjoint to the ***restriction of scalars*** S-Mod $\to R$-Mod. Restriction is exact; extension is right exact. The right adjoint to restriction is ***coextension of scalars*** $\operatorname{Hom}_R(S, \cdot)$, which is left exact. These adjunctions are expanded in Remark B.4.4.

2.4 Subobjects, quotients, and the isomorphism theorems

In an abelian category $\mathcal{A}$ we can do with subobjects and quotients everything we do in module theory. A ***subobject*** of X is a monomorphism $A \hookrightarrow X$, taken up to isomorphism over X; the subobjects of X form a poset $\operatorname{Sub}(X)$ ordered as follows: we say $(A \hookrightarrow X) \preccurlyeq (A' \hookrightarrow X)$ if there exists a monomorphism $f : A \hookrightarrow A'$ such that $A \hookrightarrow A' \hookrightarrow X$ composes to the given $A \hookrightarrow X$, in which case f is always unique.

Dually, a ***quotient*** of X is an epimorphism $X \twoheadrightarrow Q$, equivalently a subobject of X in $\mathcal{A}^{\mathrm{op}}$.

Definition 2.4.1 Given a subobject $A \hookrightarrow X$, write $X/A := \operatorname{coker}(A \to X)$; all quotients of X arise in this way.

Definition 2.4.2 Given a family of subobjects $A_i \hookrightarrow X$ with i ranging over a set I, suppose that $\mathcal{A}$ admits products (resp. coproducts) indexed by I. The ***intersection*** $\bigcap_i A_i \hookrightarrow X$ (resp. the ***sum*** $\sum_i A_i \hookrightarrow X$), as a subobject of X, is the kernel of the diagonal morphism $X \to \prod_i X/A_i$ (resp. image of the sum morphism $\coprod_i A_i \to X$). When I is finite, the assumptions hold and we write $A_1 \cap A_2$, $A_1 + A_2$, and so on.

Proposition 2.4.3 For every object X, the poset $\mathrm{Sub}(X)$ is a bounded ***modular lattice*** : meet is intersection, join is sum, with bottom 0 and top X, and the modular law $A \subset B \Rightarrow A + (C \cap B) = (A + C) \cap B$ holds.

Proof All these are standard for R-Mod. For the general case, see [10, Theorem 2.6.10]. □

Definition 2.4.4 An object X is said to be ***Noetherian*** if $\mathrm{Sub}(X)$ contains no infinite strictly ascending chain $A_1 \subsetneq A_2 \subsetneq \cdots$ of subobjects, and ***Artinian*** if it contains no infinite strictly descending chain. An object that is both Noetherian and Artinian is said to be of ***finite length*** .

For $\mathcal{A} = R$-Mod these are exactly the usual Noetherian and Artinian R-modules. Finite-length objects are the setting for Proposition 2.6.3 and Theorem 2.6.4 below.

The three ***isomorphism theorems*** of module theory carry over verbatim:

Proposition 2.4.5 (*isomorphism theorems*) Let $\mathcal{A}$ be abelian.

- For $f : X \to Y$, the induced map $X/\ker f \simeq \mathrm{im}\, f$ is an isomorphism.
- If $Z \hookrightarrow X$, the quotient $X \twoheadrightarrow X/Z$ sets up a bijection between subobjects of X containing Z and subobjects of X/Z, and for $Z \subset Y \subset X$ one has $(X/Z)/(Y/Z) \simeq X/Y$.
- For subobjects $Y, Z \hookrightarrow X$, the natural map $Y/(Y \cap Z) \to (Y+Z)/Z$ is an isomorphism.

Proof Again, the case of R-Mod is elementary. In general, the ideas are identical to the module-theoretic ones; alternatively they reduce, via the lattice structure of Proposition 2.4.3, to formal lattice theory. See [10, §2.6] for the details. □

Though the complete proof is delegated to [10], we shall use these isomorphism theorems without further comment.

Every morphism $f : X \to Y$ fits into two canonical short exact sequences

$$\begin{gathered} 0 \to \ker f \to X \to \mathrm{im}\, f \to 0, \\ 0 \to \mathrm{im}\, f \to Y \to \mathrm{coker}\, f \to 0, \end{gathered} \tag{2.4.1}$$

spliced into $0 \to \ker f \to X \xrightarrow{f} Y \to \mathrm{coker}\, f \to 0$. In particular, f is an isomorphism if and only if $\ker f = 0 = \mathrm{coker}\, f$, if and only if f is both mono and epi; another equivalent statement is that $0 \to X \xrightarrow{f} Y \to 0$ is exact.

As an application, we get a useful criterion for isomorphism: if f embeds into an exact sequence

$$L \to X \xrightarrow{f} Y \to R \tag{2.4.2}$$

then f is an isomorphism (exactness forces the two outer maps to be zero).

Definition 2.4.6 An object X is ***simple*** if $X \neq 0$ and its only subobjects are 0 and X. It is ***semisimple*** if it is a (possibly infinite) coproduct of simple objects. The category $\mathcal{A}$ is ***semisimple*** if every object is.

For example, the category of vector spaces over a field is semisimple.

Proposition 2.4.7 (*Schur's lemma*) If X is a simple object, then $\mathrm{End}(X)$ is a division ring.

Proof Same as the case of modules: every nonzero $f \in \mathrm{End}(X)$ must satisfy $\ker(f) = 0 = \mathrm{coker}(f)$, hence is an isomorphism. □

We are now ready to introduce the *cohomologies*.

Definition 2.4.8 Consider a complex $X^\bullet$ in an abelian category $\mathcal{A}$. The ***cohomology*** at X^n (where both d^{n-1} and d^n are present) is the quotient object

$$\begin{aligned} \mathrm{H}^n(X^\bullet) := \ker(d^n)/\operatorname{im}(d^{n-1}) &\simeq \operatorname{coker}[\operatorname{im}(d^{n-1}) \to \ker(d^n)] \\ &\simeq \ker[\operatorname{coker}(d^{n-1}) \to \operatorname{im}(d^n)]. \end{aligned} \tag{2.4.3}$$

For chain complexes $X_\bullet$ as in Remark 2.2.3, the corresponding notion is called the ***homology*** , denoted as $\mathrm{H}_n(X_\bullet)$.

Proposition 2.4.9 Every morphism between complexes $f : X^\bullet \to Y^\bullet$ induces $\mathrm{H}^n(f) : \mathrm{H}^n(X^\bullet) \to \mathrm{H}^n(Y^\bullet)$, for every n.

Proof The chain map f carries $\ker(d_X^n)$ into $\ker(d_Y^n)$ and $\operatorname{im}(d_X^{n-1})$ into $\operatorname{im}(d_Y^{n-1})$; passing to the quotients yields $\mathrm{H}^n(f)$. □

2.5 The diagram lemmas

We now prove the results that make homological algebra run. Throughout, $\mathcal{A}$ is a fixed abelian category. The key tool, our substitute for elementwise diagram-chasing, is the following criterion.

Lemma 2.5.1 A complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact at X iff for every morphism $h : S \to X$ with $gh = 0$, there is an epimorphism $S' \twoheadrightarrow S$ and a morphism $S' \to X'$ making the square

$$\begin{tikzcd} S' \arrow[r, two heads] \arrow[d] & S \arrow[d, "h"] \\ X' \arrow[r, "f"'] & X \end{tikzcd} \tag{2.5.1}$$

commute.

The idea is that the pullback $S' = S \times_X X'$ provides the cover: exactness at X means $X' \to \operatorname{im} f = \ker g$ is epi, which by Proposition 2.1.9 is equivalent to $S' \to S$ being epi. The converse is obtained by testing $S = \ker g$.

Theorem 2.5.2 (*snake lemma*) Consider a commutative diagram in $\mathcal{A}$ with exact rows

$$\begin{tikzcd} & \ker' \arrow[r] \arrow[d] & \ker \arrow[r] \arrow[d] & \ker'' \arrow[d] & \\ & X' \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Y'' \arrow[d] & \\ & \operatorname{coker}' \arrow[r] & \operatorname{coker} \arrow[r] & \operatorname{coker}'' & \end{tikzcd} \tag{2.5.2}$$

where the rows $X' \to X \to X'' \to 0$ and $0 \to Y' \to Y \to Y''$ are exact, the columns are the kernels and cokernels of the vertical maps, and $\ker' := \ker(X' \to Y')$, etc. Then there is a ***connecting morphism*** $\delta : \ker'' \to \operatorname{coker}'$, functorial in the diagram, such that the six-term sequence

$$\ker' \to \ker \to \ker'' \xrightarrow{\delta} \operatorname{coker}' \to \operatorname{coker} \to \operatorname{coker}'' \tag{2.5.3}$$

is exact. Moreover, if $f : X' \to X$ is mono then $\ker' \to \ker$ is mono, and if $v : Y \to Y''$ is epi then $\operatorname{coker} \to \operatorname{coker}''$ is epi.

Proof (*Sketch*) The whole difficulty is the construction of δ; once it exists, exactness is a (lengthy) application of Lemma 2.5.1. We describe δ in the language of R-modules, which by Freyd–Mitchell is harmless for any statement about exactness. Write $a : X' \to Y'$, $b : X \to Y$, $c : X'' \to Y''$ for the vertical maps.

Given $x'' \in \ker'' \subset X''$, lift it along the epi $g : X \to X''$ to some $x \in X$ (exactness of $X \to X'' \to 0$). Its image $y := b(x) \in Y$ lands in $\ker v$, since by commutativity $v(y) = vb(x) = cg(x) = c(x'') = 0$, as $x'' \in \ker c$. Exactness of $0 \to Y' \to Y$ identifies $\ker v \simeq Y'$, so y comes from a unique $y' \in Y'$. Define $\delta(x'') := \overline{y'} \in \operatorname{coker}'$. One checks:

- $\delta(x'')$ is independent of the lift x: any other lift differs by an element of $\operatorname{im} f$, whose image in Y' is killed in coker';
- δ is a homomorphism and natural in the diagram;
- $\ker \to \ker'' \xrightarrow{\delta} \operatorname{coker}'$ is a complex, and exactness at $\ker''$ and coker' follows from tracing the construction.

The remaining exactness ($\ker' \to \ker \to \ker''$ and $\operatorname{coker}' \to \operatorname{coker} \to \operatorname{coker}''$) is a direct diagram chase. See [10, Theorem 2.3.3] for the fully categorical (element-free) proof via Lemma 2.5.1. □

The connecting morphism δ is the heart of the snake lemma; it is what lets a short exact sequence of complexes produce a *long* exact sequence in cohomology (Proposition 3.3.1). In R-Mod the formula above is literally the definition of δ.

Proposition 2.5.3 (*five lemma*) Consider a commutative diagram with exact rows

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & X_3 \arrow[r] \arrow[d, "f_3"'] & X_4 \arrow[r] \arrow[d, "f_4"'] & X_5 \arrow[d, "f_5"'] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5
\end{tikzcd} \tag{2.5.4}$$

(i) if f_1 is epi and f_2, f_4 are mono, then f_3 is mono;
(ii) if f_5 is mono and f_2, f_4 are epi, then f_3 is epi;
(iii) in particular, if f_1 is epi, f_5 is mono, and f_2, f_4 are isomorphisms, then f_3 is an isomorphism.

Proof (*Sketch*) (i) and (ii) are dual; (iii) combines them. For (i), chase $s \in X_3$ with $f_3(s) = 0$: its image in X_4 is zero (since f_4 mono and the image in Y_4 vanishes), so by exactness s comes from X_2, and then from Y_1 using that f_1 is epi; since f_2 is mono, $s = 0$. □

2.6 Direct sums, splitting, and idempotents

We work within an abelian category $\mathcal{A}$. Here we record when a short exact sequence comes apart into a biproduct, and what this says about decompositions of objects.

Definition 2.6.1 A short exact sequence $0 \to X' \to X \to X'' \to 0$ ***splits*** (or is splittable) if it is isomorphic to the canonical sequence in Example 2.2.7.

Proposition 2.6.2 For a short exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$, the following are equivalent.

(i) It splits.
(ii) There is a ***section*** $s : X'' \to X$, i.e. $gs = \mathrm{id}$.
(iii) There is a ***retraction*** $r : X \to X'$, i.e. $rf = \mathrm{id}$.

Proof The proof is identical to the module-theoretic one; the biproduct identities do everything. See [10, Proposition 2.5.3]. □

Furthermore, an idempotent $e \in \mathrm{End}(X)$ ($e^2 = e$) corresponds to a direct-sum decomposition $X = \ker(e) \oplus \mathrm{im}(e) = \mathrm{im}(1-e) \oplus \mathrm{im}(e)$, so direct-sum decompositions and splittings are two faces of the same coin. See [10, Proposition 2.5.2].

Proposition 2.6.3 (*Fitting's lemma*) If $X \neq 0$ has finite length and $f \in \mathrm{End}(X)$, then for n large enough $X = \ker(f^n) \oplus \mathrm{im}(f^n)$, with f nilpotent on the first summand and invertible on the second. In particular, if X is ***indecomposable*** ($X \neq 0$ and $X = A \oplus B \Rightarrow A = 0 \;$ or $\; B = 0$), then $\mathrm{End}(X)$ is a local ring.

Theorem 2.6.4 (*Krull–Remak–Schmidt*) If $X \neq 0$ has finite length, then X decomposes as a finite direct sum $X = X_1 \oplus \cdots \oplus X_n$ of indecomposable objects, and the summands are unique up to permutation and isomorphism.

We omit the proofs of Proposition 2.6.3 and Theorem 2.6.4; both are standard for $\mathcal{A} = R$-Mod, but the general case requires some formal effort. Details may be found in [10, §2.5].

2.7 Projective and injective objects

Fix an abelian category $\mathcal{A}$. The following will be essential when we compute derived functors in Chapter 4.

Definition 2.7.1 An object $P \in \mathcal{A}$ is ***projective*** if $\mathrm{Hom}(P, \cdot)$ is exact, i.e.
if for every epimorphism $X \twoheadrightarrow Y$ the map $\mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective. Dually, $I \in \mathcal{A}$ is ***injective*** if $\mathrm{Hom}(\cdot, I)$ is exact. We say $\mathcal{A}$ has ***enough projectives*** if every object is a quotient of a projective, and ***enough injectives*** if every object embeds into an injective.

Equivalently, P is projective iff every diagram (solid part) with an exact row

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ X \arrow[r, two heads] & Y \arrow[r] & 0 \end{tikzcd} \tag{2.7.1}$$

admits a lift (dashed) $P \to X$. Dually I is injective iff every diagram (solid part) with an exact row

$$\begin{tikzcd} & I & \\ Y \arrow[ur, dashed] & X \arrow[l, hook'] \arrow[u] & 0 \arrow[l] \end{tikzcd} \tag{2.7.2}$$

admits an extension (dashed) $Y \to I$.

Proposition 2.7.2 In a short exact sequence $0 \to X' \xrightarrow{\iota} X \xrightarrow{p} X'' \to 0$, if X'' is projective the sequence splits. The dual statement holds for X' injective.

Proof It suffices to consider the projective case. The map $\mathrm{Hom}(X'', X) \to \mathrm{Hom}(X'', X'')$ is surjective by projectivity; let $s : X'' \to X$ be any preimage of $\mathrm{id}_{X''}$, and apply Proposition 2.6.2. □

Proposition 2.7.3 The product (resp. coproduct) of a family of objects, if it exists, is injective (resp. projective) if and only if every object of this family is injective (resp. projective).

Proof By duality, it suffices to consider the product $\prod_\alpha X_\alpha$ of a family of objects X_α. We have the following canonical isomorphism of functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$

$$\mathrm{Hom}_{\mathcal{A}}\left(\cdot, \prod_\alpha X_\alpha\right) \simeq \prod_\alpha \mathrm{Hom}_{\mathcal{A}}(\cdot, X_\alpha). \tag{2.7.3}$$

The assertion now follows from the elementary fact that the direct product of a family of complexes $C^\bullet_\alpha$ in Ab, formed degreewise, is exact if and only if each $C^\bullet_\alpha$ is. □

Example 2.7.4 When specialized to R-Mod, we obtain the notion of *projective and injective modules*.

- P is projective iff P is a direct summand of a free module $R^{\oplus I}$ (Proposition B.5.1). Hence free modules are projective, and R-Mod always has enough projectives as every module is a quotient of a free one (Proposition B.3.2).
- I is injective iff every R-linear map from a left ideal $\mathfrak{a} \subset R$ to I extends to R (*Baer's criterion*, Proposition B.5.2). For $R = \mathbb{Z}$, this says injective $\mathbb{Z}$-modules are exactly the ***divisible*** abelian groups ($nI = I$ for all $n \neq 0$), e.g. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.
- Finally, R-Mod has enough injectives; the proof is considerably more delicate than in the projective case, see [9, Theorem 6.9.14].

Proofs and references are collected in Section B.5, together with the companion notion of a *flat* module (Definition B.5.3).

A functor between abelian categories is said to *preserve injectives* (*resp. projectives*) if it sends injective (resp. projective) objects to injective (resp. projective) objects.

Theorem 2.7.5 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Consider a pair of adjoint functors

$$F : \mathcal{A} \rightleftarrows \mathcal{B} : G \tag{2.7.4}$$

- If F is exact, then G preserves injectives.
- If G is exact, then F preserves projectives.

Proof By duality, it suffices to prove the first assertion. Let $J \in \mathrm{Ob}(\mathcal{B})$ be injective; then

$$\mathrm{Hom}_{\mathcal{A}}(\cdot, G(J)) \simeq \mathrm{Hom}_{\mathcal{B}}(F(\cdot), J). \tag{2.7.5}$$

Since F is exact and J is injective, the functor on the right-hand side is exact. Hence $G(J) \in \mathrm{Ob}(\mathcal{A})$ is injective. □

Exercise 2.7.6 Show that $\mathbb{Z}/2\mathbb{Z}$ is not projective over $\mathbb{Z}$.

Exercise 2.7.7 Prove that a nontrivial finite abelian group is never injective as a $\mathbb{Z}$-module, while $\mathbb{Q}/\mathbb{Z}$ is injective and every abelian group embeds into a product of copies of $\mathbb{Q}/\mathbb{Z}$.

2.8 Grothendieck categories

We conclude with a class of abelian categories large enough for most applications, in which the module-like arguments of this chapter remain valid.

Definition 2.8.1 Let $\mathcal{A}$ be a preadditive category with small coproducts (see Section 1.5). An object s is a ***generator*** of $\mathcal{A}$ if every object X admits an epimorphism from a small coproduct of copies of s. If $\mathcal{A}$ has small products, the notion of cogenerators is defined in the dual way.

For example, R is a generator of R-Mod.

Definition 2.8.2 A ***Grothendieck category*** is an abelian category $\mathcal{A}$ such that
- $\mathcal{A}$ has a generator,
- $\mathcal{A}$ is cocomplete,
- small filtered colimits in $\mathcal{A}$ are exact, namely for every small filtered category I and every short exact sequence of functors $I \to \mathcal{A}$

  $$0 \to \alpha \to \beta \to \gamma \to 0, \tag{2.8.1}$$

  i.e. $0 \to \alpha(i) \to \beta(i) \to \gamma(i) \to 0$ is exact for all $i \in \mathrm{Ob}(I)$, the colimit

  $$0 \to \operatorname{colim} \alpha \to \operatorname{colim} \beta \to \operatorname{colim} \gamma \to 0 \tag{2.8.2}$$

  is again exact.

Example 2.8.3 For every ring R, the category R-Mod is a Grothendieck category. The only non-trivial condition is the exactness of filtered colimits, established in Section B.2.

Grothendieck categories behave very much like module categories: they are complete (have all limits) and, by a theorem of Grothendieck, have enough injectives. The categories R-Mod, sheaves of abelian groups on a topological space, and quasi-coherent sheaves on a scheme are all Grothendieck categories. The Gabriel–Popescu theorem [10, §A.3] expresses any Grothendieck category as a localization of R-Mod for a suitable ring R, explaining the analogy with modules. We do not pursue this further; see [10, §2.10].

Exercise 2.8.4 For any category $\mathcal{C}$, call an object s a generator of $\mathcal{C}$ if, whenever morphisms $f, g : x \to y$ satisfy $f\varepsilon = g\varepsilon$ for all morphisms $\varepsilon : s \to x$, one has $f = g$. Show that for cocomplete abelian categories this is equivalent to Definition 2.8.1.

Remark 2.8.5 More generally, a set Σ of objects of $\mathcal{C}$ is said to be generating if, whenever morphisms $f, g : x \to y$ satisfy $f\varepsilon = g\varepsilon$ for all morphisms $\varepsilon : s \to x$ with $s \in \Sigma$, one has $f = g$.

Dually, cogenerating sets for $\mathcal{C}$ are defined as generating sets of $\mathcal{C}^{\mathrm{op}}$, and we say an object s is a cogenerator if $\{s\}$ is a cogenerating set.

Chapter 3

Complexes, Homotopy, and the Mapping Cone

A *complex* is the homological-algebraic substitute for a space with a differential: it is a sequence of objects and morphisms whose composites vanish. This chapter develops the elementary theory of complexes over an additive category $\mathcal{A}$: the category $\mathsf{C}(\mathcal{A})$ of complexes, the shift functor, cohomology, chain homotopy, and the homotopy category $\mathsf{K}(\mathcal{A})$, together with three constructions that recur throughout the rest of these notes: *mapping cones*, *total complexes* of double complexes, and *truncations*. Unless otherwise specified, we work cohomologically: differentials raise degree, $d^n : X^n \to X^{n+1}$.

3.1 Complexes and cohomology

The notions of complexes over an additive category $\mathcal{A}$ and the chain maps between them are already defined in Definition 2.2.1 and Definition 2.2.2. In this section, we only consider complexes that are unbounded in both directions; a bounded complex can be made into an unbounded one by adding zeros. Such complexes over $\mathcal{A}$ form a category $\mathsf{C}(\mathcal{A})$. We denote a complex by $(X^n, d^n)_n$, $X^\bullet$ or more succinctly X. A complex concentrated in degree 0 is just an object of $\mathcal{A}$, so $\mathcal{A}$ embeds into $\mathsf{C}(\mathcal{A})$.

Proposition 3.1.1 For any additive category $\mathcal{A}$, the category $\mathsf{C}(\mathcal{A})$ is additive (Definition 2.2.2); if $\mathcal{A}$ is abelian, so is $\mathsf{C}(\mathcal{A})$, with biproducts, kernels, cokernels, images, etc. computed degreewise.

Proof Routine. □

The degreewise nature of $\mathsf{C}(\mathcal{A})$ is the key simplification: a sequence of complexes is exact (or a chain map is mono/epi/an isomorphism) iff it is so in each degree. When $\mathcal{A}$ is abelian we may take cohomology.

Definition 3.1.2 The ***shift*** (or *translation*) $X \mapsto X[n]$ of a complex X is the complex $X[1]$ with $(X[1])^n = X^{n+1}$ and differential $d^n_{X[1]} = -d^{n+1}_X$; more generally $(X[m])^n = X^{n+m}$ with $d^n_{X[m]} = (-1)^m d^{n+m}_X$. This gives an automorphism $[m]$ of $\mathsf{C}(\mathcal{A})$.

Similarly, for chain complexes (Remark 2.2.3) we define $X[1]$ by $(X[1])_n = X_{n+1}$, with the same twist on differentials.

The sign $(-1)^m$ in the shifted differential is a convention. We shall not dwell on signs, but the reader should be aware that they are ubiquitous and that there is always a Koszul-sign rationale behind them.

In the remainder of this section, we focus on complexes over an abelian category $\mathcal{A}$. For $X \in \mathsf{C}(\mathcal{A})$, the cohomology $\mathrm{H}^n(X)$ of X in degree n is defined in Definition 2.4.8. Cohomology is a functor $\mathrm{H}^n : \mathsf{C}(\mathcal{A}) \to \mathcal{A}$: a chain map $f : X \to Y$ induces $\mathrm{H}^n(f) : \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ by passing f^n to the quotient.

Definition 3.1.3 (*quasi-isomorphism*) A chain map $f : X \to Y$ is a ***quasi-isomorphism*** if $\mathrm{H}^n(f) : \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ is an isomorphism for every n.

The notion of quasi-isomorphism is central to homological algebra: it says that two complexes have "the same cohomology," even though they may look completely different degreewise. In Chapter 7 we shall *invert* all quasi-isomorphisms, passing from $\mathsf{C}(\mathcal{A})$ to the *derived category* $\mathsf{D}(\mathcal{A})$.

3.2 Chain maps, homotopy, and the homotopy category

There is a class of chain maps that induce the zero map on cohomology for trivial reasons, namely the null-homotopic maps; quotienting them out yields the homotopy category.

Definition 3.2.1 Two chain maps $f, g : X \to Y$ are ***chain homotopic*** , written $f \sim g$, if there is a family $h^n : X^n \to Y^{n-1}$ with

$$g^n - f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n \quad \text{for all } n. \tag{3.2.1}$$

A map homotopic to 0 is called ***null-homotopic*** .

Remark 3.2.2 For chain complexes $X_\bullet$ in the sense of Remark 2.2.3, a chain homotopy between chain maps $f, g : X_\bullet \to Y_\bullet$ is a family $h_n : X_n \to Y_{n+1}$ with

$$g_n - f_n = d_{n+1}^Y h_n + h_{n-1}d_n^X \quad \text{for all } n, \tag{3.2.2}$$

where d^X and d^Y denote the differentials. Under the convention $X_n = X^{-n}$ this is exactly Definition 3.2.1.

Geometrically, h is a (co)chain homotopy between f and g, analogous to a homotopy between continuous maps. The null-homotopic maps form a subgroup of each Hom-group, and a two-sided ideal: composing a null-homotopic map with any chain map (on either side) gives a null-homotopic map.

Definition 3.2.3 The ***homotopy category*** $\mathsf{K}(\mathcal{A})$ has the same objects as $\mathsf{C}(\mathcal{A})$ and morphisms

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y) := \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X, Y)/\{\text{null-homotopic maps}\}. \tag{3.2.3}$$

The quotient construction defines a functor $\mathsf{C}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$, universal among functors that kill null-homotopic maps. Now assume $\mathcal{A}$ is an abelian category. Because null-homotopic maps induce the zero map on cohomology (Proposition 3.4.2), cohomology factors through $\mathsf{K}(\mathcal{A})$:

$$\mathrm{H}^n : \mathsf{K}(\mathcal{A}) \to \mathcal{A}. \tag{3.2.4}$$

In particular a ***chain homotopy equivalence*** (an isomorphism in $\mathsf{K}(\mathcal{A})$) is a quasi-isomorphism. The converse is false in general; the K-injective and K-projective complexes of Section 4.6 are introduced precisely to control this discrepancy.

3.3 The long exact sequence in cohomology

Here is the single most useful consequence of the snake lemma. It is the reason exact sequences in homological algebra are almost always *long*.

Proposition 3.3.1 Let $\mathcal{A}$ be abelian. Every short exact sequence of complexes

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \tag{3.3.1}$$

in $\mathsf{C}(\mathcal{A})$ (i.e. exact degreewise) induces a ***long exact sequence***

$$\cdots \to \mathrm{H}^{n-1}(X) \to \mathrm{H}^{n-1}(Y) \to \mathrm{H}^{n-1}(Z) \xrightarrow{\delta^{n-1}} \mathrm{H}^{n}(X) \to \mathrm{H}^{n}(Y) \to \mathrm{H}^{n}(Z) \xrightarrow{\delta^{n}} \mathrm{H}^{n+1}(X) \tag{3.3.2}$$

natural in the short exact sequence. The maps $\delta^n : \mathrm{H}^n(Z) \to \mathrm{H}^{n+1}(X)$ are the ***connecting morphisms*** .

Proof The construction is a clean application of the snake lemma (Theorem 2.5.2). The short exact sequence of complexes is, degreewise, a short exact sequence $0 \to X^k \to Y^k \to Z^k \to 0$ for every k. The exact top row of the diagram below is the coker part of the snake lemma applied to the differentials $d^{n-2} : X^{n-2} \to X^{n-1}$ (and the Y, Z analogues), and the exact bottom row is dually the ker part applied to $d^n : X^n \to X^{n+1}$; the vertical maps are induced by d^{n-1}, where we omit the subscripts:

$$\begin{tikzcd} & \operatorname{coker}(d_X^{n-2}) \arrow[r] \arrow[d, "d^{n-1}"'] & \operatorname{coker}(d_Y^{n-2}) \arrow[r] \arrow[d, "d^{n-1}"'] & \operatorname{coker}(d_Z^{n-2}) \arrow[r] \arrow[d, "d^{n-1}"'] & 0 \\ 0 \arrow[r] & \ker(d_X^{n}) \arrow[r] & \ker(d_Y^{n}) \arrow[r] & \ker(d_Z^{n}) & \end{tikzcd} \tag{3.3.3}$$

The key observation is that each vertical map $\operatorname{coker}(d^{n-2}) \to \ker(d^n)$, induced by d^{n-1}, has *kernel* $\ker(d^{n-1})/\operatorname{im}(d^{n-2}) = \mathrm{H}^{n-1}$ and *cokernel* $\ker(d^n)/\operatorname{im}(d^{n-1}) = \mathrm{H}^n$. Applying the snake lemma to the diagram therefore yields, for every n, the six-term exact sequence

$$\mathrm{H}^{n-1}(X) \to \mathrm{H}^{n-1}(Y) \to \mathrm{H}^{n-1}(Z) \xrightarrow{\delta^{n-1}} \mathrm{H}^{n}(X) \to \mathrm{H}^{n}(Y) \to \mathrm{H}^{n}(Z), \tag{3.3.4}$$

and splicing these over all $n \in \mathbb{Z}$ produces the long exact sequence. Naturality follows from the functoriality of the snake lemma's connecting morphism. □

The connecting morphism δ is what makes cohomology useful. We shall meet it again, in a different guise, as the third arrow of a *distinguished triangle* in Chapter 7.

Proposition 3.3.2 An exact functor $F : \mathcal{A} \to \mathcal{B}$ induces an exact functor $\mathsf{C}(F) : \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{B})$, namely by applying F degreewise; it commutes with cohomology:

$$\mathrm{H}^n(\mathsf{C}(F)(X)) \simeq F(\mathrm{H}^n(X)). \tag{3.3.5}$$

In particular $\mathsf{C}(F)$ preserves quasi-isomorphisms and acyclicity; here a complex X is ***acyclic*** if $\mathrm{H}^n(X) = 0$ for all n.

Proof The proof is immediate from the degreewise description: exact F commutes with ker and im, hence with H^n. □

This result, innocuous as it looks, will be our main computational tool once we meet derived functors: it says that to compute H^n after applying an exact functor, we may exchange the order.

3.4 The Hom complex

The morphisms between two complexes carry a natural complex structure of their own, the *Hom complex*. It packages all degree-shifting maps $X \to Y[m]$ at once and makes precise the meaning of homotopy.

Definition 3.4.1 (*Hom complex*) For complexes $X, Y \in \mathsf{C}(\mathcal{A})$ over an additive category $\mathcal{A}$, the ***Hom complex*** $\mathrm{Hom}^\bullet(X,Y)$ is the complex of abelian groups with

$$\mathrm{Hom}^n(X,Y) = \prod_k \mathrm{Hom}_{\mathcal{A}}(X^k, Y^{k+n}) \tag{3.4.1}$$

in degree n, and differential

$$d^n(f) = d_Y f - (-1)^n f d_X. \tag{3.4.2}$$

Indeed, the sign $(-1)^n$ ensures $d^{n+1}d^n = 0$: expanding $d^{n+1}(d^n f)$ for f of degree n, the terms involving d_Y^2 or d_X^2 vanish, and the two cross terms cancel. It is the same Koszul sign rule that governs the shift. The zero-cycles of $\mathrm{Hom}^\bullet(X,Y)$ are exactly the chain maps $X \to Y$: the equation $d^0(f) = 0$ reads $d_Y f = f d_X$. The zero-boundaries are exactly the null-homotopic maps: a degree-(-1) element is a family $h^k : X^k \to Y^{k-1}$, and the boundary $d^{-1}(h) = d_Y h + h d_X$ is degreewise

$$f^n = d_Y^{n-1} h^n + h^{n+1} d_X^n, \tag{3.4.3}$$

which is the relation $f \sim 0$ of Definition 3.2.1. Consequently:

Proposition 3.4.2 (*homotopy classes via the Hom complex*) For all X, Y,

$$\mathrm{H}^0(\mathrm{Hom}^\bullet(X,Y)) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y), \tag{3.4.4}$$

and more generally $\mathrm{H}^n(\mathrm{Hom}^\bullet(X,Y)) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y[n])$. When $\mathcal{A}$ is abelian, null-homotopic maps induce the zero map on cohomology.

Proof For $n = 0$ this is the computation above. For general n, a cycle of degree n is a family $f^k : X^k \to Y^{k+n}$ satisfying $d_Y f = (-1)^n f d_X$, i.e. a chain map $X \to Y[n]$, since the differential of $Y[n]$ is $(-1)^n d_Y$ (Definition 3.1.2); its boundaries are the homotopies, exactly as in degree 0. Finally, if $f = d^{-1}h$ is null-homotopic and $x \in \ker d_X^n$, then $d_X^n(x) = 0$ leaves $f^n(x) = d_Y^{n-1}h^n(x)$, so f sends cycles to boundaries and induces the zero map $\mathrm{H}^n(X) \to \mathrm{H}^n(Y)$. □

If f and g are maps of degrees m and n, respectively, their composition satisfies the ***Leibniz rule***

$$d^{n+m}(g \cdot f) = (d^n g) \cdot f + (-1)^n g \cdot (d^m f), \tag{3.4.5}$$

again by expansion: the terms involving d^2 vanish and the cross terms $\pm(-1)^n g d_Y f$ cancel. This makes the assignment $(X,Y) \mapsto \mathrm{Hom}^\bullet(X,Y)$ a bifunctor, compatible with composition, valued in complexes of abelian groups. This enrichment of $\mathsf{C}(\mathcal{A})$ over complexes is the beginning of the story that culminates, in Chapter 7, in the triangulated structure on $\mathsf{K}(\mathcal{A})$.

Remark 3.4.3 For chain complexes $X_\bullet, Y_\bullet$ (Remark 2.2.3), the same construction gives a complex of abelian groups with

$$\mathrm{Hom}^n(X_\bullet, Y_\bullet) = \prod_k \mathrm{Hom}_{\mathcal{A}}(X_k, Y_{k-n}), \tag{3.4.6}$$

with differential given by the same formula $d^n(f) = d_Y f - (-1)^n f d_X$; under the convention $X_n = X^{-n}$ this is exactly Definition 3.4.1. In particular H^0 again computes chain maps modulo homotopy.

3.5 Intermezzo: Hochschild complexes

Throughout this section, $\Bbbk$ is a commutative ring and R a $\Bbbk$-algebra; abbreviate $\otimes := \otimes_\Bbbk$ and $R^{\otimes 0} := \Bbbk$. The ***enveloping algebra*** of R is

$$R^e := R \otimes R^{\mathrm{op}}. \tag{3.5.1}$$

An (R,R)-bimodule M on which the left and right $\Bbbk$-multiplications coincide is nothing but a left R^e-module with $(r \otimes r')m = rmr'$, or a right R^e-module with $m(r \otimes r') = r'mr$ (Remark B.4.2); in particular R itself is a left R^e-module.

For $n \in \mathbb{Z}_{\geq 0}$ define the (R,R)-bimodule

$$\mathsf{B}_n R := R \otimes R^{\otimes n} \otimes R = R^{\otimes(n+2)}, \quad r(r_0 \otimes \cdots \otimes r_{n+1})r' = rr_0 \otimes \cdots \otimes r_{n+1}r', \tag{3.5.2}$$

and, for $n \geq 1$, the bimodule morphism $b_n : \mathsf{B}_n R \to \mathsf{B}_{n-1} R$

$$b_n(r_0 \otimes \cdots \otimes r_{n+1}) := \sum_{k=0}^{n} (-1)^k (r_0 \otimes \cdots \otimes r_k r_{k+1} \otimes \cdots \otimes r_{n+1}), \tag{3.5.3}$$

where the k-th summand multiplies the adjacent factors r_k and r_{k+1}. Classically one writes $(r_0 \mid r_1 \mid \cdots \mid r_{n+1})$ for $r_0 \otimes \cdots \otimes r_{n+1}$, whence the name *bar*. Augment by the multiplication $b_0 : \mathsf{B}_0 R = R \otimes R \to R$. Each term of $b_{n-1}b_n$ drops two bars in exactly two ways with opposite signs, so $b^2 = 0$, and we obtain the chain complex $\mathsf{B}R$, the ***bar complex*** of R. We claim that

$$\cdots \to \mathsf{B}_1 R \to \mathsf{B}_0 R \xrightarrow{b_0} R \to 0 \tag{3.5.4}$$

is exact, i.e. a resolution of R by (R,R)-bimodules. To see this, put $\mathsf{B}_{-1} R := R$ and let $h_n : \mathsf{B}_n R \to \mathsf{B}_{n+1} R$ $(n \geq -1)$ be the $\Bbbk$-linear map inserting the unit in front. One computes

$$\begin{aligned} b_{n+1}h_n(r_0 \otimes \cdots \otimes r_{n+1}) &= (r_0 \otimes \cdots \otimes r_{n+1}) + \sum_{j=1}^{n+1} (-1)^j (1_R \otimes \cdots \otimes r_{j-1}r_j \otimes \cdots), \\ h_{n-1}b_n(r_0 \otimes \cdots \otimes r_{n+1}) &= \sum_{k=0}^{n} (-1)^k (1_R \otimes \cdots \otimes r_k r_{k+1} \otimes \cdots), \end{aligned} \tag{3.5.5}$$

and the two sums cancel term by term $(j = k+1)$, so $b_{n+1}h_n + h_{n-1}b_n = \mathrm{id}$, while $b_0 h_{-1}(r) = b_0(1_R \otimes r) = r$. Thus the identity of the augmented bar complex is null-homotopic as an endomorphism of a complex of $\Bbbk$-modules (Remark 3.2.2); since kernels and images are computed on underlying $\Bbbk$-modules, exactness follows. Note also the decomposition into left R^e-modules

$$\mathsf{B}_n R \simeq R^e \otimes R^{\otimes n}, \quad (r \otimes r') \otimes (r_1 \otimes \cdots \otimes r_n) \mapsto r \otimes r_1 \otimes \cdots \otimes r_n \otimes r'. \tag{3.5.6}$$

For an (R, R)-bimodule M, define its ***Hochschild homology*** and ***Hochschild cohomology*** by

$$\mathrm{HH}_n(M) := \mathrm{H}_n(M \otimes_{R^e} \mathsf{B}R), \quad \mathrm{HH}^n(M) := \mathrm{H}^n(\mathrm{Hom}_{R^e}(\mathsf{B}R, M)), \quad n \in \mathbb{Z}_{\geq 0}, \tag{3.5.7}$$

where $\mathrm{Hom}_{R^e}(\mathsf{B}R, M)$ is the cochain complex with degree-n term $\mathrm{Hom}_{R^e}(\mathsf{B}_n R, M)$ and differential induced by b. These are $\Bbbk$-modules, functorial in M. The case of primary interest is $M = R$: we write $\mathrm{HH}_n(R)$ and $\mathrm{HH}^n(R)$ for the Hochschild homology and cohomology of R itself.

The complexes take a concrete form: the decomposition above together with $M \otimes_{R^e} R^e \simeq M$ and $\mathrm{Hom}_{R^e}(R^e, M) \simeq M$ gives

$$\begin{aligned} M \otimes_{R^e} \mathsf{B}_n R &\simeq M \otimes R^{\otimes n} \\ m \otimes (r_0 \otimes \cdots \otimes r_{n+1}) &\mapsto r_{n+1} m r_0 \otimes r_1 \otimes \cdots \otimes r_n, \\ \mathrm{Hom}_{R^e}(\mathsf{B}_n R, M) &\simeq \mathrm{Hom}_{\Bbbk}(R^{\otimes n}, M) \\ \varphi &\mapsto [(r_1, \cdots, r_n) \mapsto \varphi(1_R, r_1, \cdots, r_n, 1_R)], \end{aligned} \tag{3.5.8}$$

and b transports to the ***Hochschild differential***

$$\begin{aligned} d_n(m \otimes r_1 \otimes \cdots \otimes r_n) &= m r_1 \otimes r_2 \otimes \cdots \otimes r_n \\ &\quad + \sum_{k=1}^{n-1} (-1)^k m \otimes \cdots \otimes r_k r_{k+1} \otimes \cdots + (-1)^n r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}, \\ (d^n f)(r_1, \cdots, r_{n+1}) &= r_1 f(r_2, \cdots, r_{n+1}) \\ &\quad + \sum_{k=1}^{n} (-1)^k f(\cdots, r_k r_{k+1}, \cdots) + (-1)^{n+1} f(r_1, \cdots, r_n) r_{n+1}. \end{aligned} \tag{3.5.9}$$

In other words, $\mathrm{HH}_n(M)$ and $\mathrm{HH}^n(M)$ are computed by the ***Hochschild complexes***

$$\begin{aligned} C_\bullet(R, M) &= [\cdots \to M \otimes R^{\otimes n} \to \cdots \to M \otimes R \to M \to 0] \\ C^\bullet(R, M) &= [0 \to M \to \mathrm{Hom}_{\Bbbk}(R, M) \to \cdots \to \mathrm{Hom}_{\Bbbk}(R^{\otimes n}, M) \to \cdots]. \end{aligned} \tag{3.5.10}$$

Exercise 3.5.1 For $M = R$, verify that
- the chain complex is $\cdots \to R^{\otimes(n+1)} \to \cdots \to R \otimes R \to R \to 0$, with d_n the signed sum of adjacent multiplications, the last term wrapping around as $(-1)^n r_n r_0$;
- $\mathrm{HH}^0(R)$ is the center of R;
- $\mathrm{HH}_0(R)$ is the quotient of R by the $\Bbbk$-submodule generated by all commutators $rr' - r'r$.

Exercise 3.5.2 Compute $\mathrm{HH}^\bullet(\Bbbk)$, $\mathrm{HH}^\bullet(\Bbbk[\varepsilon]/(\varepsilon^2))$ (the dual numbers), and $\mathrm{HH}^\bullet(\Bbbk[x])$.

3.6 Mapping cones and cylinders

Associated to every chain map is a canonical complex, its *mapping cone*, which measures the failure of the map to be a quasi-isomorphism.

Definition 3.6.1 The ***mapping cone*** of a chain map $f : X \to Y$ over an additive category $\mathcal{A}$ is the complex

$$\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n \tag{3.6.1}$$

with differential given by the matrix

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}. \tag{3.6.2}$$

The sign on d_X ensures $d^2 = 0$: in the second component of $d^{n+1}_{\mathrm{Cone}(f)} d^n_{\mathrm{Cone}(f)}$, the terms involving d^2 vanish and the cross terms cancel because $d_Y f = f d_X$. There are natural chain maps $\alpha(f) : Y \to \mathrm{Cone}(f)$ (the inclusion of the Y summand) and $\beta(f) : \mathrm{Cone}(f) \to X[1]$ (the projection onto $X[1]$), fitting into

$$Y \xrightarrow{\alpha(f)} \mathrm{Cone}(f) \xrightarrow{\beta(f)} X[1]. \tag{3.6.3}$$

The sequence in Definition 3.6.1 will become the archetypal *distinguished triangle* of the homotopy category in Chapter 7. For abelian categories $\mathcal{A}$, its algebraic shadow in $\mathsf{C}(\mathcal{A})$ is the short exact sequence

$$0 \to Y \xrightarrow{\alpha(f)} \mathrm{Cone}(f) \xrightarrow{\beta(f)} X[1] \to 0, \tag{3.6.4}$$

exact degreewise (it is just the split short exact sequence of graded objects $0 \to Y^n \to X^{n+1} \oplus Y^n \to X^{n+1} \to 0$). Applying the long exact sequence in cohomology (Proposition 3.3.1) yields:

Proposition 3.6.2 For a chain map $f : X \to Y$ over an abelian category $\mathcal{A}$, there is a long exact sequence

$$\cdots \to \mathrm{H}^n(Y) \to \mathrm{H}^n(\mathrm{Cone}(f)) \to \mathrm{H}^{n+1}(X) \xrightarrow{\mathrm{H}^{n+1}(f)} \mathrm{H}^{n+1}(Y) \to \cdots. \tag{3.6.5}$$

Since each $\mathrm{H}^{n+1}(f)$ is sandwiched between the cohomology groups of $\mathrm{Cone}(f)$, we infer that f is a quasi-isomorphism iff $\mathrm{Cone}(f)$ is acyclic.

This last equivalence is arguably the most useful characterization of quasi-isomorphisms. It reduces a question about f to a question about a single complex $\mathrm{Cone}(f)$.

Remark 3.6.3 There is a closely related construction, the ***mapping cylinder*** $\mathrm{Cyl}(f)$, endowed with canonical morphisms $X \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f)$. The maps $Y \to \mathrm{Cyl}(f) \to Y$ are mutually inverse in $\mathsf{K}(\mathcal{A})$, so in particular $Y \to \mathrm{Cyl}(f)$ is a quasi-isomorphism, and f factors as $X \to \mathrm{Cyl}(f) \to Y$. It plays, in homotopy theory, the role of the mapping cylinder of a continuous map in topology. We shall not need its explicit formula and refer to [10, §3.3, §8.10] for details.

Exercise 3.6.4 Show that a chain map $f : X \to Y$ over an abelian category $\mathcal{A}$ is null-homotopic iff it factors through the mapping cone of id_X, i.e. iff f extends along the canonical $X \to \mathrm{Cone}(\mathrm{id}_X)$. Check that $\mathrm{Cone}(\mathrm{id}_X)$ has null-homotopic identity, e.g. via the homotopy $s(x', x) := (-x, 0)$, which satisfies $ds + sd = -\mathrm{id}$; deduce that $\mathrm{Cone}(\mathrm{id}_X)$ is acyclic.

Exercise 3.6.5 Let $f : X \to Y$ be a chain map between complexes concentrated in degrees -1 and 0 (so X and Y are just morphisms $X^{-1} \to X^0$ and $Y^{-1} \to Y^0$). Compute $\mathrm{Cone}(f)$ explicitly and verify Proposition 3.6.2 by hand: write down the long exact sequence and check that f is a quasi-isomorphism iff $\mathrm{Cone}(f)$ is acyclic.

Exercise 3.6.6 Verify that the shift $X \mapsto X[1]$ is an automorphism of $\mathsf{C}(\mathcal{A})$ and of $\mathsf{K}(\mathcal{A})$, with inverse $X \mapsto X[-1]$. Show that $\mathrm{H}^n(X[1]) = \mathrm{H}^{n+1}(X)$, so that shift "rotates" the cohomology. What is the relationship between $\mathrm{Cone}(f)$ and $\mathrm{Cone}(f[1])$?

3.7 Truncation functors

Every complex can be cut, from above or from below, in a way that is well-behaved under quasi-isomorphism. These *truncations* are the elementary tools used to isolate individual cohomology groups.

Definition 3.7.1 Let $\mathcal{A}$ be an additive category. For $X \in \mathsf{C}(\mathcal{A})$ and $n \in \mathbb{Z}$, the ***brutal truncations*** of X are

$$\begin{aligned} \sigma^{\leq n}X &: \cdots \to X^{n-1} \to X^n \to 0 \to \cdots, \\ \sigma^{\geq n}X &: \cdots \to 0 \to X^n \to X^{n+1} \to \cdots. \end{aligned} \tag{3.7.1}$$

When $\mathcal{A}$ is an abelian category, the ***truncations*** $\sigma^{\leq n}$, $\tau^{\leq n}$ of X are

$$\begin{aligned} \tau^{\leq n}X &: \cdots \to X^{n-1} \to \ker d^n \to 0 \to \cdots, \\ \tau^{\geq n}X &: \cdots \to 0 \to X^n / \operatorname{im} d^{n-1} \to X^{n+1} \to \cdots. \end{aligned} \tag{3.7.2}$$

The truncations are better behaved.

- There is a canonical morphism $\tau^{\leq n}X \to X$, which induces isomorphisms between cohomologies in degrees $\leq n$ (and $\mathrm{H}^k(\tau^{\leq n}X) = 0$ for $k > n$). Similarly for $X \to \tau^{\geq n}X$.
- The composite $\tau^{\leq n}\tau^{\geq n}X$ (or $\tau^{\geq n}\tau^{\leq n}X$) is canonically isomorphic to the single object $\mathrm{H}^n(X)$ placed in degree n, i.e. $\mathrm{H}^n(X)[-n]$.
- Moreover, one can show that truncation functors descend to the homotopy category, see [10, Definition–Proposition 3.9.7].

The verifications are not difficult and are left to the reader.

There are obvious variants for chain complexes and homologies, which we omit.

3.8 Double complexes and the total complex

A *double complex* is a two-dimensional grid of objects with commuting horizontal and vertical differentials. Its *totalization* converts it into an ordinary complex and is the setting for the spectral sequences of Chapter 6.

Definition 3.8.1 A ***double complex*** $X^{\bullet,\bullet}$ over an additive category $\mathcal{A}$ is a bigraded family $X^{p,q}$ with horizontal differentials $d_{\triangleright} : X^{p,q} \to X^{p+1,q}$ and vertical differentials $d_{\triangle} : X^{p,q} \to X^{p,q+1}$ satisfying

$$\begin{aligned} d_{\triangleright}^2 = 0, \quad d_{\triangle}^2 = 0, \\ d_{\triangleright}d_{\triangle} = d_{\triangle}d_{\triangleright}. \end{aligned} \tag{3.8.1}$$

We often abbreviate $\left(X^{p,q}, d_{\triangleright}, d_{\triangle}\right)_{p,q}$ as $X^{\bullet,\bullet}$ or X. A morphism (or chain map) $f : X \to Y$ between double complexes is a family of morphisms $f^{p,q} : X^{p,q} \to Y^{p,q}$ that commute with $d_{\triangleright}$ and $d_{\triangle}$.

One can also view a double complex as a complex of complexes, either horizontally or vertically. In the same way, one can define triple complexes and so forth.

Definition 3.8.2 We denote the category of double complexes over $\mathcal{A}$ as $\mathsf{C}^2(\mathcal{A})$. Let $\mathsf{C}^2_{\mathrm{f}}(\mathcal{A})$ be the full subcategory consisting of those double complexes X such that, for all $n \in \mathbb{Z}$, the set $\{(p,q) \in \mathbb{Z}^2 : p+q=n, X^{p,q} \neq 0\}$ is finite.

The category $\mathsf{C}^2(\mathcal{A})$ is readily seen to be additive, and abelian whenever $\mathcal{A}$ is: the biproducts, kernels, cokernels, etc. are computed degreewise.

Typical examples of double complexes in $\mathsf{C}^2_{\mathrm{f}}(\mathcal{A})$ are the first-quadrant double complexes: they are the X with $X^{p,q} = 0$ whenever $p < 0$ or $q < 0$. They look like

$$\begin{tikzcd}
\vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \iddots \\
X^{0,2} \arrow[r] \arrow[u] & X^{1,2} \arrow[r] \arrow[u] & X^{2,2} \arrow[r] \arrow[u] & \cdots \arrow[u] \\
X^{0,1} \arrow[r] \arrow[u] & X^{1,1} \arrow[r] \arrow[u] & X^{2,1} \arrow[r] \arrow[u] & \cdots \arrow[u] \\
X^{0,0} \arrow[r] \arrow[u] & X^{1,0} \arrow[r] \arrow[u] & X^{2,0} \arrow[r] \arrow[u] & \cdots \arrow[u]
\end{tikzcd} \tag{3.8.2}$$

Definition 3.8.3 The ***total complex*** tot of a double complex X in $\mathsf{C}^2_{\mathrm{f}}(\mathcal{A})$ is

$$\operatorname{tot}(X)^n = \bigoplus_{p+q=n} X^{p,q}, \tag{3.8.3}$$

with differential d^n equal to $d_{\triangleright} + (-1)^p d_{\triangle}$ on the $X^{p,q}$-summand where $p+q=n$. This yields a functor $\operatorname{tot} : \mathsf{C}^2_{\mathrm{f}}(\mathcal{A}) \to \mathsf{C}(\mathcal{A})$.

Remark 3.8.4 More generally, if $\mathcal{A}$ admits countable coproducts (resp. products), then one can define

$$\operatorname{tot}^{\oplus}(X)^n = \bigoplus_{p+q=n} X^{p,q}, \quad \operatorname{tot}^{\prod}(X)^n = \prod_{p+q=n} X^{p,q} \tag{3.8.4}$$

for all double complexes X functorially, in exactly the same manner. When $X \in \operatorname{Ob}\big(\mathsf{C}^2_{\mathrm{f}}(\mathcal{A})\big)$, these two both coincide with $\operatorname{tot}(X)$ as in Definition 3.8.3.

Definition 3.8.5 Suppose $\mathcal{A}$ is an abelian category. Define the additive functors

$$\mathrm{H}_{\mathrm{I}}, \mathrm{H}_{\mathrm{II}} : \mathsf{C}^2(\mathcal{A}) \to \mathsf{C}^2(\mathcal{A}) \tag{3.8.5}$$

as follows: $\mathrm{H}_{\mathrm{I}}(X)^{p,q}$ is the degree-p cohomology of the complex $\left(X^{\bullet,q}, d_{\triangleright}\right)$ and the vertical differential $\mathrm{H}_{\mathrm{I}}(X)^{p,q} \to \mathrm{H}_{\mathrm{I}}(X)^{p,q+1}$ is induced by $d_{\triangle}$, whereas the horizontal differential $\mathrm{H}_{\mathrm{I}}(X)^{p,q} \to \mathrm{H}_{\mathrm{I}}(X)^{p+1,q}$ is zero; the case of H_{II} is obtained by switching the roles of vertical and horizontal differentials.

In brief, H_{I} takes horizontal cohomologies and H_{II} takes vertical cohomologies.

The fundamental comparison result for double complexes is the following.

Theorem 3.8.6 (*acyclic assembly lemma*) Suppose that $X \in \mathrm{Ob}\big(\mathsf{C}^2_{\mathrm{f}}(\mathcal{A})\big)$ where $\mathcal{A}$ is an abelian category, and either $\mathrm{H}_{\mathrm{I}}(X) = 0$ or $\mathrm{H}_{\mathrm{II}}(X) = 0$. Then $\mathrm{tot}(X)$ is acyclic.

Proof Without loss of generality, assume $\mathrm{H}_{\mathrm{II}}(X) = 0$, i.e. the columns are acyclic. We shall show that $\mathrm{H}^n(\mathrm{tot}(X)) = 0$ for every given n. This we will do by diagram chasing, i.e. by assuming that $\mathcal{A} = R\text{-}\mathsf{Mod}$ for some ring R. For the general case, see [10, Corollary 3.10.7].

Let $c = \big(c_{p,q}\big)_{p+q=n} \in \mathrm{tot}(X)^n$ be an n-cocycle, i.e. $d(c) = 0$. We shall find $b = \big(b_{p,q}\big)_{p+q=n-1}$ such that $d(b) = c$, that is,

$$d_{\triangleright}\big(b_{p-1,q}\big) + (-1)^p d_{\triangle}\big(b_{p,q-1}\big) = c_{p,q}, \quad p+q=n. \tag{3.8.6}$$

We may surely assume $c \neq 0$. Take p_0 such that $c_{p,n-p} = 0$ for all $p < p_0$ and $c_{p_0,n-p_0} \neq 0$. Then $d(c) = 0$ implies $d_{\triangle}\big(c_{p_0,n-p_0}\big) = 0$, hence there exists $b_{p_0,n-1-p_0}$ such that

$$(-1)^{p_0} d_{\triangle}\big(b_{p_0,n-1-p_0}\big) = c_{p_0,n-p_0}. \tag{3.8.7}$$

Let c' be c minus the d-image of $b_{p_0,n-1-p_0}$ viewed as an element of $\mathrm{tot}(X)^{n-1}$. If $c' \neq 0$, we may pass from p_0 to p_0+1 and iterate. The procedure must end in finitely many steps, since there are only finitely many p for which $X^{p,n-p} \neq 0$. □

The acyclic assembly lemma is the engine behind the two spectral sequences of a double complex (Chapter 6): it is what guarantees that both converge, to the cohomology $\mathrm{H}^n(\mathrm{tot}\,X)$ of the total complex. It also underlies the balancing of derived bifunctors in Chapter 4.

Chapter 4

Classical Derived Functors

An exact functor preserves short exact sequences, hence preserves cohomologies by Proposition 3.3.2. But the functors of everyday algebra are rarely exact: Hom is left exact, $\otimes_R$ is right exact (Example 2.3.1), and the failure of exactness is exactly what homological algebra was invented to measure. The idea of this chapter is to repair a one-sided-exact functor F by completing the truncated exact sequence it produces into a *long* exact sequence of new functors, the ***derived functors*** $\mathrm{R}^n F$ and $\mathrm{L}_n F$. We develop the classical, resolution-based theory, enough to define Ext and Tor concretely and to compute them, ahead of the derived-category reinterpretation in Chapter 7.

4.1 Resolutions

A *resolution* replaces an object or complex by a quasi-isomorphic one built from nice objects (injectives or projectives, Section 2.7) on which derived functors can be computed degreewise.

Definition 4.1.1 Let $\mathcal{A}$ be an additive category. We define the following full subcategories $\mathsf{C}^{\pm}(\mathcal{A}), \mathsf{C}^{\mathrm{b}}(\mathcal{A})$ of $\mathsf{C}(\mathcal{A})$:

- $\mathsf{C}^{+}(\mathcal{A})$ (resp. $\mathsf{C}^{-}(\mathcal{A})$) consists of objects X such that $X^n = 0$ whenever $n \ll 0$ (resp. $n \gg 0$);
- $\mathsf{C}^{\mathrm{b}}(\mathcal{A})$ is the intersection of $\mathsf{C}^{+}(\mathcal{A})$ and $\mathsf{C}^{-}(\mathcal{A})$;
- $\mathsf{C}^{\geq m}(\mathcal{A})$ (resp. $\mathsf{C}^{\leq m}(\mathcal{A})$) consists of objects X such that $X^n = 0$ whenever $n < m$ (resp. $n > m$), where $m \in \mathbb{Z}$;
- $\mathsf{C}^{[a,b]}(\mathcal{A})$ is defined analogously, where $-\infty \leq a \leq b \leq \infty$.

These subcategories are abelian when $\mathcal{A}$ is. Similarly, we can define the full subcategories $\mathsf{K}^{\pm}(\mathcal{A}), \mathsf{K}^{\mathrm{b}}(\mathcal{A})$ of the homotopy category $\mathsf{K}(\mathcal{A})$.

Definition 4.1.2 Let $\mathcal{A}$ be abelian and $X \in \mathrm{Ob}(\mathsf{C}(\mathcal{A}))$. Say that X is ***built from injectives*** if every X^n is injective, and ***built from projectives*** if every X^n is projective.

▷ ***injective resolution*** a quasi-isomorphism $X \to I$ in $\mathsf{C}(\mathcal{A})$ with $I \in \mathrm{Ob}(\mathsf{C}^{+}(\mathcal{A}))$ built from injectives.

▷ ***projective resolution*** a quasi-isomorphism $P \to X$ with $P \in \mathrm{Ob}(\mathsf{C}^{-}(\mathcal{A}))$ built from projectives.

Injective and projective resolutions are dual notions, as is easily seen by reversing arrows. The boundedness condition (C^{+} or C^{-}) is essential for their existence below; we remove it in Section 4.6.

Example 4.1.3 (*resolutions of an object*) For an object $X \in \mathcal{A}$, viewed as a complex concentrated in degree 0, an injective resolution may be truncated to live in $\mathsf{C}^{\geq 0}(\mathcal{A})$. To see this, it suffices to show $X \to I \to \tau^{\geq 0} I$ (Definition 3.7.1) composes to an injective resolution. It is clearly a quasi-isomorphism, thus it remains to show $\tau^{\geq 0} I$ is built from injectives. Consider the short exact sequence

$$0 \to \mathrm{im}(d_I^{n-1}) \to I^n \to \mathrm{coker}(d_I^{n-1}) \to 0. \tag{4.1.1}$$

Since $d_I^{n-1} = 0$ when $n \ll 0$, while exactness of I in negative degrees gives $\mathrm{coker}(d_I^{n-1}) = \mathrm{im}(d_I^n)$ when $n < 0$, one shows by induction on n (starting from $n \ll 0$, where all terms vanish) that the sequence above splits and

$$I^n \simeq \mathrm{im}(d_I^{n-1}) \oplus \mathrm{coker}(d_I^{n-1}) : \tag{4.1.2}$$

indeed, at the induction step $\mathrm{im}(d_I^{n-1}) = \mathrm{coker}(d_I^{n-2})$ is a direct summand of the injective object I^{n-1}, hence is injective by Proposition 2.7.3, so the sequence splits by Proposition 2.7.2, and both summands of I^n are again injective by Proposition 2.7.3. Taking $n = 0$ implies $\mathrm{coker}(d_I^{-1})$ is injective, hence $\tau^{\geq 0} I$ is built from injectives.

Furthermore, an injective resolution $X \to I$ with $I \in \mathsf{C}^{\geq 0}(\mathcal{A})$ can be *flattened* to an exact sequence in $\mathcal{A}$

$$0 \to X \to I^0 \to I^1 \to \cdots. \tag{4.1.3}$$

When $\mathcal{A}$ has enough injectives, such an injective resolution is built inductively: embed $X \hookrightarrow I^0$, then embed $\mathrm{coker}(X \to I^0) \hookrightarrow I^1$, and so on.

Dually, any projective resolution of X can be truncated and then flattened into an exact sequence

$$\cdots \to P_1 \to P_0 \to X \to 0 \tag{4.1.4}$$

where we adopt the chain convention $P_n := P^{-n}$.

Theorem 4.1.4 Let $\mathcal{A}^\flat$ be a full additive subcategory of $\mathcal{A}$ such that every object of $\mathcal{A}$ embeds into some object of $\mathcal{A}^\flat$ (resp. is a quotient of some object of $\mathcal{A}^\flat$). Then every $X \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$ admits a quasi-isomorphism $X \to I$ that is degreewise monic, with $I \in \mathsf{C}^+(\mathcal{A}^\flat)$ (resp. every bounded-above complex $X \in \mathsf{C}^-(\mathcal{A})$ admits a quasi-isomorphism $P \to X$ that is degreewise epic, with $P \in \mathsf{C}^-(\mathcal{A}^\flat)$). In particular, if $\mathcal{A}$ has enough injectives (resp. projectives), then every bounded-below (resp. bounded-above) complex admits an injective (resp. projective) resolution.

Proof (*Sketch*) For objects this is done in Example 4.1.3. The extension to bounded-below complexes is a degree-by-degree induction: at each step one forms a pushout of the relevant quotients and embeds it into an object of $\mathcal{A}^\flat$, extending the resolution one degree to the right and checking at every stage that the induced map on cohomology is an isomorphism. The argument uses the modular law for subobjects (Proposition 2.4.3) and the isomorphism theorems. See [10, Theorem 3.11.3] for the details. □

Even for objects X of $\mathcal{A}$, the resolutions are far from unique, but they are unique *up to homotopy*. This property ultimately makes derived functors well-defined.

Lemma 4.1.5 If $X \in \mathrm{Ob}(\mathsf{C}(\mathcal{A}))$ is acyclic and $I \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$ is built from injectives, then $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) = 0$. Dually $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P, X) = 0$ for P built from projectives and X acyclic.

Proof Given a chain map $\alpha : X \to I$ we seek a family of morphisms $h^n : X^n \to I^{n-1}$ with

$$\alpha^n = d_I^{n-1} h^n + h^{n+1} d_X^n \quad \text{for all } n. \tag{4.1.5}$$

Build h recursively from $n \ll 0$ (where both sides vanish because $I \in \mathsf{C}^+(\mathcal{A})$). Suppose …, h^{n-1}, h^n have been built. The morphism $\alpha^n - d_I^{n-1} h^n : X^n \to I^n$ annihilates $\operatorname{im}(d_X^{n-1})$, as seen by a short computation using that α is a morphism and the induction hypothesis. Since X is acyclic, $\operatorname{im}(d_X^{n-1}) = \ker(d_X^n)$, so the difference above factors through $\beta : \operatorname{im}(d_X^n) \to I^n$:

$$\begin{tikzcd}
I^n & & \\
X^n \arrow[u, "\alpha^n - d_I^{n-1} h^n"] \arrow[r, two heads] & \operatorname{im}(d_X^n) \arrow[ul, dashed, "\beta"'] \arrow[r, hook] & X^{n+1} \arrow[ull, dashed, "h^{n+1}"']
\end{tikzcd} \tag{4.1.6}$$

The dashed lift $h^{n+1} : X^{n+1} \to I^n$ exists because I^n is injective (Section 2.7). This extends the homotopy, completing the induction. □

Theorem 4.1.6 (*uniqueness of resolutions up to homotopy*) Let $\varphi : X \to Y$ be a morphism in $\mathcal{A}$, with X, Y viewed as complexes concentrated in degree 0. Let $Y \to I$ be an injective resolution with $I \in \operatorname{Ob}(\mathsf{C}^{\geq 0}(\mathcal{A}))$. For every quasi-isomorphism $X \to H$ of complexes with $H \in \operatorname{Ob}(\mathsf{C}^{\geq 0}(\mathcal{A}))$, there exists a morphism $f : H \to I$ such that

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\varphi"'] & H^0 \arrow[r] \arrow[d, "f^0"'] & H^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \\
0 \arrow[r] & Y \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd} \tag{4.1.7}$$

commutes, and such a morphism f is unique up to homotopy.

Dually, let $P \to X$ be a projective resolution with $P \in \operatorname{Ob}(\mathsf{C}^{\leq 0}(\mathcal{A}))$. For every quasi-isomorphism $J \to Y$ of chain complexes with $J \in \operatorname{Ob}(\mathsf{C}^{\leq 0}(\mathcal{A}))$, there exists a morphism $f : P \to J$ such that

$$\begin{tikzcd}
0 & X \arrow[l] \arrow[d, "\varphi"'] & P_0 \arrow[l] \arrow[d, "f_0"'] & P_1 \arrow[l] \arrow[d, "f_1"'] & \cdots \arrow[l] \\
0 & Y \arrow[l] & J_0 \arrow[l] & J_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{4.1.8}$$

commutes, and such a morphism f is unique up to homotopy.

Proof By duality, it suffices to treat the case of $P \to X$. For the existence of f, we shall construct f inductively by setting $f_{-1} = \varphi$. Suppose …, f_{n-1}, f_n have been constructed, where $n \geq -1$. Then the composite $P_{n+1} \to P_n \to J_n$ factors through $B_n(J) := \operatorname{im}[J_{n+1} \to J_n]$ (using the exactness of $0 \leftarrow Y \leftarrow J_0 \leftarrow \cdots$). We thus obtain a diagram with exact row:

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dashed, "f_{n+1}"'] \arrow[d] & \\
J_{n+1} \arrow[r] & B_n(J) \arrow[r] & 0
\end{tikzcd} \tag{4.1.9}$$

hence there exists the dashed arrow f_{n+1} making the diagram commutative, by the projectivity of P_{n+1}.

We now show the uniqueness of f up to homotopy. Let $g : P \to J$ be another candidate and let $e = f - g$. There is then a commutative diagram (solid)

$$\begin{tikzcd} 0 & 0 \arrow[l] \arrow[d, "0"'] & P_0 \arrow[l] \arrow[d, "e_0"'] \arrow[dr, dashed, "h_0"] & P_1 \arrow[l] \arrow[d, "e_1"'] \arrow[dr, dashed, "h_1"] & \cdots \arrow[l] \\ 0 & Y \arrow[l] & J_0 \arrow[l] & J_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd} \tag{4.1.10}$$

whose bottom row is exact. By Lemma 4.1.5, the sequence $(0, e_0, e_1, \dots)$ is null-homotopic (the dashed arrows, see Remark 3.2.2); this easily implies that $e : P \to J$ is null-homotopic. $\square$

Remark 4.1.7 Theorem 4.1.6 has the following variant, which applies to arbitrary complexes X, with boundedness conditions on I, resp. P. Fix a quasi-isomorphism $\alpha : X \to Y$ in $\mathsf{C}(\mathcal{A})$.

- Given $\gamma : X \to I$ with $I \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$ built from injectives, there is a unique morphism $\beta : Y \to I$ in the homotopy category $\mathsf{K}(\mathcal{A})$ making the triangle commute.
- Dually, given $\gamma : P \to Y$ with $P \in \mathrm{Ob}(\mathsf{C}^-(\mathcal{A}))$ built from projectives, there is a unique $\beta : P \to X$ in $\mathsf{K}(\mathcal{A})$.

We shall give a complete proof of these facts in Corollary 7.5.3 via triangulated categories; see also [10, Theorem 3.11.5]. The preceding Lemma 4.1.5 is the engine of its arguments.

Corollary 4.1.8 Given $f : X \to Y$ in $\mathsf{C}^+(\mathcal{A})$ and injective resolutions $X \to I$, $Y \to J$, Remark 4.1.7 (the general form of Theorem 4.1.6) provides a morphism $\beta : I \to J$, unique up to homotopy, making the square commute up to homotopy. Consequently, for every additive functor $F : \mathcal{A} \to \mathcal{B}$, the group $\mathrm{H}^n(\mathsf{C}(F)(I))$ is, up to canonical isomorphism, independent of the chosen injective resolution.

Proposition 4.1.9 (*horseshoe lemma*) Given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathsf{C}^+(\mathcal{A})$ and injective resolutions $A \to I$, $C \to K$, there is an injective resolution $B \to J$ fitting into a short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & J \arrow[r] & K \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{4.1.11}$$

whose first row is degreewise split. There is also a dual version for projective resolutions.

Proof For simplicity we assume that A, B, C are objects of $\mathcal{A}$; the general version is proved in [10, Proposition 3.11.9]. By duality, it suffices to deal with the version for projective resolutions.

Suppose the projective resolutions $\cdots \to P_1 \to P_0 \xrightarrow{\alpha} A \to 0$ and $\cdots \to R_1 \to R_0 \xrightarrow{\gamma} C \to 0$ are given. Since R_0 is projective, the augmentation γ lifts along the epimorphism $B \twoheadrightarrow C$ to a morphism $\tilde{\gamma} : R_0 \to B$. Taking the direct sum with the composite $P_0 \xrightarrow{\alpha} A \to B$, we obtain $Q_0 := P_0 \oplus R_0 \xrightarrow{\beta} B$. This yields a commutative diagram

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\alpha) \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & \ker(\gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\alpha"'] & Q_0 \arrow[r] \arrow[d, "\beta"'] & R_0 \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
\tag{4.1.12}
$$

The first and the third columns are exact; the second and the third rows are exact. Applying Theorem 2.5.2 to the two middle rows, we find that $\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to 0$ is exact, and that β is an epimorphism since $\operatorname{coker}(\alpha) = \operatorname{coker}(\gamma) = 0$; as $P_0 \to Q_0$ is monic, so is its restriction/pullback $\ker(\alpha) \to \ker(\beta)$. Hence the first row and the second column are also exact, and the construction of Q_0 is complete. We now consider the short exact sequence $0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to 0$ in $\mathcal{A}$, and the projective resolutions

$$\cdots \to P_1 \to \ker(\alpha) \to 0, \quad \cdots \to R_1 \to \ker(\gamma) \to 0. \tag{4.1.13}$$

Proceeding inductively, we obtain an epimorphism $Q_1 = P_1 \oplus R_1 \to \ker(\beta)$, and so forth. Splicing the pieces, we arrive at an exact sequence

$$\cdots \to Q_1 \to Q_0 \to B \to 0 \tag{4.1.14}$$

with $Q_n = P_n \oplus R_n$ projective (a biproduct of projectives) for all n, that is, a projective resolution of B; by construction it sits in a degreewise-split short exact sequence of complexes $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$. The injective version follows by duality. $\square$

Remark 4.1.10 (*Cartan–Eilenberg resolutions*) Let $\mathcal{A}$ have enough injectives and let $X \in \operatorname{Ob}(\mathsf{C}^+(\mathcal{A}))$. A ***Cartan–Eilenberg resolution*** of X is a morphism of double complexes $\varepsilon : X \to I$, where X is regarded as concentrated in row 0 and I is a double complex of injectives, such that the columns and the subquotients attached to X are resolved simultaneously: for every p,

- $0 \to X^p \to I^{p,0} \to I^{p,1} \to \cdots$ is an injective resolution (its differential is the vertical $d_\vartriangle$);
- $0 \to Z^p \to \ker\left(d_\triangleright^{p,0}\right) \to \ker\left(d_\triangleright^{p,1}\right) \to \cdots$ is an injective resolution of $Z^p = \ker(d^p)$;
- $0 \to B^{p+1} \to \operatorname{im}\left(d_\triangleright^{p,0}\right) \to \operatorname{im}\left(d_\triangleright^{p,1}\right) \to \cdots$ is an injective resolution of $B^{p+1} = \operatorname{im}(d^p)$;
- $0 \to \mathrm{H}^p(X) \to \mathrm{H}^p\left(I^{\bullet,0}, d_\triangleright\right) \to \mathrm{H}^p\left(I^{\bullet,1}, d_\triangleright\right) \to \cdots$ is an injective resolution of $\mathrm{H}^p(X)$.

In particular $I^{p,q} = 0$ for $q < 0$ and for $p \ll 0$ (recall $X \in \operatorname{Ob}(\mathsf{C}^+(\mathcal{A}))$). Note also that $\ker\left(d_\triangleright^{p,q}\right)$, $\operatorname{im}\left(d_\triangleright^{p,q}\right)$, and $\mathrm{H}^p(I^{\bullet,q})$ are injective, being components of injective resolutions; hence the short exact sequences linking the cycles, boundaries, and horizontal cohomology of each row of I are split, and consequently every additive functor F preserves horizontal cohomology: $\mathrm{H}_{\mathrm{I}}\left(\mathsf{C}^2(F)(I)\right)^{q,\bullet} \simeq \mathsf{C}(F)(\mathrm{H}_{\mathrm{I}}(I)^{q,\bullet})$.

Construction (sketch). Choose N with $X^n = 0$ for $n < N$, so that $B^N = 0$ and $Z^N = \mathrm{H}^N(X)$; for every p there are short exact sequences

$$0 \to Z^p \to X^p \to B^{p+1} \to 0, \quad 0 \to B^p \to Z^p \to \mathrm{H}^p(X) \to 0. \tag{4.1.15}$$

Choose injective resolutions of all $\mathrm{H}^p(X)$ and of all B^p, taking the zero resolution $0 \to 0$ where the object is zero.

- Since $Z^N = \mathrm{H}^N(X)$, Proposition 4.1.9 splices the resolutions of Z^N and B^{N+1} into a resolution $I^{N,\bullet}$ of X^N, compatible with $0 \to Z^N \to X^N \to B^{N+1} \to 0$.
- Applying Proposition 4.1.9 to $0 \to B^{N+1} \to Z^{N+1} \to \mathrm{H}^{N+1}(X) \to 0$ splices the chosen resolutions of B^{N+1} and $\mathrm{H}^{N+1}(X)$ into a resolution of Z^{N+1}.
- Applying Proposition 4.1.9 to $0 \to Z^{N+1} \to X^{N+1} \to B^{N+2} \to 0$ then gives a resolution $I^{N+1,\bullet}$ of X^{N+1}; iterate.

This fills in the columns $I^{p,\bullet}$, with $I^{p,\bullet} = 0$ for $p < N$. The construction also determines the horizontal differentials $d_{\triangleright}^{p,q} : I^{p,q} \to I^{p+1,q}$ by lifting $d^p : X^p \to X^{p+1}$ through the resolutions of $B^{p+1} = \mathrm{im}(d^p)$ and $Z^{p+1} = \ker(d^{p+1})$; the identity $d_{\triangleright}^2 = 0$ then follows from $B^{p+1} \subset Z^{p+1}$, and the remaining bookkeeping is routine. The projective (dual) version is obtained by reversing the arrows. This is the natural input for the spectral sequences of a filtered complex (Chapter 6). See [10, Theorem 3.11.10].

4.2 Derived functors

Convention 4.2.1 When we speak of the right derived functors of an additive functor $F : \mathcal{A} \to \mathcal{B}$, we tacitly assume $\mathcal{A}$ has enough injectives; dually, the left derived functors require enough projectives.

Definition 4.2.2 For an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories:

▷ **Right derived $\mathbf{R}^n F$** for $X \in \mathrm{Ob}(\mathcal{A})$ (or more generally $X \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$) with injective resolution $X \to I$, set

$$\mathrm{R}^n F(X) := \mathrm{H}^n(\mathsf{C}(F)(I)), \quad n \in \mathbb{Z}. \tag{4.2.1}$$

▷ **Left derived $\mathbf{L}_n F$** for $X \in \mathrm{Ob}(\mathcal{A})$ (or more generally $X \in \mathrm{Ob}(\mathsf{C}^-(\mathcal{A}))$) with projective resolution $P \to X$, set

$$\mathrm{L}_n F(X) := \mathrm{H}_{-n}(\mathsf{C}(F)(P)), \quad n \in \mathbb{Z}. \tag{4.2.2}$$

Here we adopt the chain convention that $P_n := P^{-n}$ and $\mathrm{H}_n := \mathrm{H}^{-n}$.

By Corollary 4.1.8 these are well-defined functors $\mathsf{C}^+(\mathcal{A}) \to \mathcal{B}$ (and, dually, $\mathsf{C}^-(\mathcal{A}) \to \mathcal{B}$), and a morphism of additive functors $F \to G$ induces morphisms $\mathrm{R}^n F \to \mathrm{R}^n G$, dually for L_n. In practice we evaluate them on $X \in \mathrm{Ob}(\mathcal{A})$.

The fundamental computational device is the long exact sequence.

Theorem 4.2.3 (*long exact sequences of derived functors*) Let $F : \mathcal{A} \to \mathcal{B}$ be additive. For a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ (or more generally $\mathsf{C}^+(\mathcal{A})$), there are natural connecting morphisms $\delta^n : \mathrm{R}^n F(Z) \to \mathrm{R}^{n+1} F(X)$ fitting into a long exact sequence

$$\cdots \to \mathrm{R}^{n-1} F(Z) \xrightarrow{\delta^{n-1}} \mathrm{R}^n F(X) \to \mathrm{R}^n F(Y) \to \mathrm{R}^n F(Z) \xrightarrow{\delta^n} \mathrm{R}^{n+1} F(X) \to \cdots. \tag{4.2.3}$$

Dually, for $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ (or more generally $\mathsf{C}^-(\mathcal{A})$) there are connecting morphisms $\partial_n : \mathrm{L}_n F(Z) \to \mathrm{L}_{n-1} F(X)$ and a long exact sequence

$$\cdots \to \mathrm{L}_{n+1} F(Z) \xrightarrow{\partial_{n+1}} \mathrm{L}_n F(X) \to \mathrm{L}_n F(Y) \to \mathrm{L}_n F(Z) \xrightarrow{\partial_n} \mathrm{L}_{n-1} F(X) \to \cdots. \quad (4.2.4)$$

Proof Apply the horseshoe lemma (Proposition 4.1.9) to obtain a degreewise-split short exact sequence of injective resolutions $0 \to I \to J \to K \to 0$. Because it is degreewise split, $0 \to \mathsf{C}(F)(I) \to \mathsf{C}(F)(J) \to \mathsf{C}(F)(K) \to 0$ is again a short exact sequence of complexes. Now apply the long exact sequence in cohomology (Proposition 3.3.1). The connecting morphisms are functorial in the short exact sequence; cf. [10, Theorem 3.12.4]. □

Proposition 4.2.4 For $F : \mathcal{A} \to \mathcal{B}$ additive, with $\mathcal{A}$ having enough injectives:

- $\mathrm{R}^n F = 0$ for $n < 0$;
- $\mathrm{R}^n F(I) = 0$ for $n > 0$ whenever I is injective;
- if F is left exact, there is a canonical isomorphism $F \simeq \mathrm{R}^0 F$.

Dually for $\mathrm{L}_n F$ (enough projectives; right-exact F gives $\mathrm{L}_0 F \simeq F$).

Proof Take the flattened resolution $0 \to X \to I^0 \to I^1 \to \cdots$; then

$$\mathrm{R}^n F(X) = \mathrm{H}^n(\cdots \to 0 \to F(I^0) \to F(I^1) \to \cdots), \quad (4.2.5)$$

from which $n < 0$ vanishes. If $X = I$ is injective, the resolution $0 \to I \xrightarrow{\mathrm{id}} I \to 0$ shows $\mathrm{R}^n F(I) = 0$ for $n > 0$. Finally, if F is left exact it preserves kernels, so $\mathrm{R}^0 F(X) = \ker(F(I^0) \to F(I^1)) \simeq F(X)$. □

Corollary 4.2.5 For a left exact $F : \mathcal{A} \to \mathcal{B}$, the following are equivalent:

(i) F is exact;
(ii) $\mathrm{R}^n F(X) = 0$ for all $X \in \mathrm{Ob}(\mathcal{A})$ and all $n > 0$;
(iii) $\mathrm{R}^1 F(X) = 0$ for all $X \in \mathrm{Ob}(\mathcal{A})$.

Dually for right exact F and $\mathrm{L}_n F$.

Proof (i) ⇒ (ii) is clear, since an exact functor F preserves the acyclicity of $I^0 \to I^1 \to \cdots$ outside degree 0.

(ii) ⇒ (iii) is trivial.

For (iii) ⇒ (i), use $\mathrm{R}^0 F \simeq F$ and read off exactness of $0 \to F(X) \to F(Y) \to F(Z) \to \mathrm{R}^1 F(X) = 0$ from the long exact sequence. □

This is the slogan: *the higher derived functors are precisely the obstruction to exactness*. In order to make the slogan precise, and to characterize derived functors without reference to resolutions, we introduce δ-functors.

Definition 4.2.6 A ***cohomological δ-functor*** from $\mathcal{A}$ to $\mathcal{B}$ is a family of additive functors $F^n : \mathcal{A} \to \mathcal{B}$ ($n \in \mathbb{Z}_{\geq 0}$) together with connecting morphisms $\delta^n : F^n(Z) \to F^{n+1}(X)$ for each short exact sequence $0 \to X \to Y \to Z \to 0$, functorial in such sequences, fitting into a long exact sequence

$$0 \to F^0(X) \to F^0(Y) \to F^0(Z) \xrightarrow{\delta^0} F^1(X) \to F^1(Y) \to F^1(Z) \xrightarrow{\delta^1} F^2(X) \to \cdots. \quad (4.2.6)$$

A ***homological δ-functor*** is defined dually (functors F_n, connecting morphisms $\partial_n : F_n(Z) \to F_{n-1}(X)$). A morphism of δ-functors is a family $\varphi^n : F^n \to G^n$ commuting with the connecting morphisms.

Definition 4.2.7 An additive functor $E : \mathcal{A} \to \mathcal{B}$ is ***effaceable*** if every $X \in \mathcal{A}$ admits a monomorphism $a : X \hookrightarrow Y$ with $E(a) = 0$ (for instance, $E(Y) = 0$). Dually, E is ***co-effaceable*** if every X admits an epimorphism $a : Y \twoheadrightarrow X$ with $E(a) = 0$.

Example 4.2.8 For $F : \mathcal{A} \to \mathcal{B}$ additive, with $\mathcal{A}$ having enough injectives (resp. projectives), Theorem 4.2.3 says that $(\mathrm{R}^n F, \delta^n)_{n\geq 0}$ (resp. $(\mathrm{L}_n F, \partial_n)_{n\geq 0}$) is a cohomological (resp. homological) δ-functor; moreover, it is effaceable (resp. co-effaceable) by Proposition 4.2.4.

For a cohomological δ-functor $(F^n, \delta^n)_{n\geq 0}$, the long exact sequence automatically forces F^0 to be left exact; dually, the F_0 in a homological δ-functor is right exact. The decisive notion is universality.

Definition 4.2.9 A cohomological δ-functor $(R^n, \delta^n)_{n\geq 0}$ is ***universal*** if for every cohomological δ-functor $F^\bullet$, every morphism $\varphi^0 : R^0 \to F^0$ extends uniquely to a morphism of δ-functors $(\varphi^n)_{n\geq 0}$. Dually for homological δ-functors.

Proposition 4.2.10 If $(R^n, \delta^n)_{n\geq 0}$ is a cohomological δ-functor whose higher pieces R^n ($n > 0$) are effaceable, then it is universal. Dually, a homological δ-functor with co-effaceable higher pieces is universal.

Proof (*Sketch*) Construct φ^n recursively on n. Given X, efface it by a short exact sequence $0 \to X \to I \to B \to 0$ with $R^n(X \to I) = 0$. The long exact sequences of $R^\bullet$ and $F^\bullet$ then present $R^n(X)$ as a cokernel, and the recursion hypothesis determines a unique φ^n:

$$\begin{tikzcd}
F^{n-1}(I) \arrow[r] & F^{n-1}(B) \arrow[r] & F^n(X) \arrow[r] & F^n(I) \\
R^{n-1}(I) \arrow[r] \arrow[u, "\varphi^{n-1}"'] & R^{n-1}(B) \arrow[r] \arrow[u, "\varphi^{n-1}"'] & R^n(X) \arrow[r] \arrow[u, dashed, "\varphi^n"'] & 0
\end{tikzcd} \tag{4.2.7}$$

One then checks, by a pushout argument, that φ^n is independent of the effacement, is functorial in X, and is compatible with the connecting morphisms. Full details are available in [10, Proposition 3.12.14]. $\square$

Corollary 4.2.11 If F is left exact and $\mathcal{A}$ has enough injectives, then $(\mathrm{R}^n F, \delta^n)_{n\geq 0}$ is the *universal* cohomological δ-functor with $\mathrm{R}^0 F \simeq F$. In particular it is determined up to unique isomorphism by F. Dually for right exact F and $(\mathrm{L}_n F, \partial_n)_{n\geq 0}$.

Proof For $n > 0$, $\mathrm{R}^n F$ is effaceable: embed X into an injective I, so $\mathrm{R}^n F(I) = 0$ by Proposition 4.2.4, hence $\mathrm{R}^n F(X \to I) = 0$. Apply Proposition 4.2.10. $\square$

We now introduce F-acyclic objects and record two practical consequences.

Convention 4.2.12 For a left-exact functor F, an object A is ***F-acyclic*** if $\mathrm{R}^n F(A) = 0$ for all $n > 0$; for a right-exact F the condition is $\mathrm{L}_n F(A) = 0$ for all $n > 0$.

By Proposition 4.2.4, every injective (resp. projective) object is F-acyclic.

Proposition 4.2.13 (*dimension shifting*) Let F be left exact and $0 \to X \to A \to B \to 0$ a short exact sequence with A being F-acyclic. Then

$$\mathrm{R}^n F(X) \simeq \begin{cases} \mathrm{R}^{n-1} F(B) \text{ if } n \geq 2 \\ \operatorname{coker}(F(A) \to F(B)) \text{ if } n = 1 \end{cases}. \tag{4.2.8}$$

Dually, for right exact F and $0 \to B \to A \to X \to 0$ with A being F-acyclic, we have

$$\mathrm{L}_n F(X) \simeq \begin{cases} \mathrm{L}_{n-1} F(B) \text{ if } n \geq 2 \\ \ker(F(B) \to F(A)) \text{ if } n = 1 \end{cases}. \tag{4.2.9}$$

Proof Read the long exact sequence: for $n \geq 1$,

$$\mathrm{R}^{n-1} F(A) \to \mathrm{R}^{n-1} F(B) \to \mathrm{R}^n F(X) \to \underbrace{\mathrm{R}^n F(A)}_{=0}, \tag{4.2.10}$$

and use $\mathrm{R}^0 F(A) \simeq F(A)$, $\mathrm{R}^0 F(B) \simeq F(B)$. □

Corollary 4.2.14 (*acyclic resolutions compute derived functors*) If $0 \to X \to A^0 \to A^1 \to \cdots$ is exact with every A^i being F-acyclic, then

$$\mathrm{R}^n F(X) \simeq \mathrm{H}^n(F(A^\bullet)). \tag{4.2.11}$$

Dually, $\cdots \to A_1 \to A_0 \to X \to 0$ with each A_i F-acyclic gives $\mathrm{L}_n F(X) \simeq \mathrm{H}_n(F(A_\bullet))$.

Proof Repeatedly apply Proposition 4.2.13 to the short exact sequences $0 \to B^i \to A^i \to B^{i+1} \to 0$ for all $i \geq 0$, where $B^i = \operatorname{im}(A^{i-1} \to A^i)$ with the convention $A^{-1} := X$. □

We will exploit the flexibility offered by Corollary 4.2.14 constantly.

4.3 Derived bifunctors and balancing

Many functors of interest are bifunctors (e.g. Hom and $\otimes$). For a bifunctor $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$, left exact in each variable, we may derive in either the first or the second variable, obtaining the bifunctors

$$\mathrm{R}^n_{\mathrm{I}} F, \mathrm{R}^n_{\mathrm{II}} F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B} \tag{4.3.1}$$

respectively; the notation here is akin to Definition 3.8.5. Dually, we have $\mathrm{L}_{\mathrm{I}_n} F$ and $\mathrm{L}_{\mathrm{II}_n} F$ when F is right exact in each variable. The remarkable fact is that, under a mild hypothesis, these two agree.

Definition 4.3.1 A bifunctor $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ between abelian categories, left exact in each variable, is ***balanced*** if $F(I_1, \cdot)$ and $F(\cdot, I_2)$ are exact whenever $I_i \in \mathcal{A}_i$ is injective. Dually for a right-exact-in-each-variable bifunctor and projectives.

Theorem 4.3.2 (*balancing of derived bifunctors*) Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ have enough injectives and $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ is balanced. Then there are canonical isomorphisms of bifunctors

$$\mathrm{R}^n_{\mathrm{I}} F(X_1, X_2) \simeq \mathrm{R}^n_{\mathrm{II}} F(X_1, X_2). \tag{4.3.2}$$

Dually, if $\mathcal{A}_1$ and $\mathcal{A}_2$ have enough projectives and F is right exact in each variable and balanced, then there are canonical isomorphisms

$$\mathrm{L}_{\mathrm{I}_n} F(X_1, X_2) \simeq \mathrm{L}_{\mathrm{II}_n} F(X_1, X_2). \tag{4.3.3}$$

Proof It suffices to prove the right derived version. Take injective resolutions $0 \to X_i \to I_i^\bullet$ for $i \in \{1, 2\}$, and form the double complex $F(I_1^\bullet, I_2^\bullet)$ (Definition 3.8.1). The inclusions $X_i \hookrightarrow I_i^0$ induce maps of double complexes

$$F(X_1, I_2^\bullet) \to F(I_1^\bullet, I_2^\bullet) \leftarrow F(I_1^\bullet, X_2) \tag{4.3.4}$$

where the leftmost (resp. rightmost) double complex is concentrated on the axis $(0, \bullet)$ (resp. $(\bullet, 0)$). We claim that both arrows induce quasi-isomorphisms after totalization. Once the claim is verified, we see that $\mathrm{tot}(F(I_1^\bullet, I_2^\bullet))$ computes both $\mathrm{R}_{\mathrm{I}}^n F(X_1, X_2)$ and $\mathrm{R}_{\mathrm{II}}^n F(X_1, X_2)$, so these two agree. Functoriality in (X_1, X_2) then follows from Theorem 4.1.6.

Let $X \in \mathrm{Ob}(\mathsf{C}_{\mathrm{f}}^2(\mathcal{B}))$ be the double complex obtained from $F(I_1^\bullet, I_2^\bullet)$ by adding the column $X^{-1,\bullet} := F(X_1, I_2^\bullet)$, the horizontal differential into column 0 being induced by $X_1 \hookrightarrow I_1^0$. Let $f : F(X_1, I_2^\bullet) \to \mathrm{tot}(F(I_1^\bullet, I_2^\bullet))$ be the chain map induced by $X_1 \hookrightarrow I_1^0$, i.e. the totalization of the left arrow above. One readily checks that $\mathrm{tot}(X)$ is the mapping cone of f (Definition 3.6.1): degreewise

$$\mathrm{tot}(X)^n = F\big(X_1, I_2^{n+1}\big) \oplus \mathrm{tot}(F(I_1^\bullet, I_2^\bullet))^n, \tag{4.3.5}$$

and $(-1)^{-1} = -1$ yields precisely the cone differential. Since $F(\cdot, I_2^q)$ is exact for all $q \geq 0$, each row of X is exact, so $\mathrm{tot}(X)$ is acyclic by Theorem 3.8.6. It follows from Proposition 3.6.2 that f is a quasi-isomorphism.

The case of $F(I_1^\bullet, X_2) \to F(I_1^\bullet, I_2^\bullet)$ can be proved similarly. Therefore the claim is justified. □

Balancing says: to compute the derived bifunctor you may resolve either variable. This is precisely what makes the Ext and Tor in Section 4.4 unambiguous.

4.4 Ext and Tor

We apply the theory to the two bifunctors of Section 2.7: Hom and $\otimes_R$, where R is a ring. We also make use of the notion of flat R-modules (Definition B.5.3).

Proposition 4.4.1 Let $\mathcal{A}$ be an abelian category. The bifunctor $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$ is balanced. Hence, if $\mathcal{A}$ has enough injectives *and* enough projectives, there are well-defined bifunctors

$$\mathrm{Ext}^n = \mathrm{Ext}^n_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}, \tag{4.4.1}$$

called the ***Ext functors*** Ext^n, with $\mathrm{Ext}^n(X, Y)$ computed either from a projective resolution of X or an injective resolution of Y.

Proof Since $\mathrm{Hom}(P, \cdot)$ is exact for P projective and $\mathrm{Hom}(\cdot, I)$ is exact for I injective (Section 2.7), and the injective objects of $\mathcal{A}^{\mathrm{op}}$ are precisely the projective objects of $\mathcal{A}$, the bifunctor Hom is balanced. Now apply Theorem 4.3.2. □

Consequently $\mathrm{Ext}^0 = \mathrm{Hom}$, and the families (Ext^n) carry long exact sequences in each variable. Specializing Corollary 4.2.5:

$$X \text{ injective} \Leftrightarrow \mathrm{Ext}^1(\cdot, X) = 0 \Leftrightarrow \mathrm{Ext}^n(\cdot, X) = 0 \quad (n > 0), \tag{4.4.2}$$

$$P \text{ projective} \Leftrightarrow \mathrm{Ext}^1(P, \cdot) = 0 \Leftrightarrow \mathrm{Ext}^n(P, \cdot) = 0 \quad (n > 0). \tag{4.4.3}$$

Remark 4.4.2 The symbol Ext abbreviates *extension*: the elements of $\mathrm{Ext}^1(X,Y)$ are in bijection with the isomorphism classes of extensions $0 \to Y \to E \to X \to 0$ (short exact sequences with end terms Y, X). We shall prove this classification in Theorem 7.6.6, where $\mathrm{Ext}^n(X,Y)$ reappears as the morphisms $X \to Y[n]$ in the derived category (Definition 7.6.4).

Convention 4.4.3 For $\mathcal{A} = R\text{-}\mathsf{Mod}$ where R is a ring, we denote the corresponding Ext bifunctors as Ext^n_R.

Proposition 4.4.4 For any ring R, the bifunctor $\otimes_R : \mathsf{Mod}\text{-}R \times R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is balanced. Hence there are well-defined bifunctors

$$\mathrm{Tor}_n = \mathrm{Tor}_n^R : \mathsf{Mod}\text{-}R \times R\text{-}\mathsf{Mod} \to \mathsf{Ab}, \tag{4.4.4}$$

called the ***Tor functors*** Tor_n^R, computed from a projective resolution of either variable (or, more generally, a flat resolution, defined below).

Proof Every projective module is flat (Proposition B.5.4), so $P \otimes_R (\cdot)$ and $(\cdot) \otimes_R P$ are exact; apply the left-derived version of Theorem 4.3.2. $\square$

A ***flat resolution*** of an R-module M is an exact sequence

$$\cdots \to F_1 \to F_0 \to M \to 0, \quad \text{each } F_i \text{ flat}. \tag{4.4.5}$$

Thus $\mathrm{Tor}_0^R(X,Y) = X \otimes_R Y$, and a module M is flat iff $\mathrm{Tor}_n^R(M,\cdot) = 0$ (iff $\mathrm{Tor}_n^R(\cdot, M) = 0$) for all $n > 0$. Because flat resolutions compute Tor (Corollary 4.2.14), Tor can be computed by resolving either variable with flat modules, which is more practical than taking projective resolutions.

Remark 4.4.5 (*bimodule structure*) Let A, B, R be rings. If X is an (A,R)-bimodule and Y an (R,B)-bimodule, then $\mathrm{Tor}_n^R(X,Y)$ is naturally an (A,B)-bimodule: each $a \in A$ acts through the endomorphism $x \mapsto ax$ of X as a right R-module, and each $b \in B$ through $y \mapsto yb$ on Y. For $n = 0$ this recovers the standard structure on $X \otimes_R Y$; in general the A-action is read degreewise on $X \otimes_R F_\bullet$ for any flat resolution $F_\bullet \to Y$, and symmetrically (resolving X) for the B-action. In particular, a ring homomorphism $R \to S$ upgrades $\mathrm{Tor}_n^R(\cdot, S)$ (resp. $\mathrm{Tor}_n^R(S, \cdot)$) to a functor $\mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$ (resp. $R\text{-}\mathsf{Mod} \to S\text{-}\mathsf{Mod}$).

Likewise, for an (R,A)-bimodule X and an (R,B)-bimodule Y, each $\mathrm{Ext}^n_R(X,Y)$ is naturally an (A,B)-bimodule: contravariance turns the right A-action on X into a left one.

When R is commutative, Tor lands in $R\text{-}\mathsf{Mod}$, and the symmetry of the tensor product gives a canonical isomorphism $\mathrm{Tor}_n^R(X,Y) \simeq \mathrm{Tor}_n^R(Y,X)$.

Remark 4.4.6 The symbol Tor abbreviates *torsion*, from the following special case. Let $t \in R$ be a non-zero-divisor, so that the left ideal Rt is canonically isomorphic to R. For any right R-module X, dimension shifting (Proposition 4.2.13) applied to $0 \to R \xrightarrow{t} R \to R/Rt \to 0$ gives

$$\mathrm{Tor}_1^R(X, R/Rt) \simeq \ker\Big(X \simeq X \otimes_R R \xrightarrow{t} X\Big) = \{x \in X : xt = 0\}, \tag{4.4.6}$$

the t-torsion submodule of X, and $\mathrm{Tor}_n^R(X, R/Rt) = 0$ for $n > 1$.

The following example is connected to the Hochschild theory expounded in Section 3.5.

Example 4.4.7 (*Hochschild (co)homology as Ext and Tor*) Consider the setting of Section 3.5, and assume that R is projective as a $\Bbbk$-module. Then so is every $R^{\otimes n}$: by the Hom–tensor adjunction (Theorem B.4.3), $\mathrm{Hom}_{\Bbbk}(P \otimes Q, \cdot) \simeq \mathrm{Hom}_{\Bbbk}(Q, \mathrm{Hom}_{\Bbbk}(P, \cdot))$ is a composition of exact functors. Consequently the functor $\mathrm{Hom}_{R^e}(\mathsf{B}_n R, \cdot)$ on left R^e-modules is, via the isomorphism $\mathrm{Hom}_{R^e}(\mathsf{B}_n R, M) \simeq \mathrm{Hom}_{\Bbbk}(R^{\otimes n}, M)$ of Section 3.5, $\mathrm{Hom}_{\Bbbk}(R^{\otimes n}, \cdot)$ precomposed with the exact forgetful functor on underlying $\Bbbk$-modules, hence exact; that is, $\mathsf{B}_n R$ is a projective left R^e-module (Section 2.7). The bar resolution is thus a projective resolution of R over R^e, and since Ext can be computed by resolving either variable (Proposition 4.4.1),

$$\mathrm{HH}^n(M) = \mathrm{H}^n(\mathrm{Hom}_{R^e}(\mathsf{B}R, M)) = \mathrm{Ext}^n_{R^e}(R, M). \tag{4.4.7}$$

Next, assume instead that R is flat as a $\Bbbk$-module. Then so is every $R^{\otimes n}$, since $(\cdot) \otimes R^{\otimes n}$ is an n-fold composition of the exact functor $(\cdot) \otimes R$. Under the isomorphism $M \otimes_{R^e} \mathsf{B}_n R \simeq M \otimes R^{\otimes n}$, the functor $(\cdot) \otimes_{R^e} \mathsf{B}_n R$ on (R, R)-bimodules is thus the exact forgetful passage to $\Bbbk$-modules followed by the exact functor $(\cdot) \otimes R^{\otimes n}$; hence $\mathsf{B}_n R$ is a flat left R^e-module (Definition B.5.3). The bar resolution is thus a flat resolution of R over R^e, and since flat resolutions compute Tor (Proposition 4.4.4, Corollary 4.2.14),

$$\mathrm{HH}_n(M) = \mathrm{H}_n(M \otimes_{R^e} \mathsf{B}R) = \mathrm{Tor}_n^{R^e}(M, R). \tag{4.4.8}$$

Hochschild theory is thereby identified with Ext and Tor over R^e, so any projective or flat resolution of R over R^e will compute it. For instance, let $R = \Bbbk[t]$, which is free over $\Bbbk$. Identifying R^e with $\Bbbk[x, y]$ turns R into the R^e-module with scalar multiplication $h(x, y) \cdot f(t) = h(t, t) f(t)$, and there is a free resolution

$$0 \to R^e \xrightarrow{x-y} R^e \to R \to 0, \tag{4.4.9}$$

exact because the kernel of $h \mapsto h(t, t)$ is the ideal generated by the non-zero-divisor $x - y$. Applying $(\cdot) \otimes_{R^e} R$ and $\mathrm{Hom}_{R^e}(\cdot, R)$ turns both differentials into multiplication by $t - t = 0$, whence

$$\begin{aligned} \mathrm{HH}_n(R) &\simeq R \simeq \mathrm{HH}^n(R) \quad (n = 0, 1), \\ \mathrm{HH}_n(R) &= 0 = \mathrm{HH}^n(R) \quad (n > 1). \end{aligned} \tag{4.4.10}$$

Finally, let C (resp. D) be a chain complex of right (resp. left) R-modules (Remark 2.2.3). The tensor double chain complex $C_\bullet \otimes_R D_\bullet$ is totalized by direct sums (cf. Definition 3.8.3) into a chain complex

$$C \otimes_R D := \left(\bigoplus_{p+q=n} C_p \otimes_R D_q \right)_{n \in \mathbb{Z}}, \tag{4.4.11}$$

with differential $d_p^C \otimes \mathrm{id} + (-1)^p \,\mathrm{id} \otimes d_q^D$ on the (p, q)-summand. If $c \in C_p$ and $e \in D_q$ are cycles, then $c \otimes e \in (C \otimes_R D)_{p+q}$ is again a cycle, and replacing c or e by a boundary changes $c \otimes e$ by a boundary. We thus obtain a canonical homomorphism

$$\kappa : \bigoplus_{p+q=n} \mathrm{H}_p(C) \otimes_R \mathrm{H}_q(D) \to \mathrm{H}_n(C \otimes_R D). \tag{4.4.12}$$

Theorem 4.4.8 (*Künneth formula*) Let C be a chain complex of right R-modules and D a chain complex of left R-modules. If C_p and $\mathrm{im}(d_p^C)$ are flat for each p, then for each n there is a natural short exact sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C) \otimes_R \mathrm{H}_q(D) &\xrightarrow{\kappa} \mathrm{H}_n(C \otimes_R D) \\ &\to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{H}_p(C), \mathrm{H}_q(D)) \to 0. \end{aligned} \tag{4.4.13}$$

Proof Put $Z_p := \ker(d_p^C) \subset C_p$ and $B_p := \mathrm{im}(d_{p+1}^C)$, and let Z and B be the complexes $(Z_p)_p$ and $(B_p)_p$ with zero differentials. Since $\mathrm{Tor}_{\geq 1}$ vanishes on flat modules (Proposition B.5.4), the long exact sequence of Tor in the first variable (Theorem 4.2.3), applied to the exact sequences

$$0 \to Z_p \to C_p \xrightarrow{d_p^C} B_{p-1} \to 0 \quad (p \in \mathbb{Z}), \tag{4.4.14}$$

shows both that each Z_p is flat and that these sequences remain exact after tensoring with any D_q. Since direct sums of modules preserve exactness, we obtain a short exact sequence of complexes

$$0 \to Z \otimes_R D \to C \otimes_R D \to B[-1] \otimes_R D \to 0, \tag{4.4.15}$$

where $B[-1]$ is the shift of Definition 3.1.2; the surjection is $(d_p^C \otimes \mathrm{id})_p$.

Now $d^Z = d^B = 0$, so $Z \otimes_R D$ and $B[-1] \otimes_R D$ are direct sums of the complexes $Z_p \otimes_R D$ and $B_{p-1} \otimes_R D$, each with differential $\mathrm{id} \otimes d^D$ up to signs, and homology commutes with direct sums. For a flat R-module F, by Proposition 3.3.2 we have $\mathrm{H}_q(F \otimes_R D) = F \otimes_R \mathrm{H}_q(D)$; the long exact homology sequence therefore reads

$$\begin{aligned} \bigoplus_{p+q=n} B_p \otimes_R \mathrm{H}_q(D) &\xrightarrow{\partial_{n+1}} \bigoplus_{p+q=n} Z_p \otimes_R \mathrm{H}_q(D) \to \mathrm{H}_n(C \otimes_R D) \\ &\to \bigoplus_{p+q=n-1} B_p \otimes_R \mathrm{H}_q(D) \xrightarrow{\partial_n} \bigoplus_{p+q=n-1} Z_p \otimes_R \mathrm{H}_q(D). \end{aligned} \tag{4.4.16}$$

Tracking the construction of the connecting morphism (a routine diagram chase, cf. [10, Theorem 3.14.12] and the discussion preceding [9, Proposition 6.8.6]), one finds that ∂_{n+1} and ∂_n are, up to the evident signs, induced degreewise by the inclusions $B_p \hookrightarrow Z_p$. Consequently:

- $\mathrm{coker}(\partial_{n+1}) \simeq \bigoplus_{p+q=n} \mathrm{H}_p(C) \otimes_R \mathrm{H}_q(D)$, since the tensor product is right exact, and the induced monomorphism into $\mathrm{H}_n(C \otimes_R D)$ is precisely κ;
- $\ker(\partial_n) \simeq \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{H}_p(C), \mathrm{H}_q(D))$, because $0 \to B_p \to Z_p \to \mathrm{H}_p(C) \to 0$ is a flat resolution of $\mathrm{H}_p(C)$, thus computes Tor by Proposition 4.4.4 and Corollary 4.2.14.

Reading $0 \to \mathrm{coker}(\partial_{n+1}) \to \mathrm{H}_n(C \otimes_R D) \to \ker(\partial_n) \to 0$ off the long exact sequence yields the claim, naturally in (C, D). □

When D is concentrated in degree 0, the short exact sequence of Theorem 4.4.8 becomes the ***universal coefficient theorem*** for homology. Over a division ring, Tor_1 vanishes (Exercise 4.4.9) and κ is an isomorphism.

We will revisit the Künneth formula in Chapter 6 by means of the spectral sequence of a double complex.

Exercise 4.4.9 Let D be a division ring. Show that $\operatorname{Ext}^n_D = 0$ and $\operatorname{Tor}^D_n = 0$ for all $n > 0$.

Exercise 4.4.10 Let R be a principal ideal domain and M, N finitely generated R-modules. Describe $\operatorname{Ext}^n_R(M, N)$ and $\operatorname{Tor}^R_n(M, N)$ for all $n \geq 0$ in terms of the invariant-factor decompositions of M and N. *Hint.* Use the structure theorem and free resolutions.

Exercise 4.4.11 Show that $\operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Q}/\mathbb{Z}, M)$ is the torsion subgroup of any abelian group M (the subgroup of elements of finite order). *Hint.* Resolve $\mathbb{Q}/\mathbb{Z}$ using $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, or use a presentation of M.

Exercise 4.4.12 Fix a ring R. Prove that Tor^R commutes with filtered colimits: for any filtered system $(X_i)_i$ we have

$$\operatorname{Tor}^R_n\Big(\operatorname*{colim}_i X_i, Y\Big) \simeq \operatorname*{colim}_i \operatorname{Tor}^R_n(X_i, Y), \tag{4.4.17}$$

and similarly in the second variable. Does Ext commute with filtered colimits? Give a counterexample or a proof.

4.5 Derived inverse limits

Our next example is of a different flavor: we derive the inverse limit functor $\varprojlim$. Regard $(\mathbb{Z}_{\geq 0}, \leq)$ as a category, so that $\mathbb{Z}^{\mathrm{op}}_{\geq 0}$ is the category $\cdots \to 2 \to 1 \to 0$. We will see that $\varprojlim$ has exactly one nonzero derived functor, namely $\lim^1$ (Theorem 4.5.4), and that the Mittag-Leffler condition of Definition 4.5.5 gives a practical criterion for its vanishing (Proposition 4.5.6).

Definition 4.5.1 The category of ***inverse systems*** in $\mathcal{A}$ is the functor category $\mathsf{InvSys}(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}^{\mathrm{op}}_{\geq 0}}$. An object is a family $(A_n, f_n)_{n \geq 0}$ with $f_n : A_{n+1} \to A_n$.

When $\mathcal{A}$ is abelian, so is $\mathsf{InvSys}(\mathcal{A})$, kernels and cokernels being formed degreewise; in particular a morphism of systems is monic (resp. epic) if and only if it is so degreewise.

Convention 4.5.2 We say an abelian category $\mathcal{A}$ has ***exact countable products*** if the product functor $\prod : \mathcal{A}^{\mathbb{Z}_{\geq 0}} \to \mathcal{A}$ is exact; being in any case left exact, it suffices that it preserve epimorphisms. Dually for countable coproducts. Every module category R-Mod has exact countable products.

Throughout this section, $\mathcal{A}$ denotes an abelian category with exact countable products; in particular $\mathcal{A}$ admits countable limits. For an inverse system $A = (A_n, f_n)_n$, define the ***translation*** $T_A : \prod_n A_n \to \prod_n A_n$ by $T_A\big((a_n)_n\big) = (f_n(a_{n+1}))_n$ and set

$$\Delta_A := T_A - \mathrm{id} : \prod_n A_n \to \prod_n A_n. \tag{4.5.1}$$

As A varies, these are the components of endomorphisms T and $\Delta = T - \mathrm{id}$ of the product functor $A \mapsto \prod_n A_n$. Then $\varprojlim A = \ker(\Delta_A)$, the equalizer of T_A and id.

Definition 4.5.3 Define functors $\mathsf{InvSys}(\mathcal{A}) \to \mathcal{A}$ by

$$\lim\nolimits^0 A := \ker(\Delta_A) = \varprojlim A, \quad \lim\nolimits^1 A := \operatorname{coker}(\Delta_A), \quad \lim\nolimits^n A := 0 \text{ for } n \geq 2. \tag{4.5.2}$$

Given $X, Y, Z \in \mathsf{InvSys}(\mathcal{A})$, applying the snake lemma (Theorem 2.5.2) to

- the degreewise short exact sequence $0 \to X \to Y \to Z \to 0$, and

- the canonical morphism Δ between the products

gives the long exact sequence

$$0 \to \lim^0 X \to \lim^0 Y \to \lim^0 Z \xrightarrow{\delta^0} \lim^1 X \to \lim^1 Y \to \lim^1 Z \to 0, \tag{4.5.3}$$

natural in the short exact sequence. Setting $\delta^n = 0$ for $n \geq 1$, we conclude that $(\lim^n, \delta^n)_{n\geq 0}$ is a cohomological δ-functor with $\lim^0 = \varprojlim$; in particular $\varprojlim$ is left exact.

We now turn to the right derived functors of $\varprojlim$, assuming in addition that $\mathcal{A}$ has enough injectives; the proof of Theorem 4.5.4 then supplies enough injectives on $\mathsf{InvSys}(\mathcal{A})$ as well, so that $\mathrm{R}^n \varprojlim$ is defined (Section 4.2).

Theorem 4.5.4 (*Eilenberg*) There are canonical isomorphisms $\mathrm{R}^1 \varprojlim \simeq \lim^1$ and $\mathrm{R}^n \varprojlim = 0$ for $n \notin \{0, 1\}$.

Proof For each $m \in \mathbb{Z}_{\geq 0}$ let $\mathrm{ev}_m : \mathsf{InvSys}(\mathcal{A}) \to \mathcal{A}$ be the exact evaluation functor $A \mapsto A_m$, and let $\mathcal{R}_m : \mathcal{A} \to \mathsf{InvSys}(\mathcal{A})$ send X to the system $\mathcal{R}_m(X)$ with

$$\mathcal{R}_m(X)_n = \begin{cases} X & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases} \tag{4.5.4}$$

and identity transition maps. A morphism $g : A \to \mathcal{R}_m(X)$ is a family of morphisms $g_n : A_n \to X$ $(n \geq m)$ with $g_{n+1} = g_n f_n$, uniquely determined by its component g_m, and every $g_m : A_m \to X$ arises in this way; in other words $\mathrm{Hom}(A, \mathcal{R}_m(X)) \simeq \mathrm{Hom}(A_m, X)$ naturally, i.e. ev_m is left adjoint to $\mathcal{R}_m$. Hence $\mathcal{R}_m$ preserves injectives by Theorem 2.7.5.

Given $A = (A_n, f_n)_n$, choose monomorphisms $u_m : A_m \hookrightarrow I_m$ into injectives, and let $\varphi : A \to R := \prod_{m\geq 0} \mathcal{R}_m(I_m)$ correspond to the family $(u_m)_m$ under

$$\mathrm{Hom}(A, R) \simeq \prod_{m\geq 0} \mathrm{Hom}(A, \mathcal{R}_m(I_m)) \simeq \prod_{m\geq 0} \mathrm{Hom}_{\mathcal{A}}(A_m, I_m). \tag{4.5.5}$$

Identifying $R_n = \prod_{m\leq n} I_m$, one computes

$$\begin{gathered} \varphi_n = (u_m f^n_m)_{m\leq n} : A_n \to R_n, \\ f^n_m := f_m f_{m+1} \cdots f_{n-1} : A_n \to A_m \quad (n \geq m; \quad f^m_m = \mathrm{id}), \end{gathered} \tag{4.5.6}$$

as in Definition 4.5.5; this is compatible with the transitions since $f^{n+1}_m = f^n_m f_n$. The component of φ_n at $m = n$ is the monomorphism u_n, so φ is a monomorphism; R being injective by Proposition 2.7.3, this also shows that $\mathsf{InvSys}(\mathcal{A})$ has enough injectives.

The map Δ_R is epimorphic: writing $r_n : R_{n+1} \to R_n$ for the transition, whose components are id for $m \leq n$ and 0 on I_{n+1}, and s_n for its section with zero I_{n+1}-component, the recursive prescription $a_0 := 0$, $a_{n+1} := s_n(a_n + b_n)$ for $b = (b_n)_n \in \prod_n R_n$ defines an endomorphism $b \mapsto a$ of $\prod_n R_n$ with

$$(\Delta_R(a))_n = r_n(a_{n+1}) - a_n = b_n \quad (n \geq 0), \tag{4.5.7}$$

so $b \mapsto a$ is a right inverse of Δ_R and $\lim^1 R = \mathrm{coker}(\Delta_R) = 0$. Although the preceding argument involves elements, it can be easily rewritten in a purely categorical manner.

All in all, every A embeds into some R with $\lim^1 R = 0$; the functor $\lim^1$ is therefore effaceable (Definition 4.2.7), and so is $\lim^n$ for $n \geq 2$ (being zero). By Proposition 4.2.10, $(\lim^n, \delta^n)_{n\geq 0}$ is a universal cohomological δ-functor; so it is canonically isomorphic to $(\mathrm{R}^n \varprojlim)_n$ by Corollary 4.2.11, and $\mathrm{R}^n \varprojlim = 0$ for $n < 0$ by Proposition 4.2.4. This concludes the proof. □

Thus $\lim^1$ is the sole obstruction to the exactness of $\varprojlim$ (Corollary 4.2.5). When does it vanish?

Definition 4.5.5 An inverse system $(X_k, f_k)_{k\geq 0}$ is ***Mittag-Leffler*** if for each k there exists $N \geq k$ with $\mathrm{im}(f_k^n) = \mathrm{im}(f_k^N)$ for all $n \geq N$, where $f_a^b : X_b \to X_a$ denotes the composite $f_a \cdots f_{b-1}$ (with $f_a^a = \mathrm{id}$).

For example, any system with epimorphic transition maps is Mittag-Leffler (take $N = k$).

Proposition 4.5.6 For $\mathcal{A} = R\text{-}\mathsf{Mod}$, every Mittag-Leffler inverse system X satisfies $\lim^1 X = 0$; equivalently, $\varprojlim$ is exact on short exact sequences $0 \to A \to B \to C \to 0$ with A Mittag-Leffler.

Proof Consider first a system with *surjective* transition maps: choosing set-theoretic sections s_n of the f_n, the recursion $a_0 := 0$, $a_{n+1} := s_n(a_n + b_n)$ defines a right inverse of Δ_A (see the proof of Theorem 4.5.4), whence $\lim^1 = 0$ for such systems.

In general, the Mittag-Leffler condition yields for each k a stable image $X'_k := \mathrm{im}(f_k^n)$, valid for all $n \geq N$; taking the thresholds N nondecreasing in k, these form a subsystem $X' \subset X$ with surjective transition maps, and every $x = (x_k)_k \in \varprojlim X$ lies in $\varprojlim X'$ as $x_k = f_k^n(x_n)$ for $n \gg k$, hence $\varprojlim X' \simeq \varprojlim X$ and $\lim^1 X' = 0$. For the quotient $Q := X/X'$, the transition composite $Q_n \to Q_k$ is zero whenever $n \gg k$, thus for all $b = (b_k)_k \in \prod_k Q_k$ the sums

$$a_k := -\sum_{s\geq 0} f_k^{k+s}(b_{k+s}) \tag{4.5.8}$$

are finite and satisfy $\big(\Delta_Q(a)\big)_k = b_k$, whence $\lim^1 Q = 0$. The long exact sequence applied to $0 \to X' \to X \to Q \to 0$ now gives $\lim^1 X \simeq \lim^1 Q = 0$. As for the equivalence of the two assertions: the forward direction is the long exact sequence, while for the converse one embeds a Mittag-Leffler A into an injective system as in Theorem 4.5.4 and applies dimension shifting (Proposition 4.2.13). See [10, Proposition 3.13.13] for an alternative argument. □

Remark 4.5.7 For a general abelian category with exact countable products, the Mittag-Leffler condition need not imply $\lim^1 = 0$; counterexamples were found by Roos [15].

Finally, take $\mathcal{A} = \mathsf{C}(R\text{-}\mathsf{Mod})$, the category of complexes of R-modules. It has exact countable products: limits of complexes are constructed degreewise, and exactness is likewise checked degreewise. In this setting, combining $\lim^1$ with the long exact sequence in cohomology gives the ***Milnor sequence*** in Exercise 4.5.8 below, which is handy for studying homotopy groups.

Exercise 4.5.8 Let $(X_k, f_k)_{k\geq 0}$ be an inverse system in $\mathsf{C}(R\text{-}\mathsf{Mod})$ such that $(X_k^n)_k$ is Mittag-Leffler for every degree n. Establish the short exact sequence

$$0 \to \lim\nolimits_k^1 \mathrm{H}^{n-1}(X_k) \to \mathrm{H}^n(\varprojlim\nolimits_k X_k) \to \varprojlim\nolimits_k \mathrm{H}^n(X_k) \to 0 \tag{4.5.9}$$

by the following steps. Write $B_k^n \subset Z_k^n \subset X_k^n$ for the boundaries and cycles, and put $H_k^n := Z_k^n / B_k^n$; view B_k, Z_k, and H_k as complexes with zero differential.

- Use the Mittag-Leffler condition on $\big(X_k^{n-1}\big)$ to deduce one on $(B_k^n) = \mathrm{im}\big(d^{n-1} : X_k^{n-1} \to X_k^n\big)$.

- By Proposition 4.5.6 applied degreewise, $\lim^1_k X_k = \lim^1_k B_k = 0$. Apply the long exact sequence of $(\lim^0, \lim^1)$ to $0 \to Z_k \to X_k \to B_k[1] \to 0$ and to $0 \to B_k \to Z_k \to H_k \to 0$ to relate the limits of Z_k, B_k, and H_k.
- Let B^n denote the boundaries of the complex $\varprojlim_k X_k$; splice the resulting sequences via the inclusions $B^n \subset \varprojlim_k B^n_k \subset \varprojlim_k Z^n_k$.

4.6 K-injective and K-projective complexes

The resolutions of Section 4.1 require boundedness: injective resolutions are guaranteed only in C^+, projective only in C^-. For unbounded complexes, which are essential for Chapter 7, one needs a broader notion introduced by Spaltenstein [16].

Definition 4.6.1 Let $\mathcal{A}$ be abelian.
- A complex $I \in \mathsf{C}(\mathcal{A})$ is ***K-injective*** if $\mathrm{Hom}^\bullet(X, I)$ is acyclic whenever X is acyclic. A ***K-injective resolution*** of A is a quasi-isomorphism $A \to I$ with I being K-injective.
- A complex P is ***K-projective*** if $\mathrm{Hom}^\bullet(P, X)$ is acyclic whenever X is acyclic. A ***K-projective resolution*** of A is a quasi-isomorphism $P \to A$ with P being K-projective.

If every complex in $\mathsf{C}(\mathcal{A})$ admits a K-injective (resp. K-projective) resolution, we say $\mathcal{A}$ has enough K-injectives (resp. K-projectives).

The two notions are dual. In practice the following reformulation is most useful.

Lemma 4.6.2 I is K-injective if and only if $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) = 0$ for every acyclic X. Dually, P is K-projective iff $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P, X) = 0$ for every acyclic X.

Proof By Proposition 3.4.2 applied in degree 0 to $(X[-n], I)$ (see Definition 3.4.1; note that $\mathrm{Hom}^n(X, I) = \mathrm{Hom}^0(X[-n], I)$),

$$\mathrm{H}^n(\mathrm{Hom}^\bullet(X, I)) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X[-n], I), \tag{4.6.1}$$

and X is acyclic iff $X[-n]$ is. So all cohomology of $\mathrm{Hom}^\bullet(X, I)$ vanishes iff all these Hom-groups vanish; the case of P is dual, using $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P, X[n])$. □

Example 4.6.3 Lemma 4.1.5 says precisely that any $I \in \mathsf{C}^+(\mathcal{A})$ built from injectives is K-injective, and dually for objects of $\mathsf{C}^-(\mathcal{A})$ built from projectives. Thus bounded injective (projective) resolutions are special cases of K-injective (K-projective) resolutions.

Remark 4.6.4 (*Dold's example*) The boundedness hypothesis is essential. Over the ring $\mathbb{Z}/4\mathbb{Z}$, the acyclic complex

$$\cdots \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \cdots \tag{4.6.2}$$

is built from free (hence projective) modules termwise, yet is *not* K-projective: otherwise, by Lemma 4.6.2, the identity would have to be null-homotopic, which is impossible since additive functors preserve null-homotopies while tensoring with $\mathbb{Z}/2\mathbb{Z}$ yields a complex with non-zero cohomology.

K-injective and K-projective resolutions both satisfy the same homotopy-uniqueness as Theorem 4.1.6; see [10, Theorem 3.15.5].

Theorem 4.6.5 If $\mathcal{A}$ has exact countable products and enough injectives, then $\mathcal{A}$ has enough K-injectives. Dually, if $\mathcal{A}$ has exact countable coproducts and enough projectives, it has enough K-projectives.

Proof (*Sketch*) For $A \in \mathsf{C}(\mathcal{A})$, resolve each truncation $\tau^{\geq -k}A$ (Definition 3.7.1) by a bounded-below injective resolution $f_k : \tau^{\geq -k}A \to I_k$, arranged via the horseshoe lemma (Proposition 4.1.9) so that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & I_2 \arrow[r] & I_1 \arrow[r] & I_0 \\ \cdots \arrow[r] & \tau^{\geq -2}A \arrow[r] \arrow[u, "f_2"] & \tau^{\geq -1}A \arrow[r] \arrow[u, "f_1"'] & \tau^{\geq 0}A \arrow[u, "f_0"] \end{tikzcd} \tag{4.6.3}$$

and the transitions $I^n_{k+1} \to I^n_k$ are split epimorphisms. The inverse limit $I = \varprojlim_k I_k$ exists since $\mathcal{A}$ has countable products, and a Mittag-Leffler argument (Proposition 4.5.6, applied degree-wise to the $\mathrm{Hom}^\bullet(X, I_k)$ with X acyclic) shows that I is K-injective. The canonical morphisms $A \to \tau^{\geq -k}A$ induce an isomorphism $A \simeq \varprojlim_k \tau^{\geq -k}A$, so the f_k furnish a morphism $A \to I$; a five-lemma check using the exactness of countable products verifies that it is a quasi-isomorphism.

The details are somewhat roundabout. See [10, Lemma 3.15.9 – Theorem 3.15.11]. □

Example 4.6.6 For any ring R, the module category R-Mod is abelian with enough injectives (resp. projectives) and exact countable products (resp. coproducts), so it has enough K-injectives (resp. enough K-projectives).

In Section 8.1, Section 8.5 and Section 8.6 we shall use K-injective and K-projective resolutions to define the *total* derived functor $\mathrm{R}\,F$ and the derived bifunctors RHom and $\overset{\mathrm{L}}{\otimes}$ on the unbounded derived category, recovering the classical $\mathrm{R}^n\,F$, Ext^n, Tor_n of this chapter as their cohomology.

Exercise 4.6.7 Let I be an object of an abelian category $\mathcal{A}$, regarded as a complex concentrated in degree 0. Show that I is an injective object of $\mathcal{A}$ if and only if it is K-injective as a complex. State and prove the dual statement. *Hint.* For the converse, view a short exact sequence $0 \to A \to B \to C \to 0$ as an acyclic complex and extend a map $A \to I$ using the K-injective property.

Chapter 5

Serre Categories and Grothendieck Groups

We gather two short topics that round out the abelian-categorical picture and point forward. The first is structural: a notion of *subcategory* stable under short exact sequences: the ***Serre subcategory*** , together with the operation of *forming the quotient* by one (Section 5.1). The second is the ***Grothendieck group*** $\mathrm{K}_0(\mathcal{A})$ of $\mathcal{A}$ (Section 5.3). Its highlight is the *Euler–Poincaré principle*, which turns the alternating sum $\sum_i (-1)^i[\mathrm{H}^i(X)]$ of the cohomology of a complex into an alternating sum of its terms (Theorem 5.3.4). Finally, the Grothendieck group and the Serre quotient are shown to be linked by a short exact sequence (Section 5.4). All these discussions follow [10, Chapter 2, §9].

5.1 Abelian subcategories and Serre subcategories

A full subcategory of an abelian category is preadditive (Definition 1.7.1), thus it is fair to ask whether it is itself abelian, and whether the inclusion is exact.

Definition 5.1.1 A full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is an ***abelian subcategory*** of $\mathcal{A}$ if $\mathcal{B}$ is abelian and the inclusion functor $\iota : \mathcal{B} \to \mathcal{A}$ is exact (Definition 2.2.9).

Proposition 5.1.2 A full subcategory $\mathcal{B} \subset \mathcal{A}$ is an abelian subcategory if and only if

- $0 \in \mathrm{Ob}(\mathcal{B})$;
- $X, Y \in \mathrm{Ob}(\mathcal{B}) \Rightarrow X \oplus Y \in \mathrm{Ob}(\mathcal{B})$ (the biproduct taken in $\mathcal{A}$);
- for every morphism $f : X \to Y$ with $X, Y \in \mathrm{Ob}(\mathcal{B})$, the objects $\ker(f)$ and $\operatorname{coker}(f)$ (taken in $\mathcal{A}$) can be taken in $\mathrm{Ob}(\mathcal{B})$.

Proof The conditions are self-dual, and the "only if" direction is clear: an exact ι preserves zero objects, biproducts, kernels, and cokernels (Lemma 2.2.8; Definition 2.2.9), so all three objects may be computed in $\mathcal{B}$.

Conversely, assume the conditions. As a full subcategory of a preadditive category (Definition 1.7.1), $\mathcal{B}$ is preadditive, and the zero object and the biproducts make it additive (Definition 1.7.3); then the inclusion functor ι is additive. If $f : X \to Y$ is a morphism of $\mathcal{B}$, then $\ker(f)$ and $\operatorname{coker}(f)$, computed in $\mathcal{A}$, lie in $\mathrm{Ob}(\mathcal{B})$ by assumption, and fullness makes them a kernel and a cokernel of f in $\mathcal{B}$ as well; hence ι preserves kernels and cokernels. Since finite limits are built from finite products and equalizers (Proposition 1.5.6), the latter being iterated biproducts and kernels of differences, respectively (Lemma 1.7.4, Definition 1.7.5), ι preserves finite limits, and dually finite colimits. Finally, $\mathcal{B}$ is abelian (Definition 2.1.3): for every morphism f of $\mathcal{B}$, the canonical morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ in Definition 2.1.1 is an isomorphism in $\mathcal{A}$, and since $\mathcal{B}$ is full it is an isomorphism in $\mathcal{B}$ as well. Hence every morphism of $\mathcal{B}$ is strict (Definition 2.1.2). $\square$

The condition we shall use most often is stronger:

Definition 5.1.3 (*J.-P. Serre*) A full subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$ is a ***Serre subcategory*** if $0 \in \mathrm{Ob}(\mathcal{T})$ and, for every short exact sequence

$$0 \to X' \to X \to X'' \to 0 \tag{5.1.1}$$

in $\mathcal{A}$, one has $X \in \mathrm{Ob}(\mathcal{T})$ if and only if $X', X'' \in \mathrm{Ob}(\mathcal{T})$.

Equivalently, $\mathcal{T}$ is closed under subobjects, quotients, and extensions. The "only if" reads off X' and X'' as a subobject and a quotient of X; the "if" is closure under extensions. A ***weak Serre subcategory***[2] weakens the condition to: for every exact $W \to X' \to X \to X'' \to Y$ with $W, X', X'', Y \in \mathcal{T}$, one has $X \in \mathcal{T}$. A weak Serre subcategory is automatically an abelian subcategory: for $X, Y \in \mathrm{Ob}(\mathcal{T})$ and a morphism $f : X \to Y$, apply the defining condition to the exact sequences

$$\begin{gathered} 0 \to X \to X \oplus Y \to Y \to 0, \\ 0 \to 0 \to \ker(f) \to X \xrightarrow{f} Y, \\ X \xrightarrow{f} Y \to \operatorname{coker}(f) \to 0 \to 0 \end{gathered} \tag{5.1.2}$$

(the first is the biproduct sequence Example 2.2.7): the outer terms all lie in $\mathcal{T}$, thus $X \oplus Y$, $\ker(f)$, and $\operatorname{coker}(f)$ belong to $\mathrm{Ob}(\mathcal{T})$, and Proposition 5.1.2 applies.

Example 5.1.4 If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, then the full subcategory

$$\mathcal{T} = \{X \in \mathcal{A} \mid F(X) = 0\} \tag{5.1.3}$$

is a Serre subcategory of $\mathcal{A}$, denoted $\ker(F)$: exactness of F turns a short exact sequence in $\mathcal{A}$ into one in $\mathcal{B}$, and the characterization is immediate.

Example 5.1.5 For a commutative ring R, the Noetherian R-modules form a Serre subcategory of R-Mod, as do the Artinian R-modules. This is standard module theory: subquotients and extensions of Noetherian (resp. Artinian) modules are Noetherian (resp. Artinian).

Exercise 5.1.6 Show that a weak Serre subcategory (Definition 5.1.3, footnote) is closed under isomorphism: if $X \in \mathcal{A}$ is isomorphic to an object of $\mathcal{T}$, then $X \in \mathcal{T}$. Apply the defining condition to $0 \to 0 \to X \simeq Y \to 0 \to 0$.

Exercise 5.1.7 Show that the torsion abelian groups form a Serre subcategory $\mathcal{T}$ of Ab, and identify the Serre quotient $\mathsf{Ab}/\mathcal{T}$. *Hint.* Tensor with $\mathbb{Q}$.

5.2 The Serre quotient

Given a Serre subcategory $\mathcal{T} \subset \mathcal{A}$, we wish to annihilate the objects of $\mathcal{T}$ universally. The result is the Serre quotient $\mathcal{A}/\mathcal{T}$.

Theorem 5.2.1 Let $\mathcal{T}$ be a Serre subcategory of an abelian category $\mathcal{A}$. There exist an abelian category $\mathcal{A}/\mathcal{T}$ and an exact, essentially surjective functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ with $\ker(Q) = \mathcal{T}$, satisfying the following universal property: for every abelian category $\mathcal{B}$ and every exact

[2] This looser variant is tailored to the derived category; see [10, Chapter 4].

functor $F : \mathcal{A} \to \mathcal{B}$ with $\ker(F) \supset \mathcal{T}$, there is a unique exact functor $G : \mathcal{A}/\mathcal{T} \to \mathcal{B}$ with $F = G \circ Q$. This is illustrated by the following (strictly) commutative diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{A}/\mathcal{T} \arrow[d, dashed, "\exists! G"] \\ & \mathcal{B} \end{tikzcd} \tag{5.2.1}$$

The functor G is faithful if and only if $\ker(F) = \mathcal{T}$. Such a category $\mathcal{A}/\mathcal{T}$, equipped with the functor Q from $\mathcal{A}$, is called the ***Serre quotient*** of $\mathcal{A}$ by $\mathcal{T}$.

Proof (*Sketch*) Let S be the class of morphisms f in $\mathcal{A}$ with $\ker(f), \operatorname{coker}(f) \in \mathcal{T}$. One checks that S is a multiplicative system (Definition 1.8.5): (S1) is clear, and (S2) follows from the exact sequences

$$\begin{gathered} 0 \to \ker(f) \to \ker(gf) \xrightarrow{f} \ker(g), \\ \operatorname{coker}(f) \xrightarrow{g} \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0, \end{gathered} \tag{5.2.2}$$

since $\mathcal{T}$ is closed under subobjects, quotients, and extensions; the Ore (S3) and cancellation (S4) conditions follow by forming pullbacks and kernels, and the axioms are self-dual, so S is a right multiplicative system as well. The Serre quotient is then the localization $\mathcal{A}[S^{-1}]$ of Definition 1.8.4, which has the same objects as $\mathcal{A}$, so Q is essentially surjective. Next, $X \in \ker(Q)$ means $\mathrm{id}_{Q(X)} = Q(\mathrm{id}_X) = 0$, which by Proposition 1.8.8 holds precisely when some zero morphism $U \to X$ belongs to S; such a morphism has cokernel X, and conversely the zero map $0 \to X$ lies in S whenever $X \in \mathrm{Ob}(\mathcal{T})$. Hence $\ker(Q) = \mathcal{T}$. If F is exact with $\ker(F) \supset \mathcal{T}$, then F inverts every $s \in S$, by exactness applied to $\ker(s) \to X \xrightarrow{s} Y \to \operatorname{coker}(s)$, so the universal property of the localization yields the unique G with $F = G \circ Q$. Using that every morphism of the quotient is a fraction $Q(a)Q(s)^{-1}$ with $s \in S$, one checks that kernels and cokernels in $\mathcal{A}/\mathcal{T}$ are computed on representatives; it follows that Q is exact, and the same computation applied to $F = G \circ Q$ shows that G is exact. Finally, an exact functor between abelian categories is faithful if and only if it annihilates no nonzero object (it kills a morphism precisely when it kills its image); in view of $F = G \circ Q$ and $\ker(Q) = \mathcal{T}$, this criterion for G says precisely that G is faithful if and only if $\ker(F) = \mathcal{T}$. For the omitted details see [10, §2.9]. □

Example 5.2.2 Let $S \subset R$ be a multiplicative subset of a commutative ring R, and let $\mathcal{T} \subset R\text{-}\mathsf{Mod}$ consist of the modules M annihilated by S (every $m \in M$ is annihilated by some $s \in S$). Then $\mathcal{T}$ is a Serre subcategory, and the Serre quotient is equivalent to the localized module category:

$$R\text{-}\mathsf{Mod}/\mathcal{T} \simeq R[S^{-1}]\text{-}\mathsf{Mod}. \tag{5.2.3}$$

Indeed the exact functor $F : M \mapsto M[S^{-1}] = M \otimes_R R[S^{-1}]$ has kernel exactly $\mathcal{T}$, so Theorem 5.2.1 supplies an exact, faithful functor $R\text{-}\mathsf{Mod}/\mathcal{T} \to R[S^{-1}]\text{-}\mathsf{Mod}$; it is an equivalence, since every $R[S^{-1}]$-module N satisfies $N \otimes_R R[S^{-1}] \simeq N$, and every morphism $M_1[S^{-1}] \to M_2[S^{-1}]$ is represented by a morphism defined on a submodule $M_0 \subset M_1$ with $M_1/M_0 \in \mathcal{T}$. This recovers localization of modules as a Serre quotient.

Because the Serre quotient $\mathcal{A}/\mathcal{T}$ is constructed as a localization, it may in principle fail to be locally small (Section 1.1), i.e. the Hom between two objects might be a proper class[3], which is forbidden per Definition 1.1.1. One set-theoretic condition on $\mathcal{A}$ suffices to rule out this pathology.

Definition 5.2.3 A category $\mathcal{C}$ is ***well-powered*** if the subobjects of every object X form a set, i.e. the poset $\mathrm{Sub}(X)$ of Section 2.4 is small; dually, it is ***well-copowered*** if $\mathcal{C}^{\mathrm{op}}$ is well-powered, i.e. the quotients of every X form a set. In an abelian category the two notions agree, since $A \hookrightarrow X \mapsto X/A$ bijects the subobjects of X with its quotients (Proposition 2.4.5).

When $\mathcal{A}$ is well-powered, the direct construction of the Serre quotient in Exercise 5.2.4 below exhibits $\mathcal{A}/\mathcal{T}$ with small Hom-sets, so the aforementioned pathology disappears; see [10, §2.9].

Exercise 5.2.4 (Direct construction of the Serre quotient.) Let $\mathcal{T}$ be a Serre subcategory of an abelian category $\mathcal{A}$. Define a new category with the same objects as $\mathcal{A}$, the morphisms from X to Y being the filtered colimit

$$\operatorname*{colim}_{\substack{X' \subset X \\ Y' \subset Y}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y') \tag{5.2.4}$$

over the pairs of subobjects $X' \subset X$ and $Y' \subset Y$ with $X/X', Y' \in \mathrm{Ob}(\mathcal{T})$.

(i) Complete the definition of composition, and show that the resulting category is equivalent to the Serre quotient $\mathcal{A}/\mathcal{T}$ of Theorem 5.2.1. *Hint.* Verify the universal property; the standard treatment is [4].

(ii) Deduce that if $\mathcal{A}$ is well-powered (Definition 5.2.3), then the Hom between any pair of objects of $\mathcal{A}/\mathcal{T}$ is indeed a set, thus the Serre quotient by $\mathcal{T}$ meets the requirements of Definition 1.1.1.

5.3 The Grothendieck group

We now attach to every abelian category a single abelian group that records the "size" of its objects up to short exact sequences.

Definition 5.3.1 Let $\mathcal{A}$ be an abelian category. The ***Grothendieck group*** $\mathrm{K}_0(\mathcal{A})$ is the abelian group with one generator $\langle X \rangle$ for each object $X \in \mathcal{A}$, whose class we write as $[X]$, subject to the relations

$$[X] = [X'] + [X''] \tag{5.3.1}$$

for every short exact sequence $0 \to X' \to X \to X'' \to 0$.

Concretely, $\mathrm{K}_0(\mathcal{A})$ is the free abelian group on $\mathrm{Ob}(\mathcal{A})$ modulo the subgroup generated by $[X] - [X'] - [X'']$ for each short exact sequence as above. The construction is functorial: an exact functor $F : \mathcal{A} \to \mathcal{B}$ preserves short exact sequences, hence induces a group homomorphism $\mathrm{K}_0(F) : \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$ given by $[X] \mapsto [F(X)]$. Strictly speaking, the free abelian group on $\mathrm{Ob}(\mathcal{A})$ makes sense only when the isomorphism classes of objects form a set; one may always pass to a small skeleton (Section 1.1), and Lemma 5.3.2 (ii) shows that this does not change K_0.

[3]With universe-based category theory, this means that the Hom-set between two objects might fail to be $\mathcal{U}$-small.

Lemma 5.3.2 The following hold in $\mathrm{K}_0(\mathcal{A})$ for every abelian category $\mathcal{A}$.

(i) $[0] = 0$.

(ii) If $X \simeq Y$ then $[X] = [Y]$.

(iii) $[X \oplus Y] = [X] + [Y]$.

(iv) For every exact sequence $0 \to X^1 \to X^2 \to \cdots \to X^n \to 0$ in $\mathcal{A}$,

$$\sum_{i=1}^{n} (-1)^i [X^i] = 0. \tag{5.3.2}$$

(v) For every filtration $X = X_0 \supset X_1 \supset \cdots \supset X_r = 0$ in $\mathcal{A}$,

$$[X] = \sum_{i=0}^{r-1} [X_i/X_{i+1}]. \tag{5.3.3}$$

Proof (i) and (ii) come from the split sequences $0 \to 0 \to 0 \to 0 \to 0$ and $0 \to X \simeq Y \to 0 \to 0$. (iii) is the split sequence $0 \to X \to X \oplus Y \to Y \to 0$. For (iv), insert the short exact sequences $0 \to \ker(d^i) \to X^i \to \mathrm{im}(d^i) \to 0$ and use $\mathrm{im}(d^i) = \ker(d^{i+1})$; the alternating sum cancels. (v) follows by induction on r from $0 \to X_1 \to X_0 \to X_0/X_1 \to 0$. □

Example 5.3.3 Take $\mathcal{A}$ to be the category of finite-dimensional vector spaces over a division ring D (say acting on the right). Every D-space is a direct sum of copies of D, so a space of dimension n is isomorphic to the n-fold direct sum, and Lemma 5.3.2 (ii)–(iii) give $[X] = n[D]$, i.e. $\mathrm{K}_0(\mathcal{A}) = \mathbb{Z} \cdot [D]$. Since $\dim_D$ is additive on short exact sequences, it descends to a homomorphism sending the generator $[D]$ to 1, whence an isomorphism

$$\dim_D : \mathrm{K}_0(\mathcal{A}) \simeq \mathbb{Z}, \quad [X] \mapsto \dim_D(X). \tag{5.3.4}$$

If infinite-dimensional spaces were allowed, K_0 would be trivial: for any D-space V, let W be a countable direct sum of copies of V; then $V \oplus W \simeq W$ (one extra summand changes nothing), so $[V] + [W] = [W]$, forcing $[V] = 0$.

Theorem 5.3.4 (*Euler–Poincaré principle*) Let $X^\bullet$ be a bounded complex in an abelian category $\mathcal{A}$. Then in $\mathrm{K}_0(\mathcal{A})$

$$\sum_i (-1)^i [\mathrm{H}^i(X^\bullet)] = \sum_i (-1)^i [X^i]. \tag{5.3.5}$$

Proof For each i the exact four-term sequence

$$0 \to \ker(d^{i-1}) \to X^{i-1} \xrightarrow{d^{i-1}} \ker(d^i) \to \mathrm{H}^i(X^\bullet) \to 0 \tag{5.3.6}$$

(with d the differential of $X^\bullet$) gives, by Lemma 5.3.2 (iv),

$$[X^{i-1}] + [\mathrm{H}^i(X^\bullet)] = [\ker(d^{i-1})] + [\ker(d^i)]. \tag{5.3.7}$$

Now multiply by $(-1)^i$ and sum over i; only finitely many terms are nonzero, so summing is legitimate. Then the right-hand side cancels to 0, and a re-indexing $i \mapsto i+1$ of the $[X^{i-1}]$ term yields the claim. □

If we regard $\sum_i (-1)^i [X^i]$ as the "class" of a bounded complex $X^\bullet$, then Theorem 5.3.4 computes the class of $X^\bullet$ from its cohomologies. When $\mathrm{K}_0(\mathcal{A})$ is a free abelian group on some given building blocks of $\mathcal{A}$, such as for the case of finite-dimensional vector spaces seen in

Example 5.3.3, the coefficients expressing each $[\mathrm{H}^i(X^\bullet)]$ in that basis are the ***Betti numbers*** : the principle then reproduces the classical Euler–Poincaré formula $\chi = \sum(-1)^i \dim \mathrm{H}^i$ in topology.

Remark 5.3.5 The Grothendieck group makes sense for any ***exact category*** in the sense of Quillen: an additive category endowed with a distinguished class of "short exact sequences" (called *admissible exact*) satisfying a few axioms; every abelian category is an example (with all short exact sequences admissible). This extra generality is essential in practice — the categories of vector bundles on a scheme, of filtered objects, or of finitely generated projective modules, are exact but rarely abelian. We refer to [2] for a friendly account.

Exercise 5.3.6 Let $\mathcal{A}$ be an abelian category whose isomorphism classes of objects form a set, in which every object has finite length (Definition 2.4.4), with simple objects $S_1, \cdots, S_r$ (up to isomorphism). Prove that $\mathrm{K}_0(\mathcal{A}) \simeq \mathbb{Z}^r$, freely generated by $[S_1], \cdots, [S_r]$. *Hint.* Jordan–Hölder ([10, Theorem 2.4.10]).

Exercise 5.3.7 Let $X^\bullet$ be a bounded complex of finite-dimensional vector spaces over a division ring D (say acting on the right). Deduce from Theorem 5.3.4 the classical identity

$$\sum_i (-1)^i \dim_D \mathrm{H}^i(X^\bullet) = \sum_i (-1)^i \dim_D(X^i). \tag{5.3.8}$$

5.4 K_0 and Serre quotients

The Grothendieck group of a Serre quotient is controlled by that of the subcategory annihilated.

Proposition 5.4.1 Let $\mathcal{T}$ be a Serre subcategory of an abelian category $\mathcal{A}$, with Serre quotient $\mathcal{B} = \mathcal{A}/\mathcal{T}$. The exact functors $\mathcal{T} \to \mathcal{A} \xrightarrow{Q} \mathcal{B}$ induce an exact sequence of abelian groups

$$\mathrm{K}_0(\mathcal{T}) \to \mathrm{K}_0(\mathcal{A}) \xrightarrow{\mathrm{K}_0(Q)} \mathrm{K}_0(\mathcal{B}) \to 0. \tag{5.4.1}$$

Proof Let I denote the image of $\mathrm{K}_0(\mathcal{T}) \to \mathrm{K}_0(\mathcal{A})$. Since Q is essentially surjective, $\mathrm{K}_0(\mathcal{B})$ is generated by classes of the form $[Q(X)]$ (Lemma 5.3.2 (ii)), so $\mathrm{K}_0(Q)$ is surjective; and $Q(X) \simeq 0$ for $X \in \mathrm{Ob}(\mathcal{T})$, thus $I \subset \ker(\mathrm{K}_0(Q))$.

Conversely, we claim that $[Y] - [Y'] - [Y''] \in I$ for every short exact sequence $0 \to Y' \xrightarrow{f} Y \xrightarrow{g} Y'' \to 0$ in $\mathcal{B}$, the brackets being computed in $\mathrm{K}_0(\mathcal{A})$, which is legitimate as $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{B})$ by the proof of Theorem 5.2.1, which constructs $\mathcal{B}$ as $\mathcal{A}[S^{-1}]$. As S is a multiplicative system, write $g = Q(v)Q(s)^{-1}$ with $s : Z \to Y$ in S and $v : Z \to Y''$, and $Q(s)^{-1} f = Q(u)Q(t)^{-1}$ with $t : W \to Y'$ in S and $u : W \to Z$. Then $gQ(s) = Q(v)$ and $fQ(t) = Q(s)Q(u)$: conjugating by the isomorphisms $Q(s)$ and $Q(t)$ turns the sequence into

$$0 \to W \xrightarrow{Q(u)} Z \xrightarrow{Q(v)} Y'' \to 0, \tag{5.4.2}$$

still exact in $\mathcal{B}$, so $Q(vu) = 0$. Now $Q(w) = 0$ holds precisely when w is annihilated on the right by some morphism of S, by Proposition 1.8.8; replacing W by the source of such a morphism, we may further assume $vu = 0$ in $\mathcal{A}$.

These replacements change brackets only within I: for $s : Z \to Y$ in S, by construction $\ker(s), \operatorname{coker}(s) \in \mathrm{Ob}(\mathcal{T})$, and Lemma 5.3.2 (iv) applied to

$$0 \to \ker(s) \to Z \xrightarrow{s} Y \to \operatorname{coker}(s) \to 0 \tag{5.4.3}$$

gives $[Y] - [Z] \in I$. Therefore it remains to show $[Z] - [W] - [Y''] \in I$. Since $vu = 0$, u factors through $\ker(v)$, and $\mathcal{A}$ contains the exact sequences

$$\begin{gathered} 0 \to \ker(v) \to Z \xrightarrow{v} Y'' \to \operatorname{coker}(v) \to 0, \\ 0 \to \ker(u) \to W \xrightarrow{u} \ker(v) \to \ker(v)/\operatorname{im}(u) \to 0, \end{gathered} \tag{5.4.4}$$

whence Lemma 5.3.2 (iv) implies

$$[Z] - [W] - [Y''] = [\ker(v)/\operatorname{im}(u)] - [\ker(u)] - [\operatorname{coker}(v)]. \tag{5.4.5}$$

Finally, Q is exact and $0 \to Q(W) \to Q(Z) \to Q(Y'') \to 0$ is exact in $\mathcal{B}$, so

$$Q(\ker(u)) = \ker(Q(u)) = 0, \quad Q(\operatorname{coker}(v)) = \operatorname{coker}(Q(v)) = 0, \tag{5.4.6}$$

and $\ker(Q(v)) = \operatorname{im}(Q(u))$ implies $Q(\ker(v)/\operatorname{im}(u)) = 0$. The three objects on the right of the displayed equation thus lie in $\mathcal{T} = \ker(Q)$, proving the claim.

The claim says that the defining relations of Definition 5.3.1 for $\mathrm{K}_0(\mathcal{B})$ hold modulo I in $\mathrm{K}_0(\mathcal{A})$; hence $[Y] \mapsto [Y] + I$ descends to a homomorphism $\theta : \mathrm{K}_0(\mathcal{B}) \to \mathrm{K}_0(\mathcal{A})/I$, whose composite with $\mathrm{K}_0(Q)$ is the quotient homomorphism (Q is the identity on objects). Therefore $\ker(\mathrm{K}_0(Q)) \subset I$, as desired. $\square$

The left end of the exact sequence in Proposition 5.4.1 cannot, in general, be extended to a short exact sequence: doing so requires a whole family of higher K-groups. Quillen answered by defining, for an exact category $\mathcal{E}$ (Remark 5.3.5), a sequence of abelian groups $\mathrm{K}_n(\mathcal{E})$ ($n \in \mathbb{Z}_{\geq 0}$) with K_0 extending the group of Definition 5.3.1, fitting into long exact sequences

$$\cdots \to \mathrm{K}_{n+1}(\mathcal{B}) \to \mathrm{K}_n(\mathcal{T}) \to \mathrm{K}_n(\mathcal{A}) \to \mathrm{K}_n(\mathcal{B}) \to \cdots \tag{5.4.7}$$

for every exact sequence of exact categories. These ***higher K-groups*** encode deep arithmetic and geometric information (e.g. K_1 recovers units and K_2 arises from the Steinberg group); their proper construction requires homotopy theory — in Quillen's Q-construction or Waldhausen's $S_\bullet$-construction, the groups are homotopy groups of a space built from $\mathcal{E}$. We can only point the reader to [7] and [17, Chapters 4 and 5].

Chapter 6

Spectral Sequences

A spectral sequence is a bookkeeping device that approximates the usually inaccessible cohomology of a complex from a succession of simpler pages, each the cohomology of the last. Introduced by J. Leray in the 1940s to understand the (co)homology of fiber bundles and systematized by J.-L. Koszul and H. Cartan, spectral sequences remain the computational engine of homological algebra and algebraic topology.

The archetypal situation is this. A complex X is too complicated to understand directly, but it often carries a *filtration* $\cdots \supset \mathrm{F}^p X \supset \mathrm{F}^{p+1} X \supset \cdots$ that we do understand. The filtration induces one on cohomology, and the spectral sequence assembles the successive subquotients $\mathrm{gr}^p X := \mathrm{F}^p X / \mathrm{F}^{p+1} X$ into a step-by-step approximation of the graded pieces of $\mathrm{H}^\bullet(X)$. This chapter is merely a *pedagogical overview*: we record the definitions, Massey's exact couples, the two standard spectral sequences (of a filtered complex and of a double complex), and the headline applications: a fresh derivation of the balancing of derived bifunctors (Theorem 4.3.2) and the Grothendieck spectral sequence for the derived functors of a composite. The messier convergence arguments we largely defer to [10, Chapter 5].

6.1 Filtrations and gradings

A $\mathbb{Z}$- ***graded object*** in a category $\mathcal{A}$ is a family $(X^p)_{p \in \mathbb{Z}}$, i.e. an object of $\mathcal{A}^{\mathbb{Z}}$; a $\mathbb{Z}^2$-graded (or **bigraded**) object is a family $(X^{p,q})_{(p,q)\in\mathbb{Z}^2}$, i.e. an object of $\mathcal{A}^{\mathbb{Z}\times\mathbb{Z}}$. If $\mathcal{A}$ is abelian then so are $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{A}^{\mathbb{Z}\times\mathbb{Z}}$, with kernels and cokernels taken degreewise. In this chapter gradings almost always come from filtrations.

Definition 6.1.1 Let $\mathcal{A}$ be additive and $X \in \mathcal{A}$.

▷ ***descending filtration*** a chain of subobjects

$$\cdots \supset \mathrm{F}^p X \supset \mathrm{F}^{p+1} X \supset \cdots \quad (p \in \mathbb{Z}). \tag{6.1.1}$$

▷ ***associated graded* (denoted by gr)** if $\mathcal{A}$ is abelian, the graded object

$$\mathrm{gr}^p X := \mathrm{F}^p X / \mathrm{F}^{p+1} X \quad (p \in \mathbb{Z}). \tag{6.1.2}$$

We say a filtration $\mathrm{F}^\bullet X$ is ***finite*** if $\mathrm{F}^a X = X$ and $\mathrm{F}^b X = 0$ for some $a \le b$.

ascending filtration $\cdots \subset \mathrm{F}_p X \subset \mathrm{F}_{p+1} X \subset \cdots$ and its associated graded $\mathrm{gr}_p X = \mathrm{F}_p X / \mathrm{F}_{p-1} X$ are defined similarly; they are related to the descending version by $\mathrm{F}_p := \mathrm{F}^{-p}$, under which $\mathrm{gr}_p X = \mathrm{gr}^{-p} X$. Throughout this chapter we use descending filtrations (the convention for cochain complexes and cohomology); the ascending version governs chain complexes and homology by duality and the recipe above.

A morphism of filtered objects $f : (X, \mathrm{F}^\bullet X) \to (Y, \mathrm{F}^\bullet Y)$ is a morphism $f : X \to Y$ such that $f(\mathrm{F}^p X) \subset \mathrm{F}^p Y$ for all p; filtered objects and their morphisms form a category.

Definition 6.1.2 Let $\mathcal{A}$ be abelian and X carry a descending filtration $\mathrm{F}^\bullet X$. We say $\mathrm{F}^\bullet X$ is:

▷ ***exhaustive*** $\bigcup_p \mathrm{F}^p X = X$ (for instance, if $\mathrm{F}^M X = X$ for some M);

▷ ***separated*** $\bigcap_p \mathrm{F}^p X = 0$;

▷ ***complete*** the natural map $X \to \varprojlim_p X/\mathrm{F}^p X$ is an isomorphism.

A technical caveat: in a general abelian category $\bigcup_p$ is interpreted as the sum of subobjects, and identities such as $K = \bigcup_p (K \cap \mathrm{F}^p X)$, used in the proof below, require either $\mathrm{F}^M X = X$ for some M or exact filtered colimits in $\mathcal{A}$ (automatic in Grothendieck categories, in particular R-Mod; Proposition B.2.2); [10, §5.1] builds this into the definition of exhaustiveness. Finite filtrations are automatically exhaustive, separated, and complete, and this is the only situation the applications below require, so the caveat will not detain us; moreover a complete filtration is automatically separated, the kernel of $X \to \varprojlim_p X/\mathrm{F}^p X$ being $\bigcap_p \mathrm{F}^p X$. The following elementary proposition is the tool that turns isomorphisms on the graded pieces into isomorphisms of filtered objects.

Proposition 6.1.3 Let $f : (X, \mathrm{F}^\bullet X) \to (Y, \mathrm{F}^\bullet Y)$ be a morphism of filtered objects.

- If $\mathrm{F}^\bullet X$ is separated and exhaustive and $\mathrm{gr}(f)$ is a monomorphism, then f is a monomorphism.
- If $\mathrm{F}^\bullet X$ is complete and exhaustive, $\mathrm{F}^\bullet Y$ is separated and exhaustive, and $\mathrm{gr}(f)$ is an isomorphism, then f is an isomorphism (and $\mathrm{F}^\bullet Y$ is then complete).

Proof For each p the short exact sequences $0 \to \mathrm{F}^{p+1} X \to \mathrm{F}^p X \to \mathrm{gr}^p X \to 0$ and likewise for Y sitting in a commutative diagram with vertical maps f. Apply the snake lemma (Theorem 2.5.2). If $\mathrm{gr}(f)$ is a monomorphism, then $K^p = \ker(\mathrm{F}^p X \to \mathrm{F}^p Y)$ satisfy $K^{p+1} \simeq K^p$ (the map being the inclusion); separatedness gives $K^p = 0$ for all p, and exhaustiveness promotes this to $\ker(f) = 0$. If $\mathrm{gr}(f)$ is an isomorphism, the same chase also gives $K^{p+1} \simeq K^p$, whence f is monic by what precedes, and cokernels satisfy $C^{p+1} \simeq C^p$; a second snake chase gives $\mathrm{F}^p X/\mathrm{F}^m X \simeq \mathrm{F}^p Y/\mathrm{F}^m Y$ for all $m > p$, and passing first to the colimit over p and then to the limit over m, using completeness of $\mathrm{F}^\bullet X$ (and separatedness of $\mathrm{F}^\bullet Y$), one can infer that f is an isomorphism. The detailed diagram chase can be found in [10, Proposition 5.1.4]. □

6.2 Spectral sequences: the definition

Spectral sequences are best defined on categories equipped with a shift. The same formalism underlies the triangulated categories of Chapter 7, so we record it once and for all.

Definition 6.2.1 A ***category with translation*** is a pair $(\mathcal{A}, T)$ where $\mathcal{A}$ is a category and $T : \mathcal{A} \to \mathcal{A}$ is an *auto-equivalence* with a prescribed quasi-inverse T^{-1}, the ***translation functor*** .

- By Remark 1.6.5 we may assume throughout that T is both a left and a right adjoint to T^{-1}. Many authors require T to be an automorphism, so that $T^{-1}T = \mathrm{id} = TT^{-1}$ on the nose; this already covers every case we need.
- A functor $(\mathcal{A}, T) \to (\mathcal{A}', T')$ between categories with translation is a functor $F : \mathcal{A} \to \mathcal{A}'$ together with a specified isomorphism $FT \simeq T'F$; we usually suppress the isomorphism. When $\mathcal{A}$ and $\mathcal{A}'$ are additive, F is required to be additive.

In this chapter $(\mathcal{A}, T)$ will be an *abelian* category with additive translation T. Since T is an equivalence, it is exact as well.

Definition 6.2.2 (*morphism of degree m*) A morphism of degree m in $(\mathcal{A}, T)$ is a morphism $f : X \to T^m Y$, written $f : X \xrightarrow{+m} Y$; by the adjunction this is equivalently a morphism $T^{-m}X \to Y$. Given $X \xrightarrow{+m} Y \xrightarrow{+n} Z$, the composite gf is the degree $m+n$ morphism $T^m(g)f$. One may speak of commutative diagrams of such morphisms.

Given $(\mathcal{A}, T)$, a ***differential object*** is a pair (E, d) with $d : E \xrightarrow{+a} E$ a degree-a endomorphism for some $a \in \mathbb{Z}$, satisfying $d^2 = 0$; its *cohomology* is

$$\mathrm{H}(E, d) := \ker(d) / \operatorname{im}(T^{-a}d), \tag{6.2.1}$$

where d is viewed as $E \to T^a E$ when taking the kernel and as $T^{-a}E \to E$ when taking the image, so that both are subobjects of E. A cochain complex (Definition 2.2.1) in an abelian category $\mathcal{B}$ is the case $\mathcal{A} = \mathcal{B}^{\mathbb{Z}}$ with translation $T(X)^n = X^{n+1}$ and $a = 1$; then $\mathrm{H}(E, d) = (\mathrm{H}^n(E))_n$. A chain complex is the case $a = -1$. We can now define spectral sequences in $(\mathcal{A}, T)$.

Definition 6.2.3 A ***spectral sequence*** $\mathcal{E} = (E_r, d_r)_{r \geq a}$ in $(\mathcal{A}, T)$ is a sequence of differential objects (E_r, d_r) with differentials $d_r : E_r \xrightarrow{+a_r} E_r$ of degrees $a_r \in \mathbb{Z}$ (which may vary with r), $r = a, a+1, \cdots$, together with specified isomorphisms $\mathrm{H}(E_r, d_r) \simeq E_{r+1}$. One calls (E_r, d_r) the r-th *page*; passing from E_r to E_{r+1} is called turning the page.

A morphism between spectral sequences $\mathcal{E} \to \mathcal{E}'$ is a sequence of morphisms $\varphi_r : E_r \to E'_r$ that commute with d_r, d'_r, and such that the morphism induced on cohomology by φ_r is identified with φ_{r+1} by the specified isomorphisms, for every r.

For simplicity, assume the pages start at E_0. Set $Z_1 := \ker(d_0)$ and $B_1 := \operatorname{im}(T^{-a_0}d_0)$, the numerator and denominator of $\mathrm{H}(E_0, d_0)$; both are subobjects of E_0, and $E_1 = Z_1/B_1$. Iterating produces a nested family of subobjects of E_0

$$0 =: B_0 \subset B_1 \subset B_2 \subset \cdots \subset Z_2 \subset Z_1 \subset Z_0 =: E_0, \tag{6.2.2}$$

with $E_r = Z_r/B_r$, where Z_r records the cycles surviving r pages and B_r the boundaries killed by then. Whenever both

$$Z_\infty := \bigcap_r Z_r, \quad B_\infty := \bigcup_r B_r \tag{6.2.3}$$

exist (recall Definition 2.4.2), the *limit page* is $E_\infty := Z_\infty / B_\infty$. If $d_r = 0$ for all $r \geq r_0$ (i.e., all further differentials vanish), we say the spectral sequence ***degenerates*** at E_{r_0}, in which case

$$Z_{r_0} = \cdots = Z_\infty, \quad B_{r_0} = \cdots = B_\infty, \quad E_{r_0} = \cdots = E_\infty. \tag{6.2.4}$$

Exercise 6.2.4 Let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a morphism of spectral sequences. Show that if $\varphi_r : E_r \to E'_r$ is an isomorphism for some r, then φ_s is an isomorphism for every $s \geq r$, and (when the limits exist) $\varphi_\infty : E_\infty \to E'_\infty$ is an isomorphism. *Hint.* Cohomology of an isomorphism is an isomorphism.

More generally, we can equip $\mathcal{A}$ with a family of commuting automorphisms $T_1, \ldots, T_n$ and require that the differentials d_r in Definition 6.2.3 are of degree $(a_{r,1}, \ldots, a_{r,n}) \in \mathbb{Z}^n$ (in the obvious sense) for all r; then Definition 6.2.3 carries over. As typical instances, we may take

$\mathcal{A} = \mathcal{B}^{\mathbb{Z}^n}$ where $\mathcal{B}$ is an abelian category, and T_i is the translation at the i-th slot $(1 \leq i \leq n)$. The most useful cases are the *graded* $(n = 1)$ and *bigraded* $(n = 2)$ ones.

Definition 6.2.5 (*graded and bigraded spectral sequences*) Let $\mathcal{A}$ be an abelian category.

- A ***cohomological graded spectral sequence*** is a spectral sequence $\mathcal{E} = (E_r, d_r)_r$ in $(\mathcal{A}^{\mathbb{Z}}, T)$ (where T is the translation) such that $d_r = (d_r^p)_p$ is of degree r, i.e.

$$d_r^p : E_r^p \to E_r^{p+r}. \tag{6.2.5}$$

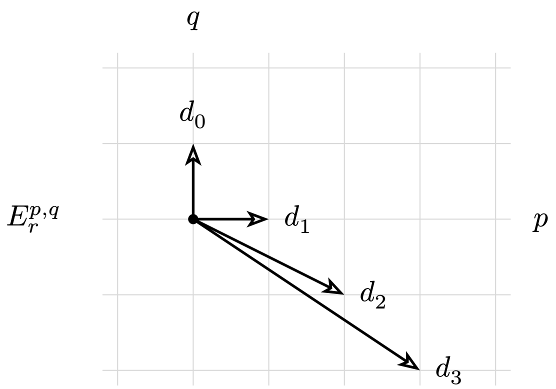
- A ***cohomological bigraded spectral sequence*** is a spectral sequence $\mathcal{E} = (E_r, d_r)_r$ in $(\mathcal{A}^{\mathbb{Z} \times \mathbb{Z}}, T_1, T_2)$ (where T_1, T_2 are the two translations) such that $d_r = (d_r^{p,q})_{p,q}$ is of bidegree $(r, 1-r)$, i.e.

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}. \tag{6.2.6}$$

In both cases the composition of morphisms with degrees still makes sense, so the requirements of Definition 6.2.3, including $d_r^2 = 0$ and the specified isomorphisms $\mathrm{H}(E_r, d_r) \simeq E_{r+1}$, carry over verbatim, as does the notion of morphism.

In the settings above, the subobjects $B_r^p \subset Z_r^p$, $B_r^{p,q} \subset Z_r^{p,q}$, as well as the limit terms E_∞^p, $E_\infty^{p,q}$, are defined as before.

What is most often encountered is the bigraded case. The bidegrees of $d_0, d_1, \cdots$ are depicted below:



Remark 6.2.6 Dually, we have a ***homological graded spectral sequence*** $(E^r, d^r)_r$ with $E^r = (E_p^r)_p$ and $d^r = (d_p^r)_p$ where $d_p^r : E_p^r \to E_{p-r}^r$. Similarly, the ***homological bigraded spectral sequence*** has differentials

$$d_{p,q}^r : E_{p,q}^r \to E_{p-r,q+r-1}^r. \tag{6.2.7}$$

The other ingredients are similar to Definition 6.2.5. The reader is invited to depict the bidegrees of $d^0, d^1, \cdots$ as before.

Some further terminology is in order. We say a page E_r is ***bounded*** if for every n there are only finitely many (p, q) with $p + q = n$ and $E_r^{p,q} \neq 0$; it lies in the ***first quadrant*** if $E_r^{p,q} \neq 0$ implies $p, q \geq 0$. Boundedness guarantees the limit exists:

Proposition 6.2.7 (*bounded pages stabilize*) If a bigraded spectral sequence has a bounded page E_r, then for each (p, q) there is $r_0(p, q)$ such that $E_s^{p,q} = E_{s+1}^{p,q}$ for all $s \geq r_0(p, q)$; in particular the limit terms exist and satisfy $E_s^{p,q} = E_\infty^{p,q}$ for all $s \geq r_0(p, q)$.

Proof Boundedness persists from page to page, since $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$. For fixed $n = p+q$ and $r \gg 0$, boundedness forces both $E_r^{p-r,q+r-1} = 0$ (so no incoming differential, $B_r^{p,q} = 0$) and $E_r^{p+r,q-r+1} = 0$ (so no outgoing differential, $Z_r^{p,q} = E_r^{p,q}$). Hence $E_r^{p,q} = E_{r+1}^{p,q}$, and the chains $B_r^{p,q}$, $Z_r^{p,q}$ stabilize, so the limit exists. $\square$

A first-quadrant page is automatically bounded. First-quadrant spectral sequences have a further virtue: information can be read off their edges.

Remark 6.2.8 (*edge morphisms and the low-degree sequence*) In a first-quadrant cohomological spectral sequence, the bottom row $E_r^{\bullet,0}$ and the left column $E_r^{0,\bullet}$ are special. Because no differential leaves the bottom row for $r \geq 2$ and none enters the left column for $r \geq 1$, one obtains ***edge morphisms*** $E_2^{p,0} \twoheadrightarrow E_\infty^{p,0}$ and $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$. Splicing these with the single possibly-nonzero differential $d : E_p^{0,p-1} \to E_p^{p,0}$ gives the *edge exact sequence* (for $p \geq 2$)

$$0 \to E_\infty^{0,p-1} \to E_p^{0,p-1} \xrightarrow{d} E_p^{p,0} \to E_\infty^{p,0} \to 0, \tag{6.2.8}$$

whose non-d maps are edge morphisms. Specializing to $p = 2$ yields the low-degree exact sequence of Corollary 6.4.5. See [10, Remark 5.2.13] for the detailed differential chase.

6.3 Exact couples

Exact couples, due to Massey, are a uniform machine for producing spectral sequences; they are more general than filtered complexes, though we shall mainly use them that way.

Definition 6.3.1 An ***exact couple*** in an abelian category $\mathcal{A}$ is an exact diagram

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \tag{6.3.1}$$

i.e. data $\mathcal{C} = (D, E, i, j, k)$ with im of each arrow equal to ker of the next.

Set $d := jk : E \to E$. Exactness gives $d^2 = jkjk = 0$, so (E, d) is a differential object with cohomology $E' := \mathrm{H}(E, d)$. The exact couple also produces $D' := i(D) \subset D$ and maps $i' := i|_{i(D)}$, plus induced maps $j' : D' \to E'$ and $k' : E' \to D'$ (both forced by exactness).

Lemma 6.3.2 (*the derived couple*) With the above notation, $\mathcal{C}' = (D', E', i', j', k')$ is again an exact couple. Iterating yields a sequence of exact couples $\mathcal{C}_r$ with $\mathcal{C}_1 = \mathcal{C}$, and $(E_r, d_r := j_r k_r)_{r \geq 1}$ is a spectral sequence.

Proof (*Sketch*) That $\mathcal{C}'$ is exact is three kernel computations; for instance

$$\ker(k') = (j(D) \cap \ker(jk))/\operatorname{im}(d) = j(D)/\operatorname{im}(d) = \operatorname{im}(j'), \tag{6.3.2}$$

and similarly for $\ker(i')$, $\ker(j')$. Iteration is then mechanical. See [10, §5.3] for details. The reader is invited to work out the concrete case $\mathcal{A} = R\text{-}\mathsf{Mod}$. $\square$

The bridge from complexes to exact couples is the following construction. Hereafter $\mathcal{A}$ is an abelian category; in this section the differentials are of degree 0, so that the translation functor of Definition 6.2.1 plays no role, and we write $\mathcal{A}_{\mathrm{d}}$ for the category of differential objects (E, d), a morphism being a morphism of the underlying objects commuting with the differentials.

Proposition 6.3.3 The category $\mathcal{A}_{\mathrm{d}}$ is abelian, and the forgetful functor $(E, d) \mapsto E$ is an exact functor $\mathcal{A}_{\mathrm{d}} \to \mathcal{A}$.

Proof Routine: kernels, cokernels, and exactness are computed on the underlying objects. The statement also follows from [10, Proposition 5.2.2]. □

As a consequence, a morphism $(E, d) \to (E', d')$ in $\mathcal{A}_{\mathrm{d}}$ is mono (resp. epi) if and only if the underlying morphism $E \to E'$ in $\mathcal{A}$ is.

Proposition 6.3.4 Let $\alpha : (D, d) \hookrightarrow (D, d)$ be a monomorphism of differential objects, and write d_α for the endomorphism induced by d on $\operatorname{coker}(\alpha)$. The short exact sequence $0 \to (D, d) \xrightarrow{\alpha} (D, d) \to (\operatorname{coker}\alpha, d_\alpha) \to 0$ induces an exact couple

$$\begin{tikzcd} \mathrm{H}(D,d) \arrow[rr, "i = \mathrm{H}(\alpha)"] & & \mathrm{H}(D,d) \arrow[dl, "j"] \\ & \mathrm{H}(\operatorname{coker}\alpha, d_\alpha) \arrow[ul, "k"] & \end{tikzcd} \tag{6.3.3}$$

with k the connecting morphism of the long exact sequence in cohomology. The spectral sequence of the couple starts at $E_0 = \operatorname{coker}(\alpha)$ with $d_0 = d_\alpha$, and the subobjects $Z_r, B_r \subset E_0$ introduced in Section 6.2 admit an explicit description: for every $r \geq 0$,

- B_{r+1} is the image of $(\alpha^r)^{-1}(dD) \subset D$ in E_0;
- Z_{r+1} is the image of $d^{-1}(\alpha^{r+1}D) \subset D$ in E_0;
- d_{r+1} is induced by the composite

$$d^{-1}(\alpha^{r+1}D) \xrightarrow{d} \alpha^{r+1}D \xrightarrow{(\alpha^{r+1})^{-1}} D, \tag{6.3.4}$$

  the second arrow being the inverse of $\alpha^{r+1} : D \simeq \alpha^{r+1}D$ viewed as an isomorphism onto its image; the composite carries $d^{-1}(\alpha^{r+1}D)$ into itself.

Consequently, there are canonical isomorphisms

$$E_{r+1} \simeq \frac{Z_{r+1}}{B_{r+1}} \simeq \frac{d^{-1}(\alpha^{r+1}D) + \alpha D}{(\alpha^r)^{-1}(dD) + \alpha D} \quad (r \geq 0). \tag{6.3.5}$$

Proof (*Sketch*) This is Proposition 3.3.1 applied to the displayed short exact sequence of differential objects, a differential object of degree 0 being nothing but a $\mathbb{Z}$-periodic complex: the long exact sequence in cohomology *is* the unrolled exact couple, with j induced by $D \twoheadrightarrow \operatorname{coker}(\alpha)$ and k the connecting morphism. The description of Z_r and B_r follows by iterating Lemma 6.3.2: the $(r+1)$-st derived couple is canonically the exact couple on $i^r\,\mathrm{H}(D, d)$ and $k^{-1}(i^r\,\mathrm{H}(D, d))/j(\ker i^r)$, where $i = \mathrm{H}(\alpha)$, and unwinding the definitions of j and k yields the stated formulas. Both the lemma and the proposition extend verbatim to differentials and morphisms carrying degrees in a category with translation; this graded version, in which the differentials $d_r = j_r k_r$ carry degrees, is what Section 6.4 uses. See [10, Lemma 5.3.3 and Proposition 5.3.4]. □

In Section 6.4, we will consider exact couples arising from filtered complexes. Exact couples are more general than filtered complexes: any monomorphism of differential objects gives one, filtration or not. But for the filtered complexes of interest to us, one can write the resulting spectral sequence down directly as in Proposition 6.4.1; exact couples are then an explanatory device rather than a necessity. For this reason we only sketch the proofs in this section.

6.4 The spectral sequence of a filtered complex

Fix an abelian category $\mathcal{A}$. A ***filtered complex*** is a complex (X, d) equipped with a filtration by subcomplexes $\cdots \supset \mathrm{F}^p X \supset \mathrm{F}^{p+1} X \supset \cdots$. Each subquotient is a complex

$$\mathrm{gr}^p X = \mathrm{F}^p X / \mathrm{F}^{p+1} X = \left((\mathrm{gr}^p X)^n, (\mathrm{gr}^p d)^n\right)_n. \tag{6.4.1}$$

Combining the *filtration degree* p and the *inner degree* n, and writing $n = p + q$, gives rise to a bigraded spectral sequence.

Proposition 6.4.1 (*the spectral sequence of a filtered complex*) Every filtered complex $(X, d, \mathrm{F}^\bullet X)$ determines a cohomological bigraded spectral sequence with

$$E_0^{p,q} = (\mathrm{gr}^p X)^{p+q}, \quad E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{gr}^p X, \mathrm{gr}^p d), \tag{6.4.2}$$

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \tag{6.4.3}$$

(see Definition 6.2.5). The first differential d_0 is the differential $\mathrm{gr}^p d$ of the subquotient complexes, and d_1 is the connecting morphism of

$$0 \to \mathrm{gr}^{p+1} X \to \mathrm{F}^p X / \mathrm{F}^{p+2} X \to \mathrm{gr}^p X \to 0. \tag{6.4.4}$$

For $r \geq 0$ the subobjects $Z_r^{p,q}, B_r^{p,q} \subset E_0^{p,q}$ are given by

$$\begin{aligned} Z_r^{p,q} &= \frac{(\mathrm{F}^p X^{p+q} \cap d^{-1} \mathrm{F}^{p+r} X^{p+q+1}) + \mathrm{F}^{p+1} X^{p+q}}{\mathrm{F}^{p+1} X^{p+q}}, \\ B_r^{p,q} &= \frac{(\mathrm{F}^p X^{p+q} \cap d\, \mathrm{F}^{p-r+1} X^{p+q-1}) + \mathrm{F}^{p+1} X^{p+q}}{\mathrm{F}^{p+1} X^{p+q}}, \end{aligned} \tag{6.4.5}$$

and each differential $d_r^{p,q}$ is induced by d via

$$d^{-1} \mathrm{F}^{p+r} X^{p+q+1} \xrightarrow{d} \mathrm{F}^{p+r} X^{p+q+1} \cap \ker(d), \tag{6.4.6}$$

whose source and target project onto $Z_r^{p,q} \subset E_0^{p,q}$ and into $Z_r^{p+r,q-r+1} \subset E_0^{p+r,q-r+1}$, respectively ([10, Proposition 5.5.2]).

Equip the graded object $\mathrm{H}^\bullet(X)$ in $\mathcal{A}$ with the filtration

$$\mathrm{F}^p \mathrm{H}^{p+q}(X) := \mathrm{im}[\mathrm{F}^p X^{p+q} \cap \ker(d) \to \mathrm{H}^{p+q}(X)]. \tag{6.4.7}$$

When $E_\infty = Z_\infty / B_\infty$ exists, $\mathrm{gr}\, \mathrm{H}^\bullet(X) = \left(\mathrm{gr}^p \mathrm{H}^{p+q}(X)\right)_{p,q}$ is canonically isomorphic to a subquotient of E_∞ as bigraded objects in $\mathcal{A}$.

Proof (*Sketch*) We summarize the approach via exact couples (Section 6.3). Set $D := (\mathrm{F}^p X)_p$, bigraded by the filtration degree p and the inner degree n; the inclusions of complexes

$$\alpha_p : (\mathrm{F}^{p+1} X, d) \hookrightarrow (\mathrm{F}^p X, d) \tag{6.4.8}$$

assemble into a monomorphism $\alpha : D \to D$ of bidegree $(-1, 0)$ (filtration degree, inner degree), while d itself has bidegree $(0, 1)$. Apply the graded version of Proposition 6.3.4: the cokernel of α is $(\mathrm{gr}^p X, \mathrm{gr}^p d)_p$, so $E_0 = (\mathrm{gr}^p X, \mathrm{gr}^p d)_p$ with $d_0 = \mathrm{gr}^p d$, and $E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{gr}^p X, \mathrm{gr}^p d)$.

The exact couple is the long exact cohomology sequence of

$$0 \to (\mathrm{F}^{p+1} X, d) \hookrightarrow (\mathrm{F}^p X, d) \to (\mathrm{gr}^p X, \mathrm{gr}^p d) \to 0 \tag{6.4.9}$$

rolled up: j is induced by $\mathrm{F}^p X \to \mathrm{gr}^p X$ and k is the connecting morphism. Consequently the differential $d_1 = j_1 k_1$ is the composite of $\mathrm{H}^{p+q}(\mathrm{gr}^p X) \xrightarrow{k} \mathrm{H}^{p+q+1}(\mathrm{F}^{p+1} X)$ with the map induced by the projection $\mathrm{F}^{p+1} X \to \mathrm{gr}^{p+1} X$, which identifies it with the connecting morphism of

$$0 \to \mathrm{gr}^{p+1} X \to \mathrm{F}^p X / \mathrm{F}^{p+2} X \to \mathrm{gr}^p X \to 0 \tag{6.4.10}$$

as asserted.

That d_r has bidegree $(r, 1-r)$ is read off from the iterated derived couple (the connecting morphism k raises the inner degree by 1, while i lowers the filtration degree by 1). The formulas for $Z_r^{p,q}$, $B_r^{p,q}$, and $d_r^{p,q}$ follow by substituting α into Proposition 6.3.4 and tracking the two degrees: the p-th component of $\alpha^r D$ is $\mathrm{F}^{p+r} X$, so $d^{-1}(\alpha^r D)$ and $(\alpha^{r-1})^{-1}(dD)$ read, in filtration degree p and inner degree $p+q$, as $\mathrm{F}^p X^{p+q} \cap d^{-1} \mathrm{F}^{p+r} X^{p+q+1}$ and $\mathrm{F}^p X^{p+q} \cap d\, \mathrm{F}^{p-r+1} X^{p+q-1}$, respectively, and $d_r^{p,q}$ is then induced by d itself.

Finally, assume E_∞ exists. Every numerator of $Z_r^{p,q}$ contains $(\mathrm{F}^p X^{p+q} \cap \ker(d)) + \mathrm{F}^{p+1} X^{p+q}$, since $\ker(d) \subset d^{-1} \mathrm{F}^{p+r} X^{p+q+1}$, while every numerator of $B_r^{p,q}$ is contained in $(\mathrm{F}^p X^{p+q} \cap \mathrm{im}(d)) + \mathrm{F}^{p+1} X^{p+q}$. Passing to the intersection and the union over r therefore realizes

$$\frac{(\mathrm{F}^p X^{p+q} \cap \ker(d)) + \mathrm{F}^{p+1} X^{p+q}}{(\mathrm{F}^p X^{p+q} \cap \mathrm{im}(d)) + \mathrm{F}^{p+1} X^{p+q}} \tag{6.4.11}$$

as a subquotient of $E_\infty^{p,q}$, and the modular law identifies this quotient with $\mathrm{gr}^p \mathrm{H}^{p+q}(X)$ (cf. Remark 6.4.2). Details can be found in [10, §§5.4–5.5]. □

Remark 6.4.2 For every (p, q) there is moreover a canonical isomorphism ([10, Lemma 5.4.6])

$$\mathrm{gr}^p \mathrm{H}^{p+q}(X) \simeq \frac{\mathrm{F}^p X^{p+q} \cap \ker(d)}{(\mathrm{F}^{p+1} X^{p+q} \cap \ker(d)) + (\mathrm{F}^p X^{p+q} \cap \mathrm{im}(d))}. \tag{6.4.12}$$

In turn, one can take the formulas of Proposition 6.4.1 as the *definition* of the spectral sequence associated with $\mathrm{F}^\bullet X$ ([10, Remark 5.4.3]), and prove its remaining assertions without resorting to exact couples. Everything is routine to check by hand when $\mathcal{A} = R$-Mod.

The second part of Proposition 6.4.1 deserves emphasis: the spectral sequence does *not* in general compute $\mathrm{H}^n(X)$ directly; it computes the graded pieces $\mathrm{gr}^p \mathrm{H}^n(X)$ of the induced filtration on cohomology. Recovering $\mathrm{H}^n(X)$ from those pieces requires solving an extension problem, which is usually hard. The definition of convergence makes precise how good this approximation is.

Definition 6.4.3 We say the spectral sequence of Proposition 6.4.1

- has ***weak convergence*** to $\mathrm{H}^\bullet(X)$ if the canonical maps of Proposition 6.4.1 are isomorphisms $E_\infty^{p,q} \simeq \mathrm{gr}^p \mathrm{H}^{p+q}(X)$ for all (p,q);
- has ***strong convergence*** to $\mathrm{H}^\bullet(X)$, written as $E_r^{p,q} \Rightarrow \mathrm{H}^{p+q}(X)$, if in addition the induced filtration $\mathrm{F}^\bullet \mathrm{H}^n(X)$ is exhaustive and complete for every n.

The hypothesis one actually checks in practice is finiteness of the filtrations, which forces both strong convergence and a finite induced filtration. The following result is sometimes called the classical convergence theorem.

Theorem 6.4.4 Let $(X, d, \mathrm{F}^\bullet X)$ be a filtered complex such that, for every degree n, the filtration $\mathrm{F}^\bullet X^n$ is exhaustive and there is $N(n)$ with $\mathrm{F}^{N(n)} X^n = 0$. Then the spectral sequence of Proposition 6.4.1 strongly converges to $\mathrm{H}^\bullet(X)$. If each $\mathrm{F}^\bullet X^n$ is finite, the induced filtration on $\mathrm{H}^n(X)$ is finite too, and the spectral sequence is bounded.

Proof (*Sketch*) Exhaustiveness plus $\mathrm{F}^{N(n)} X^n = 0$ make the filtration on $\mathrm{H}^n(X)$ exhaustive and complete (a standard lemma on induced filtrations, see [10, Lemma 5.4.5]). The remaining point is weak convergence: one must upgrade the canonical realization of $\mathrm{gr}^p \mathrm{H}^{p+q}(X)$ as a subquotient of $E_\infty^{p,q}$ (Proposition 6.4.1) to an isomorphism.

Recall the formulas of Proposition 6.4.1. Since $\mathrm{F}^{N(n+1)} X^{n+1} = 0$ and $\mathrm{F}^\bullet X^{n-1}$ is exhaustive, for $r \gg 0$ the formulas describing $Z_r^{p,q}$ and $B_r^{p,q}$ reduce to exactly the numerator and denominator describing $\mathrm{gr}^p \mathrm{H}^{p+q}(X)$. Assembling these ingredients yields the required isomorphism. See [10, Theorem 5.5.5] for the remaining details. $\square$

Corollary 6.4.5 Let $\mathrm{F}^\bullet X$ be a filtered complex with

$$\mathrm{F}^0 X^n = X^n, \quad \mathrm{F}^{n+1} X^n = 0 \quad (n \in \mathbb{Z}), \tag{6.4.13}$$

so that its spectral sequence lies in the first quadrant. There is a canonical exact sequence

$$0 \to E_2^{1,0} \to \mathrm{H}^1(X) \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to \mathrm{H}^2(X). \tag{6.4.14}$$

Proof Substituting $p = 2$ into the edge exact sequence of Remark 6.2.8 gives

$$0 \to E_\infty^{0,1} \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to E_\infty^{2,0} \to 0. \tag{6.4.15}$$

By Theorem 6.4.4 the spectral sequence converges strongly, so $E_\infty^{0,1} \simeq \mathrm{gr}^0 \mathrm{H}^1(X)$, $E_\infty^{1,0} \simeq \mathrm{gr}^1 \mathrm{H}^1(X)$, and $E_\infty^{2,0} \simeq \mathrm{gr}^2 \mathrm{H}^2(X)$. The hypotheses give

$$\begin{aligned} \mathrm{gr}^2 \mathrm{H}^2(X) = \mathrm{F}^2 \mathrm{H}^2(X), \quad \mathrm{gr}^1 \mathrm{H}^1(X) = \mathrm{F}^1 \mathrm{H}^1(X), \\ \mathrm{gr}^0 \mathrm{H}^1(X) = \mathrm{H}^1(X)/\,\mathrm{F}^1 \mathrm{H}^1(X) = \mathrm{H}^1(X)/E_\infty^{1,0}. \end{aligned} \tag{6.4.16}$$

Moreover, in the first quadrant no differential enters or leaves $E_r^{1,0}$ for $r \geq 2$ (Remark 6.2.8), so $E_2^{1,0} = E_\infty^{1,0}$. Splicing these identifications into the edge exact sequence yields the asserted five-term sequence. $\square$

There are subtler, weaker hypotheses for convergence (see [1]). Nevertheless, the finite filtrations, the setting of all our applications, already fall under Theorem 6.4.4.

Everything in this section has homological counterparts, which we omit.

Exercise 6.4.6 Let $\mathcal{E}$ be the cohomological spectral sequence associated with a filtered complex X. Assume that $\mathcal{E}$ strongly converges (Definition 6.4.3) and lies in the first quadrant with $E_2^{p,q} = 0$ unless $p \in \{0, 1\}$. Show that for every n there is a canonical short exact sequence

$$0 \to E_2^{1,n-1} \to \mathrm{H}^n(X) \to E_2^{0,n} \to 0 \tag{6.4.17}$$

(recall that $\mathrm{H}^n(X)$ carries a filtration with graded pieces $\simeq E_\infty^{p,n-p}$; cf. Definition 6.4.3). What replaces this when E_2 is nonzero only for $q \in \{0, 1\}$? Cf. [10, Chapter 5, Exercise 5].

6.5 The two spectral sequences of a double complex

The most useful spectral sequences in algebra come from double complexes (Definition 3.8.1). For a finite double complex (each diagonal has finitely many nonzero summands, Definition 3.8.2) the two totalizations $\mathrm{tot}^{\oplus}$ and $\mathrm{tot}^{\prod}$ (Remark 3.8.4) coincide; denote the common complex by tot. (Countable coproducts or products would be needed only to totalize arbitrary double complexes.)

Let $X^{\bullet,\bullet}$ be a double complex. The total complex $\mathrm{tot}(X)$ carries two natural descending filtrations, according to the column or the row in which a summand sits:

$$\begin{aligned} \mathrm{F}_{\mathrm{I}}^{p}\,(\mathrm{tot}\,X)^n &= \bigoplus_{\substack{i+j=n \\ i \geq p}} X^{i,j}, \\ \mathrm{F}_{\mathrm{II}}^{q}\,(\mathrm{tot}\,X)^n &= \bigoplus_{\substack{i+j=n \\ j \geq q}} X^{i,j}. \end{aligned} \tag{6.5.1}$$

Each determines a spectral sequence via Proposition 6.4.1; call them $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$, with pages $E_{\mathrm{I},r}^{p,q}$ and $E_{\mathrm{II},r}^{p,q}$ respectively. Both filtrations are exhaustive and separated; if X is first-quadrant, then $\mathrm{F}_{\mathrm{I}}^{0}\,(\mathrm{tot}\,X)^n = (\mathrm{tot}\,X)^n$ and $\mathrm{F}_{\mathrm{I}}^{n+1}\,(\mathrm{tot}\,X)^n = 0$, and likewise for F_{II}, so both spectral sequences lie in the first quadrant.

Proposition 6.5.1 (*the two spectral sequences of a double complex*) The first few pages of $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$ are as follows.

	$E_0^{p,q}$	$E_1^{p,q}$	$E_2^{p,q}$
$\mathcal{E}_{\mathrm{I}}$	$X^{p,q}$	$\mathrm{H}^q(X^{p,\bullet})$	$(\mathrm{H}_{\mathrm{I}}\,\mathrm{H}_{\mathrm{II}}(X))^{p,q}$
$\mathcal{E}_{\mathrm{II}}$	$X^{q,p}$	$\mathrm{H}^q(X^{\bullet,p})$	$(\mathrm{H}_{\mathrm{II}}\,\mathrm{H}_{\mathrm{I}}(X))^{q,p}$

Here d_0 of $\mathcal{E}_{\mathrm{I}}$ is the vertical differential $(-1)^p d_\triangle$ (the sign from Definition 3.8.3) and d_1 is induced by the horizontal differential $d_\triangleright$, while for $\mathcal{E}_{\mathrm{II}}$ the roles of the two differentials are exchanged, the sign now sitting on d_1: there $d_0 = d_\triangleright$ and d_1 is induced by $(-1)^q d_\triangle$. Thus $\mathcal{E}_{\mathrm{I}}$ takes vertical cohomology first and then horizontal, while $\mathcal{E}_{\mathrm{II}}$ does the reverse.

Proof (*Sketch*) One reads off the $Z_r^{p,q}$ and $B_r^{p,q}$ formulas of Proposition 6.4.1 for the two filtrations F_{I} and F_{II}; the sign on $d_\triangle$ comes from the totalization differential $d = d_\triangleright + (-1)^p d_\triangle$ of Definition 3.8.3. See [10, Proposition 5.6.1]. □

Theorem 6.5.2 If X is a finite double complex (i.e. each diagonal has finitely many nonzero summands, Definition 3.8.2), then both $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$ are bounded and strongly converge to $\mathrm{H}^\bullet(\operatorname{tot} X)$, with finite induced filtrations.

Proof The filtrations $\mathrm{F}_{\mathrm{I}}^\bullet(\operatorname{tot} X)^n$ and $\mathrm{F}_{\mathrm{II}}^\bullet(\operatorname{tot} X)^n$ are finite for every n; apply Theorem 6.4.4.□

The central computational fact of this chapter is that $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$, though with very different pages, *converge to the same thing*, namely the cohomology of the total complex.

Example 6.5.3 The comparison between $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$ is a powerful technique. We give three standard applications.

- *Re-deriving balancing.* Take abelian categories $\mathcal{A}_1$, $\mathcal{A}_2$ with enough injectives, a balanced bifunctor $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ (Definition 4.3.1), and injective resolutions $X_i \to I_i^\bullet$; the double complex $Y^{p,q} = F(I_1^p, I_2^q)$ is first-quadrant, hence finite (Definition 3.8.2), so Theorem 6.5.2 applies. For $\mathcal{E}_{\mathrm{I}}$, balancedness makes $E_{\mathrm{I},1}^{p,q} = \mathrm{H}^q(F(I_1^p, I_2^\bullet))$ vanish for $q \neq 0$ and equal $F(I_1^p, X_2)$ at $q = 0$, so the spectral sequence degenerates at $E_{\mathrm{I},2}$ with

$$E_{\mathrm{I},2}^{p,0} = \mathrm{R}_{\mathrm{I}}^p\, F(X_1, X_2), \quad \mathrm{H}^p(\operatorname{tot} Y) \simeq \mathrm{R}_{\mathrm{I}}^p\, F(X_1, X_2). \tag{6.5.2}$$

  Running $\mathcal{E}_{\mathrm{II}}$ likewise gives $\mathrm{H}^p(\operatorname{tot} Y) \simeq \mathrm{R}_{\mathrm{II}}^p\, F(X_1, X_2)$. Combining, $\mathrm{R}_{\mathrm{I}}\, F \simeq \mathrm{R}_{\mathrm{II}}\, F$, a clean re-proof of Theorem 4.3.2 ([10, Example 5.6.4]).

- *Comparison.* A morphism $f : X \to Y$ of finite double complexes inducing an isomorphism on $\mathrm{H}_{\mathrm{I}}\,\mathrm{H}_{\mathrm{II}}$ (or on $\mathrm{H}_{\mathrm{II}}\,\mathrm{H}_{\mathrm{I}}$) induces a quasi-isomorphism $\operatorname{tot}(f)$: the morphism of spectral sequences is an isomorphism from the E_2 page on, hence on E_∞ (Exercise 6.2.4), and Proposition 6.1.3 upgrades the induced isomorphisms on graded pieces to an isomorphism $\mathrm{H}^n(\operatorname{tot} f)$, the filtrations being finite (Theorem 6.5.2). See [10, Example 5.6.3].

- *The hypercohomology spectral sequence.* For abelian categories $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A}$ having enough injectives, a left-exact functor $F : \mathcal{A} \to \mathcal{B}$, and $X \in \operatorname{Ob}(\mathsf{C}^+(\mathcal{A}))$, there is a finite double complex (Definition 3.8.2) in $\mathcal{B}$ whose $\mathcal{E}_{\mathrm{II}}$ spectral sequence satisfies

$$E_{\mathrm{II},2}^{p,q} = (\mathrm{R}^p\, F)(\mathrm{H}^q(X)) \Rightarrow \mathrm{R}^{p+q}\, F(X). \tag{6.5.3}$$

  This allows one to compute the derived functors at a complex from their values at objects. The proof runs as follows; the left-derived version is obtained by duality.

  View X as a double complex concentrated in row 0, and take a Cartan–Eilenberg resolution $\varepsilon : X \to I$ (Remark 4.1.10); this is a morphism in $\mathsf{C}_{\mathrm{f}}^2(\mathcal{A})$ and $I^{p,\bullet}$ is an injective resolution of X^p, for all p. Therefore, $\mathrm{H}_{\mathrm{II}}(\varepsilon)$ is an isomorphism, hence $\mathrm{H}_{\mathrm{I}}\,\mathrm{H}_{\mathrm{II}}(\varepsilon)$ is an isomorphism; by the comparison criterion of the preceding item, $\operatorname{tot}(\varepsilon) : X = \operatorname{tot}(X) \to \operatorname{tot}(I)$ is a quasi-isomorphism. Each $\operatorname{tot}^n(I)$ is a finite direct sum of injectives, so this is an injective resolution of X.

  The convergent spectral sequences $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$ associated with the double complex $\mathsf{C}^2(F)(I)$ (Theorem 6.5.2) both converge to the same abutment

$$\mathrm{H}^{p+q} \operatorname{tot}(\mathsf{C}^2(F)(I)) = \mathrm{H}^{p+q}\, \mathsf{C}(F)(\operatorname{tot} I) = \mathrm{R}^{p+q}\, F(X). \tag{6.5.4}$$

  Since $I^{p,\bullet}$ is an injective resolution of X^p for all p, we see that

$$\begin{aligned} E^{p,q}_{\mathrm{I},1} &= \mathrm{H}^q(FI^{p,\bullet}) \simeq \mathrm{R}^q F(X^p), \\ E^{p,q}_{\mathrm{I},2} &= \mathrm{H}^p[\cdots \to \mathrm{R}^q F(X^p) \to \mathrm{R}^q F(X^{p+1}) \to \cdots]. \end{aligned} \tag{6.5.5}$$

For $\mathcal{E}_{\mathrm{II}}$, the Cartan–Eilenberg property makes the horizontal cohomology $\mathrm{H}_{\mathrm{I}}(I)^{q,\bullet}$ an injective resolution of $\mathrm{H}^q(X)$; since the short exact sequences of the Cartan–Eilenberg construction split (Remark 4.1.10), F preserves horizontal cohomology:

$$\mathrm{H}_{\mathrm{I}}\left(\mathsf{C}^2(F)(I)\right)^{q,\bullet} \simeq \mathsf{C}(F)(\mathrm{H}_{\mathrm{I}}(I)^{q,\bullet}). \tag{6.5.6}$$

Therefore $E^{p,q}_{\mathrm{II},2} = (\mathrm{R}^p F)(\mathrm{H}^q(X)) \Rightarrow \mathrm{R}^{p+q} F(X)$. This is the spectral sequence we seek.

The main application is the spectral sequence for the derived functors of a composite.

Theorem 6.5.4 Consider additive functors $\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$ between abelian categories, left exact throughout, with $\mathcal{A}$ and $\mathcal{A}'$ having enough injectives. Assume F sends injectives to F'-acyclic objects (Convention 4.2.12). Then for every $X \in \mathrm{Ob}(\mathcal{A})$ there is a first-quadrant spectral sequence, strongly convergent, with

$$E^{p,q}_2 = (\mathrm{R}^p F')(\mathrm{R}^q F)(X) \Rightarrow \mathrm{R}^{p+q}(F'F)(X). \tag{6.5.7}$$

This is called the ***Grothendieck spectral sequence*** . We also have the five-term low-degree exact sequence

$$\begin{aligned} 0 &\to (\mathrm{R}^1 F')(FX) \to \mathrm{R}^1(F'F)(X) \to F'((\mathrm{R}^1 F)(X)) \\ &\to (\mathrm{R}^2 F')(FX) \to \mathrm{R}^2(F'F)(X). \end{aligned} \tag{6.5.8}$$

Proof Choose an injective resolution $0 \to X \to I^0 \to I^1 \to \cdots$ and set $I^n := 0$ for $n < 0$. In $\mathcal{A}'$ take a Cartan–Eilenberg resolution $\mathsf{C}(F)(I) \to J$ (Remark 4.1.10), and consider the convergent spectral sequences $\mathcal{E}_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}}$ associated with the first-quadrant double complex $\mathsf{C}^2(F')(J)$. For $\mathcal{E}_{\mathrm{I}}$, Proposition 6.5.1 gives

$$\begin{aligned} E^{p,q}_{\mathrm{I},2} &= \mathrm{H}^p[\cdots \to (\mathrm{R}^q F')(FI^p) \to (\mathrm{R}^q F')(FI^{p+1}) \to \cdots] \\ &\Rightarrow \mathrm{H}^{p+q}\,\mathrm{tot}(\mathsf{C}^2(F')(J)). \end{aligned} \tag{6.5.9}$$

Since $(\mathrm{R}^q F')(FI^p) = 0$ for $q \neq 0$ by assumption, $E^{p,q}_{\mathrm{I},2} = 0$ for $q \neq 0$. Exactly as in the re-derivation of balancing (the first item of Example 6.5.3), $\mathcal{E}_{\mathrm{I}}$ therefore degenerates at E_2, with

$$E^{p,q}_{\mathrm{I},\infty} = E^{p,q}_{\mathrm{I},2} = \begin{cases} \mathrm{H}^p(F'F(I^\bullet)) = \mathrm{R}^p(F'F)(X) \text{ if } q = 0 \\ 0 \text{ if } q \neq 0 \end{cases}. \tag{6.5.10}$$

So $\mathrm{H}^p\,\mathrm{tot}(\mathsf{C}^2(F')(J)) = E^{p,0}_{\mathrm{I},\infty} = \mathrm{R}^p(F'F)(X)$.

Turning to $\mathcal{E}_{\mathrm{II}}$, the same device as in the hypercohomology spectral sequence (Example 6.5.3) shows that

- the q-th column of horizontal cohomology $\mathrm{H}_{\mathrm{I}}(J)^{q,\bullet}$ is an injective resolution of $\mathrm{H}^q(\mathsf{C}(F)(I)) = (\mathrm{R}^q F)(X)$;
- F' preserves horizontal cohomologies of J (Remark 4.1.10).

Therefore

$$E^{p,q}_{\mathrm{II},2} = (\mathrm{R}^p F')(\mathrm{R}^q F)(X) \Rightarrow \mathrm{H}^{p+q}\,\mathrm{tot}(\mathsf{C}^2(F')(J)) = \mathrm{R}^{p+q}(F'F)(X). \tag{6.5.11}$$

Thus $\mathcal{E}_{\mathrm{II}}$ is the spectral sequence of the first assertion.

For the five-term exact sequence, Corollary 6.4.5 applied to $\mathcal{E}_{\mathrm{II}}$ yields the exact sequence

$$0 \to E_{\mathrm{II},2}^{1,0} \to \mathrm{H}^1 \operatorname{tot}(\mathsf{C}^2(F')(J)) \to E_{\mathrm{II},2}^{0,1} \xrightarrow{d} E_{\mathrm{II},2}^{2,0} \to \mathrm{H}^2 \operatorname{tot}(\mathsf{C}^2(F')(J)). \qquad (6.5.12)$$

We already know $\mathrm{H}^1 \operatorname{tot}(\mathsf{C}^2(F')(J)) = \mathrm{R}^1(F'F)(X)$ and $\mathrm{H}^2 \operatorname{tot}(\mathsf{C}^2(F')(J)) = \mathrm{R}^2(F'F)(X)$ from the degeneration of $\mathcal{E}_{\mathrm{I}}$. Substituting $E_{\mathrm{II},2}^{p,q} = (\mathrm{R}^p F')(\mathrm{R}^q F)(X)$ turns this into the asserted exact sequence. □

The homological versions of these results, for right-exact functors, hold by duality.

Remark 6.5.5 (*change of rings*) Let $R \to S$ be a ring map. For a right S-module X and a left R-module Y (writing S_R, X_R for S, X viewed as right R-modules), the homological version of Theorem 6.5.4 (for right-exact functors) applied to $X \otimes_S (S \otimes_R (\cdot)) \simeq X \otimes_R (\cdot)$ gives a first-quadrant homological spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^S_p(X, \operatorname{Tor}^R_q(S_R, Y)) \Rightarrow \operatorname{Tor}^R_{p+q}(X_R, Y), \qquad (6.5.13)$$

where $\operatorname{Tor}^R_q(S_R, Y)$ is a left S-module via the (S, R)-bimodule structure of S (Remark 4.4.5).

For a left S-module X and a left R-module Y, the cohomological version applied to $\operatorname{Hom}_R({}_RX, \cdot) \simeq \operatorname{Hom}_S(X, \operatorname{Hom}_R({}_RS, \cdot))$ gives

$$E_2^{p,q} = \operatorname{Ext}^p_S(X, \operatorname{Ext}^q_R({}_RS, Y)) \Rightarrow \operatorname{Ext}^{p+q}_R({}_RX, Y), \qquad (6.5.14)$$

where ${}_RS$ and ${}_RX$ denote S and X viewed as left R-modules, and $\operatorname{Ext}^q_R({}_RS, Y)$ is a left S-module via the (R, S)-bimodule structure of S (Remark 4.4.5).

Both degenerate at the second page when S has global dimension ≤ 1 (Definition 8.3.4). In the cohomological case this gives the universal-coefficient-type sequence

$$0 \to \operatorname{Ext}^1_S(X, \operatorname{Ext}^{n-1}_R({}_RS, Y)) \to \operatorname{Ext}^n_R({}_RX, Y) \to \operatorname{Hom}_S(X, \operatorname{Ext}^n_R({}_RS, Y)) \to 0. \qquad (6.5.15)$$

The homological one degenerates under the weaker hypothesis $\operatorname{Tor}^S_p(X, \cdot) = 0$ for $p > 1$, and then yields the short exact sequence of Exercise 6.5.8. See [10, Examples 5.6.7 and 5.6.8].

Exercise 6.5.6 Assuming the reader is familiar with sheaf theory, apply Theorem 6.5.4 with $F = \pi_*$ (direct image along π) and $F' = \Gamma$ (global sections, whose derived functors are sheaf cohomology) to state the *Leray spectral sequence* $\mathrm{H}^p(Y, \mathrm{R}^q \pi_* \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F})$ for a map $\pi : X \to Y$ of topological spaces. (Take the existence of enough injectives in sheaf categories as given.) What is the low-degree exact sequence?

Exercise 6.5.7 Suppose a strongly convergent spectral sequence $E_r^{p,q} \Rightarrow \mathrm{H}^{p+q}$ associated to a filtered complex X (so $\mathrm{H}^n = \mathrm{H}^n(X)$) has only finitely many nonzero terms at some page E_r, with all terms finite-dimensional over a field $\Bbbk$. Prove that

$$\sum_n (-1)^n \dim_{\Bbbk} \mathrm{H}^n = \sum_{p,q} (-1)^{p+q} \dim_{\Bbbk} E_r^{p,q}. \qquad (6.5.16)$$

Hint. The alternating sum of dimensions is unchanged by taking cohomology. Cf. Theorem 5.3.4; the K_0-valued version is [10, Proposition 5.5.8].

Exercise 6.5.8 Let $R \to S$ be a ring map, X a right S-module such that $\operatorname{Tor}_p^S(X, \cdot) = 0$ for $p > 1$, and Y a left R-module. Deduce from Remark 6.5.5 a natural short exact sequence

$$0 \to X \otimes_S \operatorname{Tor}_n^R(S_R, Y) \to \operatorname{Tor}_n^R(X_R, Y) \to \operatorname{Tor}_1^S\big(X, \operatorname{Tor}_{n-1}^R(S_R, Y)\big) \to 0, \quad (6.5.17)$$

with S_R, X_R as in Remark 6.5.5 and the convention $\operatorname{Tor}_{-1}^R = 0$. Cf. [10, Example 5.6.8].

6.6 Other topics

We close with three directions, sketched only.

6.6.1 Künneth, revisited

The universal coefficient theorem, i.e. Theorem 4.4.8 with D concentrated in degree 0, is an immediate consequence of the homological counterpart of Theorem 6.5.2. Let C be a chain complex of right R-modules, flat in each degree and bounded below, and let D be a left R -module, regarded as a complex concentrated in degree 0; resolve D by projectives $P_\bullet$ and form the double chain complex $C_\bullet \otimes_R P_\bullet$, which is finite (Definition 3.8.2). The associated homological spectral sequence $\mathcal{E}_{\mathrm{II}}$ satisfies

$$\left(E_{\mathrm{II}}^2\right)_{p,q} = \operatorname{Tor}_p^R\big(\mathrm{H}_q(C), D\big) \Rightarrow \mathrm{H}_{p+q}(C \otimes_R D). \quad (6.6.1)$$

Assume now, as in Theorem 4.4.8, that the boundaries $\operatorname{im}\left(d_n^C\right)$ are flat; then the cycles $\ker\left(d_n^C\right)$ are flat as well, by the first step of the proof of Theorem 4.4.8, and the short exact sequence

$$0 \to \operatorname{im}\left(d_{q+1}^C\right) \to \ker\left(d_q^C\right) \to \mathrm{H}_q(C) \to 0, \quad (6.6.2)$$

which is a flat resolution of $\mathrm{H}_q(C)$, implies $\left(E_{\mathrm{II}}^2\right)_{p,q} = 0$ unless $p \in \{0, 1\}$.

Therefore the spectral sequence degenerates at E^2 and, only two columns surviving, yields for every n the short exact sequence

$$0 \to \mathrm{H}_n(C) \otimes_R D \to \mathrm{H}_n(C \otimes_R D) \to \operatorname{Tor}_1^R(\mathrm{H}_{n-1}(C), D) \to 0 \quad (6.6.3)$$

of Theorem 4.4.8 (by the homological version of Exercise 6.4.6); cf. [10, Chapter 5, Exercise 8].

For arbitrary bounded complexes C and D this generalizes to the ***Künneth spectral sequence***

$$\left(E^2\right)_{p,q} = \bigoplus_{p'+p''=q} \operatorname{Tor}_p^R\big(\mathrm{H}_{p'}(C), \mathrm{H}_{p''}(D)\big) \Rightarrow \mathrm{H}_{p+q}\left(C \overset{\mathrm{L}}{\otimes}_R D\right), \quad (6.6.4)$$

valid with no flatness hypothesis; here the abutment is the derived tensor product (Definition 8.6.1), which agrees with $C \otimes_R D$ when C is degreewise flat.

6.6.2 Multiplicative structures

For simplicity, we consider complexes of R-modules, where R is a commutative ring, so that $\otimes := \otimes_R$ makes sense.

When the complex carries a product compatible with the filtration, the spectral sequence inherits one. Concretely, a ***differential graded algebra*** (dg-algebra) is a complex (X, d) with a multiplication $X \otimes X \to X$ satisfying $X^p \cdot X^{p'} \subset X^{p+p'}$ and the Leibniz rule $d(xy) = (dx)y +$

$(-1)^p x(dy)$ for $x \in X^p$. The de Rham complex of a manifold, with wedge product and exterior derivative, is the geometric prototype with $R = \mathbb{R}$; its cohomology is the de Rham cohomology *ring*.

If a dg-algebra X is filtered with $\mathrm{F}^p X^n \cdot \mathrm{F}^{p'} X^{n'} \subset \mathrm{F}^{p+p'} X^{n+n'}$, then the spectral sequence of Proposition 6.4.1 carries a canonical multiplication on every page, and the convergence isomorphism $E_\infty \simeq \operatorname{gr} \mathrm{H}(X, d)$ is one of graded algebras. This is the device that makes spectral sequences compute cohomology *rings*, not merely groups, indispensable in topology. We refer to [10, §5.7] for the details.

6.6.3 The Lyndon–Hochschild–Serre spectral sequence

Our final example comes from group cohomology, whose rudiments we recall; the proofs and much more can be found in [10, §§6.1–6.2]. Fix a commutative ring $\Bbbk$ and a group G. A ***G-module*** is a $\Bbbk$-module endowed with a linear left G-action, or what amounts to the same, a left module over the group algebra $\Bbbk[G]$; the *trivial* G-module is $\Bbbk$ with $gm = m$. For a G-module M, its ***invariants*** and ***coinvariants*** are

$$\begin{aligned} M^G &= \{m \in M : gm = m \quad \text{for all } g \in G\}, \\ M_G &= M / \langle gm - m : g \in G, m \in M \rangle, \end{aligned} \tag{6.6.5}$$

the largest submodule and the largest quotient of M on which G acts trivially. They define $\Bbbk$ -linear functors $(\cdot)^G$ and $(\cdot)_G$ from G-modules to $\Bbbk$-modules, left and right exact respectively. Their derived functors are ***group cohomology*** and ***group homology*** : for all $n \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{H}^n(G, M) &:= \big(\mathrm{R}^n (\cdot)^G\big)(M), \\ \mathrm{H}_n(G, M) &:= (\mathrm{L}_n (\cdot)_G)(M). \end{aligned} \tag{6.6.6}$$

Under the identifications $(\cdot)^G \simeq \operatorname{Hom}_{\Bbbk[G]}(\Bbbk, \cdot)$ and $(\cdot)_G \simeq \Bbbk \otimes_{\Bbbk[G]} (\cdot)$, these are simply Ext and Tor:

$$\begin{aligned} \mathrm{H}^n(G, M) &\simeq \operatorname{Ext}^n_{\Bbbk[G]}(\Bbbk, M), \\ \mathrm{H}_n(G, M) &\simeq \operatorname{Tor}^{\Bbbk[G]}_n(\Bbbk, M). \end{aligned} \tag{6.6.7}$$

Both are computed at one stroke by a canonical resolution. Let L_n be the free $\Bbbk$-module with basis G^{n+1}, with G acting diagonally, $g(g_0, \cdots, g_n) = (gg_0, \cdots, gg_n)$ (so that each L_n is a free $\Bbbk[G]$ -module). The two alternating sums of face maps

$$\partial_n(g_0, \cdots, g_n) := \sum_{i=0}^{n} (-1)^{n-i} (g_0, \cdots, \hat{g}_i, \cdots, g_n), \quad \partial'_n := (-1)^n \partial_n \tag{6.6.8}$$

assemble, together with the augmentation $\mathsf{L}_0 \to \Bbbk$ sending $(g_0) \mapsto 1$, into free resolutions $\mathsf{L} \to \Bbbk$ of the trivial $\Bbbk[G]$-module. The choice between ∂_n and ∂'_n only changes the complex by the degree-dependent sign $(-1)^n$ (equivalently, by the flip $(g_0, \cdots, g_n) \mapsto (g_n, \cdots, g_0)$), but it decides the shape of the differentials: applying $\operatorname{Hom}_{\Bbbk[G]}(\mathsf{L}, M)$, formed with ∂_n (the sign $(-1)^{n+1}$ of Definition 3.4.1 is then exactly compensated), and $\mathsf{L} \otimes_{\Bbbk[G]} M$, formed with ∂'_n and with L viewed as a complex of right $\Bbbk[G]$-modules via $(g_0, \cdots, g_n)g = (g_0 g, \cdots, g_n g)$, yields the ***standard complexes*** with the standard differentials: their degree-n terms are the $\Bbbk$-modules

$$C^n(G,M) = \{\text{maps } f : G^n \to M\}, \quad C_n(G,M) = \bigoplus_{G^n} M, \tag{6.6.9}$$

with the ***bar differentials***

$$\begin{aligned}(df)(g_1,\cdots,g_{n+1}) &= g_1 f(g_2,\cdots,g_{n+1}) + \sum_{k=1}^{n}(-1)^k f(\cdots,g_k g_{k+1},\cdots)\\ &\quad +(-1)^{n+1} f(g_1,\cdots,g_n),\\ d((g_1\mid\cdots\mid g_n)\otimes x) &= (g_2\mid\cdots\mid g_n)\otimes x + \sum_{k=1}^{n-1}(-1)^k(\cdots\mid g_k g_{k+1}\mid\cdots)\otimes x\\ &\quad +(-1)^n(g_1\mid\cdots\mid g_{n-1})\otimes g_n x,\end{aligned} \tag{6.6.10}$$

in exactly the shape of the Hochschild complexes of Section 3.5; here $(g_1 \mid \cdots \mid g_n)\otimes x$ denotes the element of $\bigoplus_{G^n} M$ supported at $(g_1,\cdots,g_n)$, with value $x \in M$. Their (co)homology is the group (co)homology:

$$\mathrm{H}^n(C^\bullet(G,M)) \simeq \mathrm{H}^n(G,M), \quad \mathrm{H}_n(C_\bullet(G,M)) \simeq \mathrm{H}_n(G,M). \tag{6.6.11}$$

The resolution $\mathsf{L} \to \Bbbk$ itself will resurface as a bar construction in Example 9.7.9.

Group (co)homology is moreover functorial in the group: every homomorphism $\varphi : H \to G$ induces an exact functor $\varphi^* : \Bbbk[G]\text{-}\mathsf{Mod} \to \Bbbk[H]\text{-}\mathsf{Mod}$ (same underlying $\Bbbk$-module, H acting through φ) and canonical homomorphisms

$$\mathrm{H}^n(G,M) \to \mathrm{H}^n(H,\varphi^* M), \quad \mathrm{H}_n(H,\varphi^* M) \to \mathrm{H}_n(G,M), \tag{6.6.12}$$

making group cohomology contravariant and group homology covariant in the group variable; for the inclusion of a subgroup (resp. a quotient projection) these are the classical restriction (resp. inflation) maps. See [10, §6.5].

Coming back to spectral sequences, the most celebrated single example is the ***Lyndon–Hochschild–Serre spectral sequence*** in this context: for a group extension $1 \to N \to G \to Q \to 1$ and a G-module M (conjugation makes each $\mathrm{H}^q(N,M)$ a Q-module), there is a first-quadrant spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(Q,\mathrm{H}^q(N,M)) \Rightarrow \mathrm{H}^{p+q}(G,M), \tag{6.6.13}$$

obtained in either of two ways (see [10, §6.9]):

(i) as a special case of the Grothendieck spectral sequence (Theorem 6.5.4), applied to $(\cdot)^G = (\cdot)^Q \circ (\cdot)^N$ (invariants of the normal subgroup N, then of Q);

(ii) more directly and canonically by
 (a) passing to the normalized standard complex $\overline{C}^\bullet(G,M) \subset C^\bullet(G,M)$, consisting (in degree n) of all $f : G^n \to M$ such that $f(g_1,\cdots,g_n) = 0$ whenever $g_i = 1_G$ for some $1 \le i \le n$, which does not alter the cohomology,
 (b) taking the filtration defined by $f \in \mathrm{F}^p\,\overline{C}^n(G,M)$ if and only if $f(g_1,\cdots,g_n) = 0$ whenever $(g_1,\cdots,g_n)$ contains at least $n-p+1$ entries in N (normality of N is what keeps the filtration stable under d), so that $\mathrm{F}^p\,\overline{C}^n(G,M) = 0$ for $p > n$; Proposition 6.4.1 then applies, and since the filtration is exhaustive and bounded above, Theorem 6.4.4 yields the claimed convergence.

Route (ii) also yields the multiplicative structure of Subsection 6.6.2, the Hochschild–Serre filtration being compatible with the cup product.

We stop here; [10, Chapter 6] develops group cohomology systematically. The example is the prototype for how the machinery of this chapter organizes real computations.

Chapter 7

Triangulated and Derived Categories

The derived category was created by A. Grothendieck and J.-L. Verdier in the 1960s to study duality theorems in algebraic geometry; subsequent developments in geometry, topology, and representation theory have confirmed it as an efficient and flexible language. The derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ records information intermediate between a complex and its cohomology, and is best expressed in the language of *triangulated categories* developed in Section 7.1 and Section 7.2: $\mathsf{D}(\mathcal{A})$ carries a translation automorphism $X \mapsto X[1]$ together with a class of ***exact* (*or distinguished*) *triangles***

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1], \quad \text{abbreviated } X \to Y \to Z \xrightarrow{+1}, \tag{7.0.1}$$

subject to a family of axioms (TR0)–(TR5). For instance a triangle can be rotated into

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]. \tag{7.0.2}$$

Exact triangles replace the short exact sequences of $\mathsf{C}(\mathcal{A})$ — though the more apt analogy is with the *cofiber sequences* of homotopy theory.

The passage from $\mathsf{C}(\mathcal{A})$ to $\mathsf{D}(\mathcal{A})$ (Section 7.5) proceeds in two steps. The first replaces $\mathsf{C}(\mathcal{A})$ by the homotopy category $\mathsf{K}(\mathcal{A})$, equipped with its translation functor $X \mapsto X[1]$ and a triangulated structure whose exact triangles are the mapping-cone triangles

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1] \tag{7.0.3}$$

up to isomorphism; the tools for this were prepared in Section 3.6. The second step formally inverts the quasi-isomorphisms (equivalently, annihilates the acyclic complexes) via *Verdier localization* (Section 7.4). The resulting category $\mathsf{D}(\mathcal{A})$ has the same objects as $\mathsf{C}(\mathcal{A})$, and its triangulated structure is inherited from $\mathsf{K}(\mathcal{A})$.

Cohomology descends to a cohomological functor $\mathrm{H}^0 : \mathsf{D}(\mathcal{A}) \to \mathcal{A}$, with $\mathrm{H}^n(X) := \mathrm{H}^0(X[n])$. The tautological functor $\mathcal{A} \to \mathsf{D}(\mathcal{A})$ is fully faithful, identifying $\mathcal{A}$ with the full subcategory $\mathsf{D}^{\leq 0}(\mathcal{A}) \cap \mathsf{D}^{\geq 0}(\mathcal{A})$; and, when $\mathcal{A}$ has enough injectives or projectives,

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n]) \simeq \mathrm{Ext}^n_{\mathcal{A}}(X, Y). \tag{7.0.4}$$

These are the themes of Section 7.6, where we also reconcile Ext^n with the classical *n-extensions* of Yoneda.

The derived functors are then revisited in Chapter 8: total derived functors as Kan extensions along the localization, existence via resolutions and the Grothendieck spectral sequence, and the derived bifunctors RHom, $\mathrm{R}\lim$, and $\overset{\mathrm{L}}{\otimes}$.

7.1 Triangulated categories: the definition

A triangulated category is a category equipped with a translation functor and a class of distinguished triangles. We begin with the translation: throughout, $(\mathcal{D}, T)$ denotes a category with translation in the sense of Definition 6.2.1, with T an auto-equivalence carrying a prescribed quasi-inverse T^{-1}, made into an adjoint equivalence; functors $(\mathcal{D}, T) \to (\mathcal{D}', T')$ carry a specified isomorphism $FT \simeq T'F$. We shall only ever meet *additive* categories with translation. The prototypical example is the complex category.

Example 7.1.1 For an additive category $\mathcal{A}$, the complex category $\mathsf{C}(\mathcal{A})$ with the shift $T : X \mapsto X[1]$ (Definition 3.1.2) is a category with translation, as is the homotopy category $\mathsf{K}(\mathcal{A})$ (Definition 3.2.3) with the inherited shift; the bounded variants $\mathsf{C}^\star(\mathcal{A})$, $\mathsf{K}^\star(\mathcal{A})$ for $\star \in \{+, -, \mathrm{b}\}$ likewise.

It is convenient to shift the degree of a morphism: we freely use morphisms $X \xrightarrow{+m} Y$ of degree m and their calculus, as recorded in Definition 6.2.2 (Chapter 6).

A ***triangle*** in $(\mathcal{D}, T)$ is a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ of morphisms. A morphism of triangles is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & TX \arrow[d, "T\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & TX'
\end{tikzcd} \tag{7.1.1}$$

with both rows being triangles. The abbreviation $X \to Y \to Z \xrightarrow{+1}$, with the last arrow being a morphism of degree $+1$, is often the cleanest.

Definition 7.1.2 A ***triangulated category*** is an additive category with translation $(\mathcal{D}, T)$ together with a class $\mathcal{H}$ of triangles, the ***exact (or distinguished) triangles*** , satisfying the following axioms.

- ▷ **(TR0)** Any triangle isomorphic to an exact triangle is exact.
- ▷ **(TR1)** For every $X \in \mathcal{D}$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$ is exact.
- ▷ **(TR2)** Every morphism $f : X \to Y$ embeds in an exact triangle $X \xrightarrow{f} Y \to Z \to TX$; the object Z is customarily called a *cone* of f, unique up to isomorphism (Corollary 7.2.2).
- ▷ **(TR3)** The triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is exact iff its counter-clockwise *rotation*

$$Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY \tag{7.1.2}$$

is exact.
- ▷ **(TR4)** Given the solid commutative diagram with both rows exact,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, dashed, "\gamma"'] & TX \arrow[d, dashed, "T\alpha"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX'
\end{tikzcd} \tag{7.1.3}$$

there exists $\gamma : Z \to Z'$ making the whole diagram commute.

▷ **(TR5)** Given exact triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to TX, \quad Y \xrightarrow{g} Z \xrightarrow{k} X' \to TY, \quad X \xrightarrow{gf} Z \xrightarrow{m} Y' \to TX, \tag{7.1.4}$$

there is an exact triangle $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} TZ'$ fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, "u"] & TX \arrow[d, equal] \\
X \arrow[r, "gf"] \arrow[d, "f"'] & Z \arrow[r, "m"] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "v"] & TX \arrow[d, "Tf"] \\
Y \arrow[r, "g"] \arrow[d, "h"'] & Z \arrow[r, "k"] \arrow[d, "m"'] & X' \arrow[r] \arrow[d, equal] & TY \arrow[d, "Th"] \\
Z' \arrow[r, "u"] & Y' \arrow[r, "v"] & X' \arrow[r, "w"] & TZ'
\end{tikzcd} \tag{7.1.5}$$

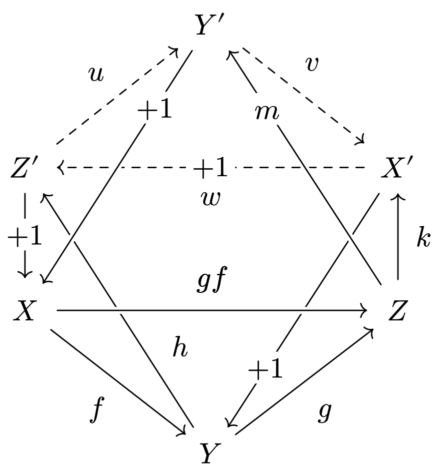
in which every row is an exact triangle.

A datum $(\mathcal{D}, T, \mathcal{H})$ satisfying only (TR0)–(TR4) is also called a ***pre-triangulated category*** . A ***triangulated functor*** $F : \mathcal{D} \to \mathcal{D}'$ is a functor of categories with translation carrying exact triangles to exact triangles; this defines when two triangulated functors are quasi-inverse, hence what it means for triangulated categories to be *equivalent*.

(TR5) is also known as the ***octahedral axiom*** .

Remark 7.1.3 Given signs ε, ζ, η, the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is isomorphic to $X \xrightarrow{\varepsilon f} Y \xrightarrow{\zeta g} Z \xrightarrow{\eta h} TX$ whenever $\varepsilon\zeta\eta = 1$ (rescale the middle two vertices). Thus the signs of morphisms in (TR3) may equally be $(-{+}{+})$, $(+{-}{+})$, or $(-{-}{-})$.

Remark 7.1.4 In the notation of Definition 6.2.2, the axiom (TR5) can be repackaged as an *octahedron*:



(7.1.6)

The dashed arrows are those whose existence is asserted; the arrows marked $+1$, as well as w, are of degree $+1$. Four of the eight faces of the octahedron are exact triangles, namely those with vertex sets (X, Y, Z'), (Y, Z, X'), (X, Z, Y'), and (Z', Y', X'); the remaining four faces commute. The details are left to the reader. This is the origin of the name "octahedral axiom".

One can also regard (TR5) as the triangulated analogue of the isomorphism theorem $(X/Z)/(Y/Z) \simeq X/Y$ for abelian categories.

Remark 7.1.5 The opposite of a (pre-)triangulated category is (pre-)triangulated. Regard T^{-1} as an endofunctor S of $\mathcal{D}^{\mathrm{op}}$; for every exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$, the opposites of its morphisms form a triangle

$$Z \to Y \to X \to SZ \tag{7.1.7}$$

in $(\mathcal{D}^{\mathrm{op}}, S)$, the last map being the opposite of $-T^{-1}h$. These triangles are the exact triangles of a (pre-)triangulated structure on $(\mathcal{D}^{\mathrm{op}}, S)$, the construction being involutive. This is the formal content of duality arguments involving $\mathcal{D}^{\mathrm{op}}$.

The first real consequence of the axioms is that exact triangles behave like short exact sequences when probed by additive functors.

Definition 7.1.6 Let $\mathcal{D}$ be pre-triangulated and $\mathcal{A}$ abelian. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is a ***cohomological functor*** if for every exact triangle $X \to Y \to Z \xrightarrow{+1}$ the sequence $HX \to HY \to HZ$ is exact.

Remark 7.1.7 Rotation (TR3) extends the exact sequence of Definition 7.1.6 to a long exact sequence

$$\cdots \to HT^{-1}Z \to HX \to HY \to HZ \to HTX \to HTY \to \cdots. \tag{7.1.8}$$

Rotation introduces signs, which do not affect exactness.

Lemma 7.1.8 If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is exact, then $gf = 0$.

Proof Axioms (TR1) and (TR4) applied to the diagram with first row $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ and second row $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ (with $\alpha = \mathrm{id}_X$, $\beta = f$) supply $\gamma : 0 \to Z$, the zero morphism; an easy chase yields $gf = 0$. □

The basic example of a cohomological functor is Hom itself.

Proposition 7.1.9 For a pre-triangulated $\mathcal{D}$ and any object S, the functors $\mathrm{Hom}(S, \cdot) : \mathcal{D} \to \mathsf{Ab}$ and $\mathrm{Hom}(\cdot, S) : \mathcal{D}^{\mathrm{op}} \to \mathsf{Ab}$ are cohomological. If $\mathcal{D}$ is $\Bbbk$-linear (Remark 1.7.2), replace Ab by $\Bbbk$-Mod.

Proof By duality (Remark 7.1.5), it suffices to treat $\mathrm{Hom}(S, \cdot)$. For an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$, Lemma 7.1.8 gives $g_* f_* = (gf)_* = 0$. If $\varphi \in \mathrm{Hom}(S, Y)$ satisfies $g_*\varphi = g\varphi = 0$, we apply (TR4) to the solid diagram whose first row is the exact triangle $S \to 0 \to TS \xrightarrow{-\mathrm{id}} TS$ and whose second row is $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$ (with $\alpha = \varphi$, $\beta = 0$); express the filler $TS \to TX$ as Tk, then $k : S \to X$ satisfies $fk = \varphi$, exhibiting φ in the image of f_*. □

Corollary 7.1.10 If $X \to Y \to 0 \xrightarrow{+1}$ is exact, then $X \to Y$ is an isomorphism.

Proof For every S, Proposition 7.1.9 gives an exact sequence $\mathrm{Hom}(S, T^{-1}0) \to \mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, 0)$ with the two end groups zero, so $\mathrm{Hom}(S, X) \xrightarrow{\sim} \mathrm{Hom}(S, Y)$. As S is arbitrary, $X \to Y$ is an isomorphism. □

Exercise 7.1.11 Let $(\mathcal{D}, T, \mathcal{H})$ be (pre-)triangulated. Show that reversing the sign of the last map of every triangle

$$\mathcal{H}^- := \left\{ \left[X \to Y \to Z \xrightarrow{-h} TX \right] : \left[X \to Y \to Z \xrightarrow{h} TX \right] \in \mathcal{H} \right\} \tag{7.1.9}$$

defines another (pre-)triangulated structure $(\mathcal{D}, T, \mathcal{H}^-)$, and that $(\mathcal{D}, T, \mathcal{H})$ and $(\mathcal{D}, T, \mathcal{H}^-)$ are equivalent as (pre-)triangulated categories.

7.2 Basic properties

Fix a (pre-)triangulated category $\mathcal{D}$. We record the structural facts needed later. The first is a five-lemma for triangles.

Proposition 7.2.1 (*two-out-of-three*) Consider a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & TX \arrow[d, "T\alpha"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd} \tag{7.2.1}$$

with both rows exact. If two of α, β, γ are isomorphisms, so is the third.

Proof Rotating the morphism of triangles by (TR3) — the equivalence T preserves and reflects isomorphisms — we may assume α and γ are isomorphisms. Fix $S \in \mathrm{Ob}(\mathcal{D})$: Proposition 7.1.9 applied to $\mathrm{Hom}(S, \cdot)$ gives a commutative diagram with exact rows, and the five lemma (Proposition 2.5.3) implies that $\mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, Y')$ is an isomorphism. Since S was arbitrary, β is an isomorphism. □

Corollary 7.2.2 For any morphism $f : X \to Y$, the object Z in an exact triangle $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ is unique up to (non-canonical) isomorphism.

Proof Given two such, (TR4) supplies a morphism of triangles $(\mathrm{id}_X, \mathrm{id}_Y, \gamma)$; Proposition 7.2.1 makes γ an isomorphism. □

The isomorphism in Corollary 7.2.2 is *not canonical* — a recurring theme.

One can show that any functor $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ between pre-triangulated categories that carries exact triangles to exact triangles is automatically additive, hence a triangulated functor. See [10, Corollary 4.2.7].

Proposition 7.2.3 Let $F : \mathcal{D} \dashv \mathcal{D}' : G$ be an adjunction between pre-triangulated categories. Then F is triangulated iff G is; in that case the unit and counit are compatible with the translations, and F, G give an equivalence whenever they are quasi-inverse.

Proof (*Sketch*) Assume F triangulated. The right adjoint of $T'FT^{-1} \simeq F$ is unique up to isomorphism, giving $GT' \simeq TG$; then a five-lemma argument with $\mathrm{Hom}(B, \cdot)$, for arbitrary

$B \in \mathcal{D}$ (as in Proposition 7.2.1), shows G carries exact triangles to exact triangles. See [10, Proposition 4.2.8, Corollary 4.2.9]. □

(Pre-)triangulated subcategories are detected by the cone condition.

Definition 7.2.4 A full additive subcategory $\mathcal{D}' \subset \mathcal{D}$, closed under T, is a ***triangulated subcategory*** (resp. ***pre-triangulated subcategory***) if it admits the triangulated (resp. pre-triangulated) structure for which the inclusion is triangulated. A (pre-)triangulated subcategory is ***saturated*** if it is closed under isomorphisms. Intersections of saturated subcategories remain saturated.

Proposition 7.2.5 A full additive subcategory $\mathcal{D}'$, closed under T, is a (pre-)triangulated subcategory if and only if the following condition holds: whenever $X \to Y \to Z \xrightarrow{+1}$ is exact in $\mathcal{D}$ and two of X, Y, Z lie in $\mathcal{D}'$, the third is isomorphic to an object of $\mathcal{D}'$. The structure is then unique: its exact triangles are precisely the triangles of $\mathcal{D}'$ that are exact in $\mathcal{D}$.

Proof Suppose first the structure exists. In an exact triangle $X \to Y \to Z \xrightarrow{+1}$ with two of X, Y, Z in $\mathcal{D}'$, rotate by (TR3) so that $X, Y \in \mathcal{D}'$ (two rotations may be needed; T restricts to an auto-equivalence of $\mathcal{D}'$, so this is harmless). By (TR2) in $\mathcal{D}'$ there is an exact triangle $X \to Y \to Z' \to TX$; it is exact in $\mathcal{D}$ because the inclusion is triangulated, so Corollary 7.2.2 gives $Z \simeq Z'$.

Conversely, assume the condition, and call a triangle of $\mathcal{D}'$ *exact* when it is exact in $\mathcal{D}$. Note first that T restricts to an auto-equivalence of $\mathcal{D}'$: it is fully faithful because $\mathcal{D}'$ is full, and for $X \in \mathcal{D}'$, rotating the (TR1)-triangle of $T^{-1}X$ and transporting along $TT^{-1}X \simeq X$ shows that $T^{-1}X$ is isomorphic to an object of $\mathcal{D}'$. Axioms (TR0), (TR1), (TR3), and (TR4) are inherited from $\mathcal{D}$; in (TR4) the filler provided by $\mathcal{D}$ is a morphism of $\mathcal{D}'$ by fullness. In (TR2), embed a morphism of $\mathcal{D}'$ into an exact triangle of $\mathcal{D}$: the condition identifies its third vertex with an object of $\mathcal{D}'$, and (TR0) transports the triangle into $\mathcal{D}'$. When $\mathcal{D}$ is triangulated, (TR5) is inherited as well: the octahedron furnished by (TR5) in $\mathcal{D}$ involves only the six given objects, and fullness keeps its morphisms in $\mathcal{D}'$.

The structure so obtained is unique, since for any structure with triangulated inclusion the exact triangles are precisely those exact in $\mathcal{D}$: one inclusion is clear, the other follows from (TR2), Corollary 7.2.2, fullness, and (TR0). □

Cohomological functors cut out saturated subcategories.

Proposition 7.2.6 Let $H : \mathcal{D} \to \mathcal{A}$ be cohomological and $\mathcal{T} \subset \mathcal{A}$ a Serre subcategory. Then the full subcategory $\mathcal{D}_{H,\mathcal{T}}$ of objects X with $H(T^n X) \in \mathcal{T}$ for all $n \in \mathbb{Z}$ is a saturated (pre-)triangulated subcategory. Taking $\mathcal{T} = \{0\}$ gives the saturated subcategory $\mathcal{N}_H$ of objects with $H(T^n X) = 0$ for all $n \in \mathbb{Z}$.

Proof The subcategory is full, additive, saturated, and closed under T and T^{-1}, since H is additive and $\mathcal{T}$ is Serre. Given an exact triangle $X \to Y \to Z \xrightarrow{+1}$ with two of X, Y, Z in it, rotate the triangle so that its first and third vertices belong to it; for every $n \in \mathbb{Z}$, Remark 7.1.7 yields an exact sequence $H(T^n X) \to H(T^n Y) \to H(T^n Z)$ with outer terms in $\mathcal{T}$, and the Serre property forces $H(T^n Y) \in \mathcal{T}$. Now apply Proposition 7.2.5. □

The following extension result holds in triangulated categories and depends crucially on the octahedral axiom; it will be applied in Section 7.4.

Lemma 7.2.7 (*J.-L. Verdier*) Every commutative square in a triangulated category extends to a diagram whose rows and columns are exact triangles. Precisely, let

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"'] & Y \end{tikzcd} \tag{7.2.2}$$

be a commutative square, i.e. $fu = vf'$. Then it embeds into a diagram

$$\begin{tikzcd} TX' \arrow[r, "Tf'"] & TY' \arrow[r, "Tg'"] & TZ' \arrow[r, "-Th'"] & T^2X' \\ X'' \arrow[u, "o"] \arrow[r, "f''"] & Y'' \arrow[u, "p"] \arrow[r, "g''"] & Z'' \arrow[u, "q"] \arrow[r, "h''"] \arrow[ur, phantom, "\star"] & TX'' \arrow[u, "-To"'] \\ X \arrow[u, "r"] \arrow[r, "f"] & Y \arrow[u, "s"] \arrow[r, "g"] & Z \arrow[u, "t"] \arrow[r, "h"] & TX \arrow[u, "Tr"'] \\ X' \arrow[u, "u"] \arrow[r, "f'"'] & Y' \arrow[u, "v"] \arrow[r, "g'"'] & Z' \arrow[u, "w"] \arrow[r, "h'"'] & TX' \arrow[u, "Tu"'] \end{tikzcd} \tag{7.2.3}$$

in which every row and every column is an exact triangle. Every square commutes, except the one marked $\star$, which anti-commutes: $(To)h'' = -(Th')q$.

Proof (*Sketch*) The construction rests on three applications of (TR5). Begin by applying (TR2) to u, v, f, f', and to the composite $fu = vf'$, calling A the cone of $X' \to Y$. Applying (TR5) to the factorization through Y' (resp. through X) connects the cones Z' and Y'' (resp. X'' and Z) through A, in exact triangles $Z' \to A \to Y'' \to TZ'$ and $X'' \to A \to Z \to TX''$. A third application of (TR5), to the two triangles with vertex A and the triangle of the induced morphism $X'' \to Y''$ between the cones of u and v, produces the exact triangle $Z \to Z'' \to TZ' \to TZ$, and rotating it supplies the third column (the rotation sign being absorbed into w, as in Remark 7.1.3). The top row and the right column are obtained from the bottom row and the left column by (TR3), up to the signs of Remark 7.1.3. The three applications of (TR5) then verify every square, the marked one picking up the rotation sign and anticommuting. See [10, Lemma 4.2.15]. □

Exercise 7.2.8 Show that every monomorphism $u : X \to Y$ in a (pre-)triangulated category has a left inverse, and deduce that a (pre-)triangulated category that is abelian must be semi-simple. *Hint.* Place u in an exact triangle $X \to (u)Y \to Z \xrightarrow{w} TX$. First deduce $w = 0$ from $u \circ T^{-1}w = 0$ (Lemma 7.1.8, (TR3), and the faithfulness of T^{-1}). Then lift id_X through u using the contravariant part of Proposition 7.1.9, applied to $\mathrm{Hom}(\cdot, X)$. Finally, note that a left-invertible monomorphism splits.

7.3 Localization of triangulated categories

The localization of categories from Section 1.8, applied to a multiplicative system, carries over to the triangulated setting. Throughout, $(\mathcal{D}, T)$ is pre-triangulated and $S \subset \mathrm{Mor}(\mathcal{D})$ is a multiplicative system (Definition 1.8.5).

Definition 7.3.1 A multiplicative system S is ***compatible with the triangulation*** if

▷ **(ST1)** $s \in S$ iff $Ts \in S$;

▷ **(ST2)** in the solid commutative diagram with both rows exact,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, dashed, "\gamma"'] & TX \arrow[d, dashed, "T\alpha"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd} \tag{7.3.1}$$

if $\alpha, \beta \in S$, there is a filler $\gamma \in S$.

It may be said that (ST2) refines (TR4) by demanding $\gamma \in S$. Localization then inherits a triangulation, in the following sense.

Proposition 7.3.2 If S is compatible with the triangulation, then $\mathcal{D}[S^{-1}]$ carries a unique pre-triangulated structure for which $Q : \mathcal{D} \to \mathcal{D}[S^{-1}]$ is triangulated; up to isomorphism, its exact triangles are the images of those of $\mathcal{D}$. If $\mathcal{D}$ is triangulated, so is $\mathcal{D}[S^{-1}]$.

Proof (*Sketch*) First, the localization of an additive category at a multiplicative system is additive, with Q additive: two left fractions can be brought to a common apex by (S3) and (S4), where they are added by adding numerators, $[U; s, a] + [U; s, b] := [U; s, a + b]$ (Theorem 1.8.6); see [10, Theorem 1.9.17]. Next, QT and QT^{-1} invert S by (ST1), so the universal property of Definition 1.8.4 extends each to a unique functor of $\mathcal{D}[S^{-1}]$, and uniqueness makes the two extensions mutually inverse. Denote them again by T and T^{-1}; then $(\mathcal{D}[S^{-1}], T)$ is an additive category with translation.

Uniqueness. A triangulated Q must carry exact triangles of $\mathcal{D}$ to exact triangles of $\mathcal{D}[S^{-1}]$. Conversely, every morphism of $\mathcal{D}[S^{-1}]$ is a fraction $Q(a)Q(s)^{-1}$ (Theorem 1.8.6), and precomposition with the isomorphism $Q(s)$ turns it into a morphism of $\mathcal{D}$. Thus, given an exact triangle $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ of $\mathcal{D}[S^{-1}]$, adjust f in this way, embed the adjusted morphism in an exact triangle by (TR2), and conclude by Corollary 7.2.2 that the given triangle is isomorphic to the image of an exact triangle of $\mathcal{D}$: the pre-triangulated structure, if it exists, is unique, and its exact triangles are as described.

Existence. Declare a triangle of $\mathcal{D}[S^{-1}]$ exact when it is isomorphic to the image of an exact triangle of $\mathcal{D}$. Axioms (TR0), (TR1), and (TR3) are then immediate since the translation of $\mathcal{D}[S^{-1}]$ was built to commute with Q, and (TR2) follows from the adjustment above. For (TR4), adjust the given commutative square so that both rows and all horizontal arrows come from $\mathcal{D}$; a routine refinement of spans based on Theorem 1.8.6 then produces the solid part of a diagram in $\mathcal{D}$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & TX \\ A \arrow[u, "s"'] \arrow[r] \arrow[d, "a"'] & B \arrow[u, "t"'] \arrow[r, dashed] \arrow[d, "b"'] & C \arrow[u, dashed, "u"] \arrow[r, dashed] \arrow[d, dashed, "c"'] & TA \arrow[u, dashed, "Ts"] \arrow[d, dashed, "Ta"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r] & TX' \end{tikzcd} \tag{7.3.2}$$

with $s, t \in S$, $\alpha = Q(a)Q(s)^{-1}$, $\beta = Q(b)Q(t)^{-1}$, and the two outer rows the given exact triangles. Embed the morphism $A \to B$ in an exact triangle $A \to B \to C \xrightarrow{+1}$ by (TR2); (ST2) supplies $u : C \to Z$ in S, and (TR4) applied in $\mathcal{D}$ supplies $c : C \to Z'$. One checks from the commutativities of the diagram that $\gamma := Q(c)Q(u)^{-1}$ fills the required square. Finally, when $\mathcal{D}$ is triangulated, (TR5) for $\mathcal{D}[S^{-1}]$ reduces to $\mathcal{D}$ by a similar adjustment. See [10, Proposition 4.3.2]. □

7.4 The Verdier quotient

Throughout this section, $(\mathcal{D}, T)$ is triangulated. Every saturated triangulated subcategory $\mathcal{N}$ gives a compatible multiplicative system by the construction below; the resulting Verdier quotient will be the triangulated analogue of the Serre quotient (Section 5.2) of abelian categories.

Definition 7.4.1 For a saturated triangulated subcategory $\mathcal{N} \subset \mathcal{D}$, set

$$S\mathcal{N} := \left\{ s \in \mathrm{Mor}(\mathcal{D}) : \exists \text{ an exact triangle } \left[X \xrightarrow{s} Y \to Z \xrightarrow{+1} \right] \text{ with } Z \in \mathrm{Ob}(\mathcal{N}) \right\} \tag{7.4.1}$$

The ***Verdier quotient*** is the triangulated category $\mathcal{D}/\mathcal{N} := \mathcal{D}[(S\mathcal{N})^{-1}]$.

Proposition 7.4.2 For every saturated triangulated subcategory $\mathcal{N} \subset \mathcal{D}$, the class $S\mathcal{N}$ is a multiplicative system compatible with the triangulation (Definition 7.3.1). Consequently the Verdier quotient $\mathcal{D}/\mathcal{N}$ is triangulated, with triangulated localization functor Q (Proposition 7.3.2).

Proof (*Sketch*) Axiom (ST1) follows by rotating triangles, since $\mathcal{N}$ is closed under T. For (ST2), embed the commutative square of Definition 7.3.1 into the diagram supplied by Lemma 7.2.7, in which all rows and columns are exact triangles: the cones over α and β are isomorphic to objects of $\mathcal{N}$ (Corollary 7.2.2), hence lie in $\mathcal{N}$ by saturation, and the two-out-of-three property of triangulated subcategories (Proposition 7.2.5) forces the cone over γ into $\mathcal{N}$ as well.

The multiplicative-system axioms. (S1) is (TR1) with third vertex $0 \in \mathcal{N}$; (S2) follows from the octahedral axiom (TR5), since the cones of f, g, and gf assemble into an exact triangle, to which two-out-of-three applies; (S3) is a chase with (TR2) and (TR4); and (S4) follows by considering cohomological functors $\mathrm{Hom}(X, \cdot)$ for various X. For details, see [10, Proposition 4.3.3]. □

Theorem 7.4.3 The localization $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is the universal triangulated functor annihilating $\mathcal{N}$. Namely:

- for every morphism f in $\mathcal{D}$, we have $Qf = 0$ if and only if f factors through an object of $\mathcal{N}$; in particular Q annihilates $\mathcal{N}$;
- any triangulated functor $F : \mathcal{D} \to \mathcal{E}$ annihilating $\mathcal{N}$ factors uniquely through a triangulated functor $\mathcal{D}/\mathcal{N} \to \mathcal{E}$;
- any cohomological functor $H : \mathcal{D} \to \mathcal{A}$ annihilating $\mathcal{N}$ factors uniquely through a cohomological functor $\mathcal{D}/\mathcal{N} \to \mathcal{A}$.

Proof (*Sketch*) (i) Let $N \in \mathrm{Ob}(\mathcal{N})$. In the rotation of the (TR1)-triangle of N, the morphism $N \to 0$ has cone $TN \in \mathcal{N}$, hence lies in $S\mathcal{N}$; so $Q(N \to 0)$ is invertible. Being also the zero morphism (Q is additive), it yields $\mathrm{id}_{QN} = 0$. This implies that Q annihilates $\mathcal{N}$, and $Qf = 0$ follows whenever f factors through an object of $\mathcal{N}$. Conversely, $Qf = 0$ gives, by Proposition 1.8.8 applied to f and 0, a morphism $s : M \to X$ in $S\mathcal{N}$ with $fs = 0$. Embed s in an exact triangle $M \xrightarrow{s} X \to N \xrightarrow{+1}$ with $N \in \mathcal{N}$: the contravariant part of Proposition 7.1.9 renders $\mathrm{Hom}(N, Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(M, Y)$ exact, so f, belonging to the kernel of the second arrow, factors as $X \to N \to Y$.

(ii) Let $F : \mathcal{D} \to \mathcal{E}$ be triangulated and annihilate $\mathcal{N}$. For $s \in S\mathcal{N}$, embedded in an exact triangle $X \xrightarrow{s} Y \to N \xrightarrow{+1}$ with $N \in \mathcal{N}$, the image $FX \xrightarrow{Fs} FY \to FN \xrightarrow{+1}$ is exact with FN a zero object, so Fs is an isomorphism (Corollary 7.1.10). Hence F inverts $S\mathcal{N}$, and Definition 1.8.4 yields the unique factorization $F = \overline{F}Q$; that $\overline{F}$ commutes with the translation and preserves exact triangles is routine, since by Proposition 7.3.2 the exact triangles of $\mathcal{D}/\mathcal{N}$ are the images of those of $\mathcal{D}$.

(iii) The argument parallels (ii). For $s \in S\mathcal{N}$ as above, the segment

$$H(T^{-1}N) \to H(X) \xrightarrow{H(s)} H(Y) \to H(N) \tag{7.4.2}$$

of the long exact sequence (Remark 7.1.7) has vanishing outer terms since $T^{-1}N \in \mathcal{N}$, hence $H(s)$ is an isomorphism, H inverts $S\mathcal{N}$, and Definition 1.8.4 gives the unique factorization $H = \overline{H}Q$. The descended $\overline{H}$ is cohomological because, once more, the exact triangles of the quotient are the images of exact triangles of $\mathcal{D}$. See [10, Theorem 4.3.4]. □

Cohomological functors give compatible multiplicative systems, describable in two equivalent ways below.

Definition 7.4.4 For a cohomological functor $H : \mathcal{D} \to \mathcal{A}$, we say

- a morphism f of $\mathcal{D}$ is an ***H-quasi-isomorphism*** if $H(T^n f)$ is an isomorphism for all $n \in \mathbb{Z}$;
- an object X of $\mathcal{D}$ is ***H-acyclic*** if $H(T^n X) = 0$ for all n.

Proposition 7.4.5 For a cohomological functor $H : \mathcal{D} \to \mathcal{A}$, let

- S_H be the class of all H-quasi-isomorphisms (Definition 7.4.4),
- $\mathcal{N}_H$ be the triangulated subcategory of all H-acyclic objects, and
- $S\mathcal{N}_H$ be as in Definition 7.4.1.

Then $S\mathcal{N}_H = S_H$.

Proof Embed a morphism $f : X \to Y$ of $\mathcal{D}$ in an exact triangle

$$X \xrightarrow{f} Y \to N \xrightarrow{+1} \tag{7.4.3}$$

(TR2). The object N is unique up to isomorphism (Corollary 7.2.2), and $\mathcal{N}_H$ is saturated (Proposition 7.2.6), so it makes sense to ask whether N belongs to $\mathcal{N}_H$. For every $n \in \mathbb{Z}$, the long exact sequence of Remark 7.1.7 is

$$\cdots \to H(T^{n-1}N) \to H(T^n X) \xrightarrow{H(T^n f)} H(T^n Y) \to H(T^n N) \to H(T^{n+1}X) \to \cdots \tag{7.4.4}$$

It follows readily that f is an H-quasi-isomorphism if and only if $N \in \mathcal{N}_H$; by Definition 7.4.1, applied to $\mathcal{N} = \mathcal{N}_H$, the right-hand side is exactly the condition $f \in S\mathcal{N}_H$. See [10, Proposition 4.3.6]. □

Summing up, localizing at S_H is the same as quotienting by $\mathcal{N}_H$, i.e. inverting the H-quasi-isomorphisms is the same as annihilating the H-acyclic objects.

Finally, we relate localization to subcategories; the result below powers the resolutions of Section 8.2. Let $\mathcal{N}$ and $\mathcal{I}$ be triangulated subcategories of $\mathcal{D}$, with $\mathcal{N}$ saturated. Then $\mathcal{N} \cap \mathcal{I}$ is a saturated triangulated subcategory of $\mathcal{I}$, and the universal property of Theorem 7.4.3 supplies a triangulated functor

$$i : \frac{\mathcal{I}}{\mathcal{N} \cap \mathcal{I}} \to \frac{\mathcal{D}}{\mathcal{N}}. \tag{7.4.5}$$

Proposition 7.4.6 In the situation above, assume the *resolution condition*: for every $X \in \mathcal{D}$ there exist $U \in \mathcal{I}$ and a morphism $U \to X$ in $S\mathcal{N}$ (resp. $X \to U$ in $S\mathcal{N}$). Then i is an equivalence of triangulated categories.

Proof (*Sketch*) Every QX is isomorphic to some QU, so i is essentially surjective. Full faithfulness follows by refining left fractions through the resolution condition, using the identity

$$S(\mathcal{N} \cap \mathcal{I}) = S\mathcal{N} \cap \mathrm{Mor}(\mathcal{I}) : \tag{7.4.6}$$

the inclusion $\subset$ is clear, and if $f \in \mathrm{Mor}(\mathcal{I})$ embeds in an exact triangle $X \xrightarrow{f} Y \to N \xrightarrow{+1}$ in $\mathcal{D}$ with $N \in \mathrm{Ob}(\mathcal{N})$, then embedding f in a triangle inside $\mathcal{I}$ and applying Corollary 7.2.2 together with the saturation of $\mathcal{N}$ exhibits N as an object of $\mathcal{N} \cap \mathcal{I}$. The upgrade to a triangulated equivalence is routine. See [10, Proposition 4.3.7]. □

7.5 The derived category $\mathsf{D}(\mathcal{A})$

Fix an abelian category $\mathcal{A}$. We saw in Example 7.1.1 that $\mathsf{C}(\mathcal{A})$ and $\mathsf{K}(\mathcal{A})$ are categories with translation. The homotopy category is in fact triangulated, as explicated below.

Theorem 7.5.1 The homotopy category $\mathsf{K}(\mathcal{A})$, with translation $T : X \mapsto X[1]$, is a triangulated category whose exact triangles are those isomorphic to a mapping-cone triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Cone}(f) \xrightarrow{\beta(f)} X[1] \tag{7.5.1}$$

(Definition 3.6.1). The bounded variants $\mathsf{K}^\star(\mathcal{A})$, $\star \in \{+, -, \mathrm{b}\}$, are triangulated subcategories.

Proof (*Sketch*) (TR0) and (TR2) hold by definition. (TR1) follows by rotating (TR3) the mapping-cone triangle of the zero map $0 \to X$, which reads $0 \to X \xrightarrow{\mathrm{id}} X \to 0 \xrightarrow{+1}$ once $\mathrm{Cone}(0 \to X)$ is identified with X so that $\alpha(0) = \mathrm{id}_X$.

For (TR4), let $f : X \to Y$ and $f' : X' \to Y'$ be chain maps, and let $\alpha : X \to X'$, $\beta : Y \to Y'$ satisfy $\beta f \sim f'\alpha$ (Definition 3.2.1). Choose a homotopy h from $f'\alpha$ to βf and define, degree-wise,

$$\gamma^n := \begin{pmatrix} \alpha^{n+1} & 0 \\ h^n & \beta^n \end{pmatrix} : \mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n \to (X')^{n+1} \oplus (Y')^n = \mathrm{Cone}(f')^n. \quad (7.5.2)$$

Upon expansion, the chain-map identity for γ is precisely the homotopy relation for h (the remaining terms cancel because α and β are chain maps), and the two right-hand squares in (TR4) commute already in $\mathsf{C}(\mathcal{A})$; thus (α, β, γ) is a morphism of triangles in $\mathsf{K}(\mathcal{A})$.

The rotation (TR3) and the octahedral axiom (TR5) are further matrix computations with cones. For (TR3), we need a homotopy equivalence $\mathrm{Cone}(\alpha(f)) \cong X[1]$ relating the rotation of a mapping-cone triangle to the mapping-cone triangle of $\alpha(f)$. For (TR5), we need a homotopy equivalence $\mathrm{Cone}(u) \cong \mathrm{Cone}(g)$ for $u = \left(\mathrm{id}_{X[1]}, g\right) : \mathrm{Cone}(f) \to \mathrm{Cone}(gf)$. We refer to [10, Theorem 4.4.1] for details. Finally, the bounded variants are triangulated subcategories: $X, Y \in \mathrm{Ob}(\mathsf{C}^\star(\mathcal{A}))$ if and only if $\mathrm{Cone}(f) \in \mathrm{Ob}(\mathsf{C}^\star(\mathcal{A}))$, so Proposition 7.2.5 applies. □

Remark 7.5.2 The passage through $\mathsf{K}(\mathcal{A})$ is essential: at the level of $\mathsf{C}(\mathcal{A})$ the mapping cone does *not* yield a triangulated category.

We can now close the promise made in Remark 4.1.7; by Example 4.6.3 the same argument also proves the homotopy-uniqueness of K-injective and K-projective resolutions stated in Section 4.6.

Corollary 7.5.3 Let $\alpha : X \to Y$ be a quasi-isomorphism and I a K-injective complex. Then $\alpha^* : \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(Y, I) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I)$, and dually for a K-projective source. In particular, resolutions are unique up to a unique homotopy class.

Proof The mapping cone $\mathrm{Cone}(\alpha)$ is acyclic (Proposition 3.6.2), hence $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\cdot, I)$ being cohomological (Proposition 7.1.9) gives an exact sequence

$$\begin{aligned} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{Cone}(\alpha), I) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(Y, I) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) \\ \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{Cone}(\alpha)[-1], I); \end{aligned} \quad (7.5.3)$$

K-injectivity makes both ends zero (Lemma 4.6.2). Dually for K-projectives. □

Take $H = \mathrm{H}^0$ in Proposition 7.4.5: the H^0-quasi-isomorphisms are exactly the quasi-isomorphisms of Definition 3.1.3, and $\mathcal{N}_{\mathrm{H}^0}$ is the saturated triangulated subcategory of acyclic complexes. Since the multiplicative systems S_{H^0} and $S\mathcal{N}_{\mathrm{H}^0}$ coincide (Proposition 7.4.5), inverting the one is annihilating the other.

Denote the multiplicative system of quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ by qis; it satisfies (ST1) and (ST2) of Definition 7.3.1, by Proposition 7.4.5 together with Proposition 7.4.2.

Definition 7.5.4 Let $\mathcal{A}$ be an abelian category. The ***derived category*** of $\mathcal{A}$ is the triangulated category

$$\mathsf{D}(\mathcal{A}) := \mathsf{K}(\mathcal{A})[\mathrm{qis}^{-1}] = \mathsf{K}(\mathcal{A})/\mathcal{N}, \tag{7.5.4}$$

where $\mathcal{N}$ is the saturated triangulated subcategory of acyclic complexes. For $\star \in \{+, -, \mathrm{b}\}$ set $\mathsf{D}^\star(\mathcal{A}) := \mathsf{K}^\star(\mathcal{A})/\mathcal{N}^\star$ with $\mathcal{N}^\star := \mathcal{N} \cap \mathsf{K}^\star(\mathcal{A})$: the *bounded below* $(+)$, *bounded above* $(-)$, and *bounded* (b) derived categories.

Remark 7.5.5 The passage through $\mathsf{K}(\mathcal{A})$ is not strictly necessary: if Qis denotes the class of quasi-isomorphisms in $\mathsf{C}(\mathcal{A})$, then $\mathsf{C}(\mathcal{A})[\mathrm{Qis}^{-1}]$ is equivalent to $\mathsf{D}(\mathcal{A})$. The point is that any functor $G : \mathsf{C}(\mathcal{A}) \to \mathcal{E}$ inverting Qis also annihilates homotopies. Namely, a homotopy from f to g (where $f, g : X \rightrightarrows Y$) determines a morphism $\tilde{h} : \mathrm{Cyl}(X) \to Y$, where $\mathrm{Cyl}(X) := \mathrm{Cyl}(\mathrm{id}_X)$ is the mapping cylinder, with $f = \tilde{h}i_0$ and $g = \tilde{h}i_1$; the chain maps $i_0, i_1 : X \to \mathrm{Cyl}(X)$ satisfy $ji_0 = ji_1$ for a quasi-isomorphism $j : \mathrm{Cyl}(X) \to X$. As Gj is invertible, $Gi_0 = Gi_1$ and thus $Gf = Gg$. The detour through $\mathsf{K}(\mathcal{A})$ is nevertheless preferable, since the triangulated structure lives on $\mathsf{K}(\mathcal{A})$ and Qis is not a multiplicative system in $\mathsf{C}(\mathcal{A})$. See [10, Remark 4.4.14, Proposition 3.3.12] and Remark 3.6.3.

Proposition 7.5.6 Cohomology factors through the derived category: for every $n \in \mathbb{Z}$ there is a cohomological functor

$$\mathrm{H}^n : \mathsf{D}(\mathcal{A}) \to \mathcal{A}, \quad \mathrm{H}^n(X) = \mathrm{H}^0(X[n]). \tag{7.5.5}$$

Furthermore, each short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathsf{C}(\mathcal{A})$ extends canonically to an exact triangle $X \to Y \to Z \xrightarrow{+1}$ in $\mathsf{D}(\mathcal{A})$, so the long exact sequence in cohomology (Proposition 3.3.1) is the cohomological-functor sequence of Remark 7.1.7.

Proof Each $\mathrm{H}^n : \mathsf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological: for the mapping-cone triangle $X \xrightarrow{f} Y \to \mathrm{Cone}(f) \xrightarrow{+1}$, the exactness of $\mathrm{H}^n(X) \to \mathrm{H}^n(Y) \to \mathrm{H}^n(\mathrm{Cone}(f))$ is part of the long exact sequence of Proposition 3.6.2. Since H^n annihilates $\mathcal{N}$, it descends to a cohomological functor on the quotient $\mathsf{D}(\mathcal{A}) = \mathsf{K}(\mathcal{A})/\mathcal{N}$ (Theorem 7.4.3).

Now let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathsf{C}(\mathcal{A})$, and set $\Phi := (0, g) : \mathrm{Cone}(f) \to Z$. Because $gf = 0$ and g commutes with the differentials, Φ is a chain map, and there is a sequence of complexes

$$0 \to \mathrm{Cone}(\mathrm{id}_X) \xrightarrow{(\mathrm{id}, f)} \mathrm{Cone}(f) \xrightarrow{(0,g)} Z \to 0, \tag{7.5.6}$$

exact degreewise: $(0, g)$ is surjective, with kernel $X^{n+1} \oplus \mathrm{im}(f^n)$, the image of (id, f). The identity of $\mathrm{Cone}(\mathrm{id}_X)$ is null-homotopic (Exercise 3.6.4), so $\mathrm{Cone}(\mathrm{id}_X)$ is acyclic and the long exact sequence (Proposition 3.3.1) shows Φ to be a quasi-isomorphism. Consequently the triangle $X \to Y \to Z \xrightarrow{+1}$, in which $Z \to X[1]$ is the composite $Z \xrightarrow{\Phi^{-1}} \mathrm{Cone}(f) \xrightarrow{\beta(f)} X[1]$, is isomorphic in $\mathsf{D}(\mathcal{A})$ to the mapping-cone triangle of f, hence exact. The induced long exact sequence (Remark 7.1.7) is that of Proposition 3.3.1, up to the sign of the connecting morphism. □

Proposition 7.5.7 For each $\star \in \{+, -, \mathrm{b}\}$ the functor $\mathsf{D}^\star(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is a fully faithful embedding of triangulated subcategories. An object $X \in \mathrm{Ob}(\mathsf{D}(\mathcal{A}))$ lies in the essential image of $\mathsf{D}^+(\mathcal{A})$ (resp. $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) iff $\mathrm{H}^n(X) = 0$ for $n \ll 0$ (resp. $n \gg 0$, $|n| \gg 0$).

Proof We treat $\star = +$; the case $\star = -$ is dual, and b follows by truncating on both sides. For the description of the image, use the truncation functors (Definition 3.7.1): every X with $\mathrm{H}^n(X) = 0$ for $n \ll 0$ is quasi-isomorphic to $\tau^{\geq m}X$ for $m \ll 0$, which lies in $\mathsf{C}^+(\mathcal{A})$, through the canonical morphism $X \to \tau^{\geq m}X$. For full faithfulness, recall that a morphism $X \to Y$ in $\mathsf{D}(\mathcal{A})$ between objects of $\mathsf{K}^+(\mathcal{A})$ is a fraction as^{-1} with $s : X' \to X$ a quasi-isomorphism; replacing X' by $\tau^{\geq m}X'$ for $m \ll 0$, for which the composite $\tau^{\geq m}X' \to X' \to X$ is still a quasi-isomorphism, represents the same morphism within $\mathsf{D}^+(\mathcal{A})$, and the same replacement settles the equivalence between two fractions. See [10, Proposition 4.4.9]. □

In fact, the truncation functors of Definition 3.7.1 descend to $\mathsf{K}(\mathcal{A})$, and then to $\mathsf{D}(\mathcal{A})$ since quasi-isomorphisms are preserved by truncation.

Definition 7.5.8 For $-\infty \leq s \leq t \leq +\infty$, let $\mathsf{D}^{[s,t]}(\mathcal{A})$ be the full subcategory of X with $\mathrm{H}^n(X) = 0$ for $n \notin [s,t]$; write $\mathsf{D}^{\geq s}(\mathcal{A}) = \mathsf{D}^{[s,\infty]}(\mathcal{A})$ and $\mathsf{D}^{\leq t}(\mathcal{A}) = \mathsf{D}^{[-\infty,t]}(\mathcal{A})$.

Proposition 7.5.9 For every $n \in \mathbb{Z}$ the truncations $\tau^{\leq n} : \mathsf{D}(\mathcal{A}) \to \mathsf{D}^{\leq n}(\mathcal{A})$ and $\tau^{\geq n} : \mathsf{D}(\mathcal{A}) \to \mathsf{D}^{\geq n}(\mathcal{A})$ are right and left adjoint to the inclusions, and there is a canonical exact triangle

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{+1} . \tag{7.5.7}$$

Proof We establish the first adjunction; the second is symmetric. Let $i_X : \tau^{\leq n}X \to X$ be the canonical morphism (Definition 3.7.1): it is natural in X, and a quasi-isomorphism exactly when $X \in \mathsf{D}^{\leq n}(\mathcal{A})$. The natural transformation i is initially defined at the level of complexes; in fact it descends to $\mathsf{D}(\mathcal{A})$, the naturality being a routine computation with fractions.

For $X \in \mathrm{Ob}(\mathsf{D}^{\leq n}(\mathcal{A}))$ and $Y \in \mathrm{Ob}(\mathsf{D}(\mathcal{A}))$, the mutually inverse bijections

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y) \simeq \mathrm{Hom}_{\mathsf{D}^{\leq n}(\mathcal{A})}(X, \tau^{\leq n}Y) \tag{7.5.8}$$

read $f \mapsto \tau^{\leq n}fi_X^{-1}$ and $g \mapsto i_Y g$: indeed $i_Y\tau^{\leq n}fi_X^{-1} = f$ is the naturality of i, while $\tau^{\leq n}i_Y = \mathrm{id}$ and $i_{\tau^{\leq n}Y} = \mathrm{id}$ (for $\tau^{\leq n}$ fixes complexes in $\mathsf{C}^{\leq n}(\mathcal{A})$) give $\tau^{\leq n}(i_Y g)i_X^{-1} = \tau^{\leq n}gi_X^{-1} = g$.

As for the triangle, the sequence $0 \to \tau^{\leq n}X \to X \to X/\tau^{\leq n}X \to 0$ is exact degreewise, hence extends to an exact triangle (Proposition 7.5.6), and the canonical morphism $X/\tau^{\leq n}X \to \tau^{\geq n+1}X$ (quotient map in degree $n+1$, identity in degrees $\geq n+2$, zero in degree n) is a quasi-isomorphism: both complexes have cohomology $\mathrm{H}^k(X)$ in degrees $k \geq n+1$ and none below. See [10, Proposition 4.4.13]. □

Remark 7.5.10 The pair $(\mathsf{D}^{\leq 0}(\mathcal{A}), \mathsf{D}^{\geq 0}(\mathcal{A}))$ is a ***t-structure*** on $\mathsf{D}(\mathcal{A})$ with ***heart*** $\mathsf{D}^{\leq 0}(\mathcal{A}) \cap \mathsf{D}^{\geq 0}(\mathcal{A})$; Theorem 7.6.3 identifies this heart with $\mathcal{A}$ itself. The formalism of t-structures generalizes this to settings (perverse sheaves, stability conditions) with no obvious abelian category in sight, but we do not pursue them here.

Exercise 7.5.11 (Pre-triangulated K_0.) For a (pre-)triangulated $\mathcal{D}$, define $\mathrm{K}_0(\mathcal{D})$ as the free abelian group on $\mathrm{Ob}(\mathcal{D})$ modulo $[X] - [Y] + [Z] = 0$ for every exact triangle $X \to Y \to Z \xrightarrow{+1}$.

(i) Show $[0] = 0$, $[TX] = -[X]$, and $[X \oplus Y] = [X] + [Y]$.

(ii) Show every triangulated functor $\mathcal{D} \to \mathcal{D}'$ induces $\mathrm{K}_0(\mathcal{D}) \to \mathrm{K}_0(\mathcal{D}')$.

(iii) For $\mathcal{D} = \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, construct mutually inverse homomorphisms $\mathrm{K}_0(\mathcal{A}) \rightleftarrows \mathrm{K}_0(\mathsf{D}^{\mathrm{b}}(\mathcal{A}))$ given by $[A] \mapsto [A]$ and $[X] \mapsto \bigoplus_n (-1)^n[\mathrm{H}^n(X)]$. *Hint.* One direction is clear; the other uses truncation (Proposition 7.5.9).

7.6 Morphisms and extensions: Ext in $\mathsf{D}(\mathcal{A})$

We now identify $\mathcal{A}$ inside $\mathsf{D}(\mathcal{A})$ and compute the morphism groups. The key technical input is that K-injective (resp. K-projective) complexes compute morphisms in the derived category.

Proposition 7.6.1 If $X \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$ is built from injectives, or more generally X is a K-injective complex (Definition 4.6.1), then

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\cdot, X) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(\cdot, X). \tag{7.6.1}$$

Dually, K-projective X gives $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, \cdot) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, \cdot)$.

Proof Assume X is K-injective. A morphism in $\mathsf{D}(\mathcal{A})$ is represented by a left fraction $Y \leftarrow S \to X$ with $S \to Y$ a quasi-isomorphism, and Corollary 7.5.3 makes precomposition with such an $S \to Y$ an isomorphism $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(Y, X) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(S, X)$. Hence every left fraction $[S; s, a]$ equals the class of the single morphism $f := (s^*)^{-1} a : Y \to X$, and if two morphisms $f, g : Y \to X$ become equal in $\mathsf{D}(\mathcal{A})$ then $fs = gs$ in $\mathsf{K}(\mathcal{A})$ for some quasi-isomorphism s, whence $f = g$. The case of K-projectives is dual. □

The morphism groups respect the cohomological degree, in the following sharp sense.

Proposition 7.6.2 (*orthogonality*) Let $a, b, n \in \mathbb{Z}$ with $a < b$.

- If $X \in \mathsf{D}^{\leq a}(\mathcal{A})$ and $Y \in \mathsf{D}^{\geq b}(\mathcal{A})$, then $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y) = 0$.
- If $X \in \mathsf{D}^{\leq n}(\mathcal{A})$ and $Y \in \mathsf{D}^{\geq n}(\mathcal{A})$, then H^n gives a canonical isomorphism $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(X), \mathrm{H}^n(Y))$.

Proof (i) A morphism $f : X \to Y$ in $\mathsf{D}(\mathcal{A})$ is a left fraction $X \xleftarrow{s} Z \xrightarrow{g} Y$ with s a quasi-isomorphism (Theorem 1.8.6). Since $\mathrm{H}^k(Z) \simeq \mathrm{H}^k(X) = 0$ for $k > a$, the truncation $\tau^{\leq a} Z \to Z$ is again a quasi-isomorphism, so we may assume $Z \in \mathrm{Ob}(\mathsf{C}^{\leq a}(\mathcal{A}))$. Post-composing the numerator with the quasi-isomorphism $Y \to \tau^{\geq b} Y$ likewise represents the composite $X \to Y \to \tau^{\geq b} Y$, whose numerator is now a chain map with source in $\mathsf{C}^{\leq a}(\mathcal{A})$ and target in $\mathsf{C}^{\geq b}(\mathcal{A})$. When $a < b$ such a chain map is degreewise zero, so the composite vanishes; as $Y \to \tau^{\geq b} Y$ is invertible in $\mathsf{D}(\mathcal{A})$, this forces $f = 0$.

(ii) The same reductions with $a = n = b$ leave us with complexes $X \in \mathrm{Ob}(\mathsf{C}^{\leq n}(\mathcal{A}))$ and $Y \in \mathrm{Ob}(\mathsf{C}^{\geq n}(\mathcal{A}))$ (the truncation morphisms being quasi-isomorphisms that induce isomorphisms on H^n). Consider the canonical maps

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(X), \mathrm{H}^n(Y)) \xleftarrow{\mathrm{H}^n} \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X, Y) \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y). \tag{7.6.2}$$

The first arrow is bijective: a chain map $X \to Y$ is nonzero only in degree n, where the chain condition forces it to carry $\mathrm{im}\, d_X^{n-1}$ to zero and to land in $\ker d_Y^n$, that is, to descend to a map $\mathrm{H}^n(X) \to \mathrm{H}^n(Y)$. The second arrow is bijective because a homotopy between chain maps $X \to Y$ is a family of maps $X^k \to Y^{k-1}$ (Definition 3.2.1), and no such family exists other than zero, $k \leq n$ and $k - 1 \geq n$ being incompatible. Finally, in any fraction $X \xleftarrow{s} Z \xrightarrow{g} Y$ we may, as in (i), assume $Z \in \mathrm{Ob}(\mathsf{C}^{\leq n}(\mathcal{A}))$; the previous steps then identify $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(Z, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(Z), \mathrm{H}^n(Y))$, and $\mathrm{H}^n(s)$ transports this back to $\mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^n(X), \mathrm{H}^n(Y)) \simeq \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y)$. In other words, every fraction equals the class of a single chain map $X \to Y$, and two chain maps $f, g : X \to Y$ that become equal in $\mathsf{D}(\mathcal{A})$ satisfy

$fs = gs$ in $\mathsf{K}(\mathcal{A})$ for some quasi-isomorphism s whose source lies in $\mathsf{C}^{\leq n}(\mathcal{A})$ (refine as in (i)), whence $f = g$; the third arrow is bijective as well. $\square$

Theorem 7.6.3 Placing an object of $\mathcal{A}$ in degree 0 gives an additive equivalence

$$\mathcal{A} \simeq \mathsf{D}^{\leq 0}(\mathcal{A}) \cap \mathsf{D}^{\geq 0}(\mathcal{A}), \tag{7.6.3}$$

with quasi-inverse H^0. More generally $A \mapsto A[-n]$ identifies $\mathcal{A}$ with $\mathsf{D}^{\geq n}(\mathcal{A}) \cap \mathsf{D}^{\leq n}(\mathcal{A})$, quasi-inverse H^n.

Proof (*Sketch*) Proposition 7.6.2 (with $n = 0$) gives full faithfulness. For essential surjectivity, truncate any $X \in \mathsf{D}^{\leq 0}(\mathcal{A}) \cap \mathsf{D}^{\geq 0}(\mathcal{A})$: the maps $X \to \tau^{\geq 0}X \leftarrow \tau^{\leq 0}\tau^{\geq 0}X \simeq \mathrm{H}^0(X)$ are all quasi-isomorphisms. See [10, Theorem 4.5.4]. $\square$

Henceforth we identify $\mathcal{A}$ with $\mathsf{D}^{\leq 0}(\mathcal{A}) \cap \mathsf{D}^{\geq 0}(\mathcal{A})$, viewing its objects as complexes concentrated in degree 0. Shifting the degree gives the derived-categorical version of Ext.

Definition 7.6.4 For $X, Y \in \mathcal{A}$ and $n \in \mathbb{Z}$, set

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n]). \tag{7.6.4}$$

This is an additive bifunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$; no assumption on enough injectives or projectives is needed.

Proposition 7.6.5 The family of bifunctors $(\mathrm{Ext}^n)_{n \in \mathbb{Z}}$ satisfies, for all $X, Y \in \mathcal{A}$:

(i) $\mathrm{Ext}^n(X, Y) = 0$ for $n < 0$;
(ii) $\mathrm{Ext}^0(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$;
(iii) every pair of short exact sequences $0 \to X' \to X \to X'' \to 0$ and $0 \to Y' \to Y \to Y'' \to 0$ induces long exact sequences in either variable (with functorial connecting morphisms);
(iv) $\mathrm{Ext}^1(X, \cdot) = 0$ implies X is projective; $\mathrm{Ext}^1(\cdot, Y) = 0$ implies Y is injective.

Proof (i) and (ii) are Proposition 7.6.2 (with the shifted degree). For (iii), extend the short exact sequence to an exact triangle (Proposition 7.5.6) and apply the cohomological functor Hom (Proposition 7.1.9). For (iv), apply the long exact sequence of (iii) in the second variable to any short exact sequence $0 \to Y' \to Y \to Y'' \to 0$: if $\mathrm{Ext}^1(X, \cdot) = 0$ then $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y'')$ is surjective, so X is projective; dually, $\mathrm{Ext}^1(\cdot, Y) = 0$ forces Y injective. $\square$

The composition of morphisms in $\mathsf{D}(\mathcal{A})$ gives Ext a graded structure: for $f \in \mathrm{Ext}^m(Y, Z)$ and $g \in \mathrm{Ext}^n(X, Y)$ the **composition**

$$fg := f[n] \circ g \in \mathrm{Ext}^{m+n}(X, Z) \tag{7.6.5}$$

is associative and bilinear, so $\mathrm{Ext}(X) := \bigoplus_n \mathrm{Ext}^n(X, X)$ is a graded ring with unit $\mathrm{id}_X \in \mathrm{Ext}^0(X, X)$, called the ***Ext-algebra*** of X, and $\mathrm{Ext}(X, Y) := \bigoplus_n \mathrm{Ext}^n(X, Y)$ a graded $(\mathrm{Ext}(Y), \mathrm{Ext}(X))$-bimodule.

The symbol "Ext" abbreviates "extension". For $n \geq 1$, an ***n-extension*** of X by Y is an exact sequence

$$\mathcal{E}: \quad 0 \to Y \to E^1 \to \cdots \to E^n \to X \to 0 \tag{7.6.6}$$

in $\mathcal{A}$; two n-extensions are said to be equivalent, written as $\mathcal{E} \sim \mathcal{E}'$, if they are connected by a chain of commutative diagrams fixing X and Y. For $n = 1$ the five lemma (Proposition 2.5.3) shows that equivalence amounts to isomorphism of short exact sequences over X and Y, so in that case we simply speak of isomorphic extensions.

There are several operations on n-extensions. First, an m-extension of Y by Z and an n-extension of X by Y can be spliced at the shared object Y into an $(m+n)$-extension of X by Z: the ***Yoneda product*** . Second, there are pull-back (along $X' \to X$) and push-out (along $Y \to Y'$) of n-extensions of X by Y, which commute. Finally, we can take the ***Baer sum*** $\dotplus$ of two n-extensions of X by Y: pull back along $X \hookrightarrow X \oplus X$ and push out along $Y \oplus Y \twoheadrightarrow Y$, or in reverse order. One can show that the classes of n-extensions form an abelian group under $\dotplus$. Its zero element is the class of the split extension $0 \to Y \to X \oplus Y \to X \to 0$ when $n = 1$, and that of

$$0 \to Y \xrightarrow{\mathrm{id}} Y \to 0 \to \cdots \to 0 \to X \xrightarrow{\mathrm{id}} X \to 0 \tag{7.6.7}$$

otherwise, the case $n = 2$ reading $0 \to Y \xrightarrow{\mathrm{id}} Y \xrightarrow{0} X \xrightarrow{\mathrm{id}} X \to 0$.

Indeed, under the bijection of Theorem 7.6.6 these classes map to $0 \in \mathrm{Ext}^n(X, Y)$: the split extension because id_X lifts through it, its class being the image of id_X under the connecting morphism of the long exact sequence (Corollary 8.5.3), while for $n \geq 2$ the displayed sequence is the Yoneda product of a 1-extension of the zero object with an $(n-1)$-extension by the zero object, and since the composition on Ext is bilinear, any composite with a zero factor vanishes $(\mathrm{Ext}^{\cdot}(0, \cdot) = 0 = \mathrm{Ext}^{\cdot}(\cdot, 0))$.

Theorem 7.6.6 (*Yoneda*) For all $X, Y \in \mathcal{A}$ and $n \geq 1$ there is a canonical bijection

$$\{n\text{-extensions of } X \text{ by } Y\}/\sim \;\xleftrightarrow{1:1}\; \mathrm{Ext}^n(X, Y) \tag{7.6.8}$$

matching the Yoneda product with composition and the Baer sum $\dotplus$ with the group law. In particular $\mathrm{Ext}^1(X, Y)$ classifies extensions $0 \to Y \to E \to X \to 0$ up to isomorphism.

Proof (*Sketch*) An n-extension $0 \to Y \to E^1 \to \cdots \to E^n \to X \to 0$ gives a left fraction $X \leftarrow Z \to Y[n]$ (with Z the complex $[Y \to E^1 \to \cdots \to E^n]$ in degrees $[-n, 0]$, the right leg being the identity on Y in degree $-n$) whose left leg $Z \to X$ is a quasi-isomorphism, representing a class in $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$. This induces the rightward map.

For surjectivity: any left fraction $X \leftarrow S \to Y[n]$ is, by truncating S to $\mathsf{C}^{[-n,0]}$ and then pushing out at the degree $(-n)$ term (as for push-outs of extensions, so that the map to $Y[n]$ becomes the identity on Y), reduced to such a resolution. Injectivity is an application of the calculus of fractions in $\mathsf{K}(\mathcal{A})$ and truncation functors, but the details are somewhat lengthy. See [10, Theorem 4.5.13]. ☐

Exercise 7.6.7 Let $X \in \mathrm{Ob}(\mathsf{D}(\mathcal{A}))$. Prove

$$X \in \mathrm{Ob}(\mathsf{D}^{\leq 0}(\mathcal{A})) \text{ iff } \forall Z \in \mathrm{Ob}(\mathsf{D}^{\geq 1}(\mathcal{A})),\ \ \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Z) = 0, \tag{7.6.9}$$

and the dual statement for $\mathsf{D}^{\geq 0}(\mathcal{A})$. *Hint.* The forward implication is Proposition 7.6.2. For the converse, apply the hypothesis to $Z = \tau^{\geq 1}X$ in the truncation triangle $\tau^{\leq 0}X \to X \xrightarrow{u} \tau^{\geq 1}X \xrightarrow{+1}$ (Proposition 7.5.9): the map u vanishes, yet the long exact cohomology sequence shows it induces isomorphisms $\mathrm{H}^n(X) \simeq \mathrm{H}^n(\tau^{\geq 1}X)$ for all $n \geq 1$ (the cohomology of $\tau^{\leq 0}X$ lives in degrees ≤ 0); conclude that $\mathrm{H}^n(X) = 0$ for $n \geq 1$.

Chapter 8

Derived functors revisited

Chapter 4 constructed the derived functors $\mathrm{R}^n F$ classically, by resolving and taking cohomology. The derived categories of Chapter 7 now allow us to package resolution and cohomology together into a single triangulated functor. Here and below the superscript $\star$ records the boundedness condition: $+$ for bounded below, $-$ for bounded above, b for bounded, or nothing for the unbounded case, as fixed by the context. When it exists, the total derived functor (right $\mathrm{R}\,F$ or left $\mathrm{L}\,F$) of an additive functor $F : \mathcal{A} \to \mathcal{A}'$ is a triangulated functor $\mathsf{D}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A}')$, best regarded as the left (resp. right) Kan extension (Definition 1.8.1) of $Q' \cdot \mathsf{K}^\star F$ along the localization $\mathsf{K}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A})$ (Section 8.1). It exists once one can resolve by injectives or projectives, or by K-injectives or K-projectives (the general resolution theorems of Section 8.2 and Section 4.6), and its cohomology recovers the classical functors $\mathrm{R}^n F$ (resp. $\mathrm{L}_n F$) of Section 4.2. Composition of derived functors is governed by the abstract machinery that, on cohomology, recovers the ***Grothendieck spectral sequence*** of Theorem 6.5.4 (Section 8.3). The Kan-extension formalism extends to bifunctors (Section 8.4); we illustrate it (Section 8.5) with the fundamental derived bifunctor RHom, and interpret the derived functor $\mathrm{R}\lim$ as a homotopy limit. A second fundamental derived bifunctor, the derived tensor product $\overset{\mathrm{L}}{\otimes}$, is constructed in Section 8.6 out of K-flat resolutions.

Finally we record some *shortcomings* of $\mathsf{D}(\mathcal{A})$: exact triangles are unique only up to non-canonical isomorphism; triangulated categories generally lack kernels and cokernels, so limits and colimits are awkward; and derived functors resist the clean effaceability characterization of universal cohomological δ-functors. The root cause is that constructing $\mathsf{D}(\mathcal{A})$ crudely erases the homotopy relation. The remedy is a higher, and perhaps more natural, object of which the derived category is but a shadow (Section 8.7).

8.1 Derived functors as Kan extensions

For any abelian category $\mathcal{A}$, we have the triangulated category $\mathsf{K}^\star(\mathcal{A})$ given by Theorem 7.5.1 and its localization $Q : \mathsf{K}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A})$ (Definition 7.5.4), where the superscript $\star$ records the boundedness condition fixed at the beginning of this chapter.

Let $F : \mathcal{A} \to \mathcal{A}'$ be additive. Since F preserves mapping cones, it induces a triangulated functor $\mathsf{K}^\star F : \mathsf{K}^\star(\mathcal{A}) \to \mathsf{K}^\star(\mathcal{A}')$. Consider the following diagram:

$$\begin{tikzcd}
\mathsf{K}^\star(\mathcal{A}) \arrow[r, "\mathsf{K}^\star F"] \arrow[d, "Q"'] & \mathsf{K}^\star(\mathcal{A}') \arrow[d, "Q'"] \\
\mathsf{D}^\star(\mathcal{A}) & \mathsf{D}^\star(\mathcal{A}')
\end{tikzcd} \tag{8.1.1}$$

Unless F is exact, no filler $\mathsf{D}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A}')$ can make the diagram commutative. Instead, we seek a triangulated functor making it commute up to a 2-cell, in two possible ways:

$$\begin{tikzcd}
\mathsf{K}^\star(\mathcal{A}) \arrow[r, "\mathsf{K}^\star F"] \arrow[d, "Q"'] & \mathsf{K}^\star(\mathcal{A}') \arrow[d, "Q'"] \arrow[dl, Rightarrow] & \mathsf{K}^\star(\mathcal{A}) \arrow[r, "\mathsf{K}^\star F"] \arrow[d, "Q"'] & \mathsf{K}^\star(\mathcal{A}') \arrow[d, "Q'"] \\
\mathsf{D}^\star(\mathcal{A}) \arrow[r, dashed] & \mathsf{D}^\star(\mathcal{A}') & \mathsf{D}^\star(\mathcal{A}) \arrow[r, dashed] \arrow[ur, Rightarrow] & \mathsf{D}^\star(\mathcal{A}')
\end{tikzcd} \tag{8.1.2}$$

Moreover, we want to do so in a universal manner. The correct notion is left and right Kan extensions (Definition 1.8.1), corresponding respectively to the right and left derived functors of F.

Definition 8.1.1 For $\star \in \{+, -, \mathrm{b}, \ \}$,

- the ***right derived functor*** $\mathrm{R}\, F : \mathsf{D}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A}')$ is the left Kan extension $\mathrm{Lan}_Q(Q' \cdot \mathsf{K}^\star F)$, whenever it exists *as a triangulated functor*;
- dually, the ***left derived functor*** $\mathrm{L}\, F$ is the right Kan extension $\mathrm{Ran}_Q(Q' \cdot \mathsf{K}^\star F)$, whenever it exists *as a triangulated functor*.

Uniqueness holds up to unique isomorphism by Proposition 1.8.3. When they exist, set

$$\mathrm{R}^n\, F := \mathrm{H}^n \circ \mathrm{R}\, F, \quad \mathrm{L}_n\, F := \mathrm{H}^{-n} \circ \mathrm{L}\, F. \tag{8.1.3}$$

Remark 8.1.2 When the relevant derived functor exists, the definition of Kan extension includes morphisms

$$Q' \mathsf{K}^\star F \to (\mathrm{R}\, F)Q, \quad (\mathrm{L}\, F)Q \to Q' \mathsf{K}^\star F. \tag{8.1.4}$$

Therefore, the derived functor comes equipped with comparison morphisms to and from the underived $\mathsf{K}^\star F$, playing the role of the resolutions of Section 4.2.

Theorem 8.1.3 (*long exact sequence of derived functors*) If $\mathrm{R}\, F$ (resp. $\mathrm{L}\, F$) exists, then every exact triangle $X \to Y \to Z \xrightarrow{+1}$ in $\mathsf{D}^\star(\mathcal{A})$ yields a canonical long exact sequence

$$\cdots \to \mathrm{R}^{n-1}\, F(Z) \to \mathrm{R}^n\, F(X) \to \mathrm{R}^n\, F(Y) \to \mathrm{R}^n\, F(Z) \to \mathrm{R}^{n+1}\, F(X) \to \cdots \tag{8.1.5}$$

(resp. for $\mathrm{L}_n\, F$).

Proof $\mathrm{R}\, F$ is triangulated by definition; apply the cohomological functor H^0 (Remark 7.1.7).□

Example 8.1.4 If F is exact, $\mathsf{K}^\star F$ preserves acyclic complexes; it is then easy to show that the functor supplied by the universal property of localization is both $\mathrm{R}\, F$ and $\mathrm{L}\, F$. In particular $\mathrm{R}^0\, F(X) \simeq F(X)$ and $\mathrm{R}^n\, F(X) = 0$ for $n \neq 0$ and all $X \in \mathrm{Ob}(\mathcal{A})$.

8.2 Bounded derived functors and resolutions

To guarantee existence and to *compute* the derived functors, one resolves. The abstract device is the notion of an F-injective (or F-projective) subcategory.

Definition 8.2.1 A triangulated subcategory $\mathcal{I} \subset \mathsf{K}^\star(\mathcal{A})$ (resp. $\mathcal{P} \subset \mathsf{K}^\star(\mathcal{A})$) is called ***F*-injective** (resp. ***F*-projective**) if the following two properties hold:

▷ ***resolution*** for every $X \in \mathrm{Ob}(\mathsf{K}^\star(\mathcal{A}))$ there is a quasi-isomorphism $X \to I$ (resp. $P \to X$) with $I \in \mathrm{Ob}(\mathcal{I})$ (resp. $P \in \mathrm{Ob}(\mathcal{P})$);

▷ ***acyclicity*** $X \in \mathrm{Ob}(\mathcal{I})$ (resp. $X \in \mathrm{Ob}(\mathcal{P})$) is acyclic implies $\mathsf{K}^\star F(X)$ is acyclic.

The key existence-and-computation result goes as follows. Let $\mathcal{N}^\star \subset \mathsf{K}^\star(\mathcal{A})$ be the saturated triangulated subcategory of acyclic complexes (Definition 7.5.4).

Proposition 8.2.2 If $\mathcal{I}$ is F-injective, then $\mathrm{R}\,F$ exists; moreover the canonical functor

$$i : \mathcal{I}/(\mathcal{I} \cap \mathcal{N}) \to \mathsf{D}^\star(\mathcal{A}) \tag{8.2.1}$$

is a triangulated equivalence, and for any resolution $X \to I$ with $I \in \mathrm{Ob}(\mathcal{I})$ one has

$$\mathrm{R}\,F(QX) \simeq Q'\mathsf{K}^\star F(I). \tag{8.2.2}$$

Dually, an F-projective $\mathcal{P}$ gives $\mathrm{L}\,F(QX) \simeq Q'\mathsf{K}^\star F(P)$ for $P \to X$ with $P \in \mathrm{Ob}(\mathcal{P})$.

Proof The resolution condition makes $\mathcal{I}/(\mathcal{I} \cap \mathcal{N}) \to \mathsf{D}^\star(\mathcal{A})$ an equivalence by Proposition 7.4.6. The acyclicity condition makes $Q'\mathsf{K}^\star F$ invert the quasi-isomorphisms in $\mathcal{I}$ by considering mapping cones. Hence $Q'\mathsf{K}^\star F$ descends to $\mathcal{I}/(\mathcal{I} \cap \mathcal{N})$. The resulting composite satisfies the universal property of Definition 8.1.1, hence is $\mathrm{R}\,F$ (see [10, Proposition 4.6.4]).□

Concrete subcategories come from the auxiliary notion of an "I-type" subcategory.

Definition 8.2.3 An additive full subcategory $\mathcal{A}^\flat \subset \mathcal{A}$ is said to be of ***I-type*** relative to F if

▷ **(I1)** every $X \in \mathrm{Ob}(\mathcal{A})$ embeds into some $I \in \mathrm{Ob}(\mathcal{A}^\flat)$;

▷ **(I2)** for a short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X', X \in \mathrm{Ob}(\mathcal{A}^\flat)$, also $X'' \in \mathrm{Ob}(\mathcal{A}^\flat)$, and $0 \to FX' \to FX \to FX'' \to 0$ is exact.

Dually, $\mathcal{A}^\flat$ is of ***P-type*** if

▷ **(P1)** every $X \in \mathrm{Ob}(\mathcal{A})$ is a quotient of some $P \in \mathrm{Ob}(\mathcal{A}^\flat)$;

▷ **(P2)** for a short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X, X'' \in \mathrm{Ob}(\mathcal{A}^\flat)$, also $X' \in \mathrm{Ob}(\mathcal{A}^\flat)$ and $0 \to FX' \to FX \to FX'' \to 0$ is exact.

Proposition 8.2.4 If $\mathcal{A}^\flat$ is of I-type (resp. P-type) relative to F, then $\mathsf{K}^+(\mathcal{A}^\flat)$ is F-injective (resp. $\mathsf{K}^-(\mathcal{A}^\flat)$ is F-projective).

Proof We treat the I-type case. Since $\mathcal{A}^\flat$ is additive, $\mathsf{K}^+(\mathcal{A}^\flat)$ is a full triangulated subcategory of $\mathsf{K}^+(\mathcal{A})$. Its resolution condition is (I1), by Theorem 4.1.4, so only the acyclicity remains. Let $X \in \mathrm{Ob}(\mathsf{K}^+(\mathcal{A}^\flat))$ be acyclic. The degreewise short exact sequences $0 \to \ker d_X^n \to X^n \to \ker d_X^{n+1} \to 0$ start from $\ker d_X^n = X^n = 0$ for $n \ll 0$; by (I2) and induction on n, every $\ker d_X^n$ lies in $\mathrm{Ob}(\mathcal{A}^\flat)$ and

$$0 \to F(\ker d_X^n) \to F(X^n) \to F\big(\ker d_X^{n+1}\big) \to 0 \tag{8.2.3}$$

is exact in $\mathcal{A}'$. Since Fd_X^n factors as $F(X^n) \twoheadrightarrow F(\ker d_X^{n+1}) \hookrightarrow F(X^{n+1})$ with epi and mono both furnished by (I2), splicing yields $\ker(Fd_X^n) = F(\ker d_X^n) = \operatorname{im}(Fd_X^{n-1})$ for every n; thus $F(X)$ is acyclic. $\square$

Example 8.2.5 If $\mathcal{A}$ has enough injectives, the injective objects form an I-type subcategory for *every* F since a short exact sequence with injective first term splits. Dually enough projectives give a P-type subcategory for every F.

Theorem 8.2.6 (*derived functors via resolutions*) Suppose $\mathcal{A}^\flat$ is of I-type (resp. P-type) relative to F. Then $\mathrm{R}\,F$ (resp. $\mathrm{L}\,F$) exists, and:

(i) every $X \in \mathrm{Ob}(\mathsf{K}^+(\mathcal{A}))$ (resp. $X \in \mathrm{Ob}(\mathsf{K}^-(\mathcal{A}))$) admits a quasi-isomorphism $X \to I$ (resp. $P \to X$) with $I \in \mathrm{Ob}(\mathsf{K}^+(\mathcal{A}^\flat))$ (resp. $P \in \mathrm{Ob}(\mathsf{K}^-(\mathcal{A}^\flat))$);

(ii) for any such quasi-isomorphism, the comparison morphism of Remark 8.1.2 gives

$$\mathrm{R}\,F(QX) \simeq \mathrm{R}\,F(QI) \simeq Q'\mathsf{K}^+F(I) \tag{8.2.4}$$

(resp. $\mathrm{L}\,F(QX) \simeq \mathrm{L}\,F(QP) \simeq Q'\mathsf{K}^-F(P)$);

(iii) $\mathrm{R}\,F$ restricts to a functor $\mathsf{D}^{\geq 0}(\mathcal{A}) \to \mathsf{D}^{\geq 0}(\mathcal{A}')$ (resp. $\mathrm{L}\,F$ restricts to $\mathsf{D}^{\leq 0}(\mathcal{A}) \to \mathsf{D}^{\leq 0}(\mathcal{A}')$);

(iv) if F is left exact (resp. right exact), then for every $X \in \mathrm{Ob}(\mathcal{A})$ the comparison morphism gives $FX \simeq \mathrm{R}^0\,F(QX)$ (resp. $FX \simeq \mathrm{L}_0\,F(QX)$);

(v) for every $X \in \mathrm{Ob}(\mathcal{A}^\flat)$ we have $\mathrm{R}^n\,F(X) = 0$ (resp. $\mathrm{L}_n\,F(X) = 0$) whenever $n > 0$.

Proof We treat the I-type case only; the P-type case is dual. By Proposition 8.2.4 the subcategory $\mathsf{K}^+(\mathcal{A}^\flat)$ is F-injective, so (i)–(ii) restate Proposition 8.2.2.

For (iii), represent $X \in \mathrm{Ob}(\mathsf{D}^{\geq 0}(\mathcal{A}))$ by a complex in $\mathsf{C}^{\geq 0}(\mathcal{A})$ via truncation (Definition 3.7.1): the construction of Theorem 4.1.4 produces a quasi-isomorphism $X \to I$ with $I \in \mathrm{Ob}(\mathsf{C}^{\geq 0}(\mathcal{A}^\flat))$, and (ii) yields $\mathrm{R}\,F(QX) \simeq Q'\mathsf{K}^+F(I) \in \mathrm{Ob}(\mathsf{D}^{\geq 0}(\mathcal{A}'))$.

For (v), resolve $X \in \mathrm{Ob}(\mathcal{A}^\flat)$ by itself: then $\mathrm{R}\,F(QX) = Q'FX$ is concentrated in degree 0.

For (iv), flatten a resolution of $X \in \mathrm{Ob}(\mathcal{A})$ into an exact sequence $0 \to X \to I^0 \to I^1 \to \cdots$ with $I^n \in \mathrm{Ob}(\mathcal{A}^\flat)$ (Example 4.1.3); by (ii) and left exactness, $\mathrm{R}^0\,F(QX) \simeq \ker(Fd_I^0) \simeq FX$. $\square$

Corollary 8.2.7 Suppose F is left exact (resp. right exact) and $\mathcal{A}$ has enough injectives (resp. projectives). Then $\mathrm{R}^n\,F|_{\mathcal{A}}$ (resp. $\mathrm{L}_n\,F|_{\mathcal{A}}$) agrees with the classical derived functors of Definition 4.2.2.

Proof Example 8.2.5 and (ii) compute both sides from one injective (resp. projective) resolution. $\square$

We have the following slightly more general version.

Proposition 8.2.8 Let F be left exact (resp. right exact) and suppose $\mathcal{A}^\flat$ is of I-type (resp. P-type) relative to F. Then $(\mathrm{R}^n\,F|_{\mathcal{A}})_{n\geq 0}$, with connecting morphisms induced by Theorem 8.1.3, is the universal cohomological δ-functor with $\mathrm{R}^0\,F \simeq F$ (Definition 4.2.9); dually, $(\mathrm{L}_n\,F|_{\mathcal{A}})_{n\geq 0}$ is the universal homological δ-functor.

Proof Theorem 8.2.6 implies that $(\mathrm{R}^n\,F|_{\mathcal{A}})_{n\geq 0}$ is a cohomological δ-functor (Definition 4.2.6) with $\mathrm{R}^0\,F|_{\mathcal{A}} \simeq F$. For $n > 0$, the functor $\mathrm{R}^n\,F$ is effaceable in the sense of Definition 4.2.7: embed X into some $I \in \mathrm{Ob}(\mathcal{A}^\flat)$ by (I1); by (v) we deduce $\mathrm{R}^n\,F(I) = 0$. Universality follows from Proposition 4.2.10. $\square$

Corollary 8.2.9 In the setting of Theorem 8.2.6, with F left exact (resp. right exact), let $\mathcal{A}^{\natural}$ be the additive full subcategory of $\mathcal{A}$ on the F-acyclic objects (Convention 4.2.12). Then $\mathcal{A}^{\flat} \subset \mathcal{A}^{\natural}$, the subcategory $\mathcal{A}^{\natural}$ is again of I-type, and every quasi-isomorphism $X \to I$ with $I \in \mathrm{Ob}(\mathsf{K}^{+}(\mathcal{A}^{\natural}))$ computes $\mathrm{R}\,F(QX) \simeq Q'\mathsf{K}^{+}F(I)$. Dually for P-type and $\mathrm{L}\,F$.

Proof The inclusion and (I1) for $\mathcal{A}^{\natural}$ follow from (v). For (I2), let $0 \to X' \to X \to X'' \to 0$ be exact with $X', X \in \mathrm{Ob}(\mathcal{A}^{\natural})$: the long exact sequence of Theorem 8.1.3 reads

$$\underbrace{\mathrm{R}^n F(X)}_{=0} \to \mathrm{R}^n F(X'') \to \underbrace{\mathrm{R}^{n+1} F(X')}_{=0}, \quad n \geq 1, \tag{8.2.5}$$

so X'' is F-acyclic; its $n = 0$ portion, with $\mathrm{R}^0 F$ identified with F by (iv), gives the exactness of $0 \to FX' \to FX \to FX'' \to 0$. Now apply the theorem to $\mathcal{A}^{\natural}$. □

On objects of $\mathcal{A}$, the above recovers Corollary 4.2.14, in view of Proposition 8.2.8. This is the reconciliation promised in Section 4.2: the abstract $\mathrm{R}\,F$ of Definition 8.1.1, applied to an object of $\mathcal{A}$ and then H^n, recovers the classical $\mathrm{R}^n F$ computed from an injective resolution. The same holds, dually, for $\mathrm{L}\,F$ via projectives. One may even allow K-injective and K-projective resolutions and obtain similar reconciliations (Example 4.6.3, Corollary 7.5.3).

8.3 Composition and the Grothendieck spectral sequence

Derived functors of a composite are related to the composite of derived functors by a canonical map, which is an isomorphism under an acyclicity hypothesis.

Theorem 8.3.1 Consider additive functors $\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$.

(i) If the relevant right derived functors exist, there is a canonical morphism

$$\mathrm{R}(F'F) \to (\mathrm{R}\,F')(\mathrm{R}\,F). \tag{8.3.1}$$

Dually, for left derived functors, $(\mathrm{L}\,F')(\mathrm{L}\,F) \to \mathrm{L}(F'F)$.

(ii) Suppose $\mathcal{A}^{\flat} \subset \mathcal{A}$ is I-type for F, $(\mathcal{A}')^{\flat} \subset \mathcal{A}'$ is I-type for F', and F sends the objects of $\mathcal{A}^{\flat}$ to F'-acyclic objects. Then $\mathcal{A}^{\flat}$ is I-type for $F'F$, and the canonical map above is an isomorphism

$$\mathrm{R}(F'F) \simeq (\mathrm{R}\,F')(\mathrm{R}\,F). \tag{8.3.2}$$

Dually with P-type subcategories for the left-derived version.

Proof (*Sketch*) We only consider the right derived case. The morphism in (i) comes from the universal property of the Kan extension: the composite $\mathrm{R}\,F' \circ \mathrm{R}\,F$ receives a 2-cell from $Q'' \mathsf{K}\, F' \mathsf{K}\, F$ through those of $\mathrm{R}\,F$ and $\mathrm{R}\,F'$, and $\mathrm{R}(F'F)$ is universal among such recipients.

As for (ii), a resolution of X by objects of $\mathcal{A}^{\flat}$ is sent by F to a resolution of FX by F' -acyclic objects, hence an F'-injective one (Corollary 8.2.9); in particular $\mathcal{A}^{\flat}$ is I-type for $F'F$: (I1) is shared, and (I2) follows from Corollary 8.2.9 applied to $(\mathcal{A}')^{\flat} \subset (\mathcal{A}')^{\natural}$. Finally, Proposition 8.2.2 computes both $\mathrm{R}(F'F)(X)$ and $(\mathrm{R}\,F')(\mathrm{R}\,F(X))$ from one and the same complex. □

Remark 8.3.2 Take H^n of the isomorphism $\mathrm{R}(F'F) \simeq (\mathrm{R}\,F')(\mathrm{R}\,F)$ of Theorem 8.3.1, in the situation of its assertion (ii), and let $X \to I$ be a resolution with I built from $\mathcal{A}^{\flat}$. The left side is $\mathrm{R}^n(F'F)(X)$; the right side is the cohomology of the object $(\mathrm{R}\,F')(\mathrm{R}\,F(X))$, where $\mathrm{R}\,F(X)$

is represented by the complex $F(I)$, with $\mathrm{H}^q(\mathrm{R}\,F(X)) = \mathrm{R}^q\,F(X)$, concentrated in degrees $q \geq 0$ (Theorem 8.2.6 (iii)). The Grothendieck spectral sequence of Theorem 6.5.4 makes this concrete: take a Cartan–Eilenberg resolution $F(I) \to J$ in $\mathcal{A}'$ (Remark 4.1.10), so that both sides are computed by the total complex $\operatorname{tot} \mathsf{C}^2 F'(J)$, and filter the latter by rows and columns (Proposition 6.5.1):

- $\mathcal{E}_{\mathrm{I}}$ has $E_{\mathrm{I},1}^{p,q} = (\mathrm{R}^q\,F')(F(I^p))$, which vanishes for $q \neq 0$ by the F'-acyclicity of $F(I^p)$: it degenerates at E_2 and leads to $\mathrm{H}^n(\operatorname{tot} \mathsf{C}^2 F'(J)) \simeq \mathrm{R}^n(F'F)(X)$;
- $\mathcal{E}_{\mathrm{II}}$ has, at its second page, $E_{\mathrm{II},2}^{p,q} = (\mathrm{R}^p\,F')(\mathrm{R}^q\,F)(X)$, because the horizontal cohomology of J resolves $\mathrm{H}^q(F(I)) = (\mathrm{R}^q\,F)(X)$ while F' preserves it (Remark 4.1.10).

Both converge strongly with finite induced filtrations (Theorem 6.5.2). Thus the Grothendieck spectral sequence is a concrete incarnation of the triangulated isomorphism $\mathrm{R}(F'F) \simeq (\mathrm{R}\,F')(\mathrm{R}\,F)$; it is strictly more informative, since it also records the induced filtration on each $\mathrm{R}^n(F'F)(X)$.

Derived functors have a notion of size and dimension.

Definition 8.3.3 We say the ***amplitude*** of a triangulated functor $R : \mathsf{D}^\star(\mathcal{A}) \to \mathsf{D}^\star(\mathcal{A}')$ is contained in $[a,b]$, for $-\infty \leq a \leq b \leq \infty$, if R restricts to a functor $\mathcal{A} \to \mathsf{D}^{[a,b]}(\mathcal{A}')$.

For a left-exact (resp. right-exact) F with $\mathrm{R}\,F$ (resp. $\mathrm{L}\,F$) as in Theorem 8.2.6, the amplitude is contained in $[0,d]$ (resp. $[-d,0]$) for some $d \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ by item (iii) of that theorem; the infimum of such d is called the ***dimension*** of $\mathrm{R}\,F$ (resp. $\mathrm{L}\,F$).

Definition 8.3.4 Let $X \in \mathrm{Ob}(\mathcal{A})$.

- If $\mathcal{A}$ has enough injectives, the ***injective dimension*** $\operatorname{inj.dim}(X)$ is the dimension of $\mathrm{R}(\mathrm{Hom}(\cdot\,,X))$.
- If $\mathcal{A}$ has enough projectives, the ***projective dimension*** $\operatorname{proj.dim}(X)$ is that of $\mathrm{R}(\mathrm{Hom}(X,\cdot))$.
- When both premises hold, the ***global dimension*** of $\mathcal{A}$ is

$$\begin{aligned} \operatorname{gl.dim}(\mathcal{A}) &:= \sup_X \operatorname{inj.dim}(X) \\ &= \inf\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}^{\geq n+1}(\cdot,\cdot) = 0\} = \sup_X \operatorname{proj.dim}(X), \end{aligned} \tag{8.3.3}$$

  with the convention $\inf \varnothing := \infty$; the equalities will be justified after the Proposition 8.3.5 below.

Proposition 8.3.5 Suppose $\mathcal{A}$ has enough injectives. For $X \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{Z}_{\geq 0}$, the following are equivalent:

(i) $\operatorname{inj.dim}(X) \leq n$;
(ii) $\mathrm{Ext}^{\geq n+1}(\cdot, X) = 0$;
(iii) $\mathrm{Ext}^{n+1}(\cdot, X) = 0$;
(iv) X has an injective resolution of length $\leq n$, i.e. an exact sequence $0 \to X \to I^0 \to I^1 \to \cdots \to I^n \to 0$ with every I^k injective.

Dually, with enough projectives, the following are equivalent:

(i)' $\operatorname{proj.dim}(X) \leq n$;
(ii)' $\mathrm{Ext}^{\geq n+1}(X, \cdot) = 0$;
(iii)' $\mathrm{Ext}^{n+1}(X, \cdot) = 0$;
(iv)' X has a projective resolution of length $\leq n$.

Proof By duality, we treat the injective case only. We will compute $\mathrm{Ext}^\bullet(Y, X)$ by taking injective resolutions of X, i.e. by viewing it as the value of $\mathrm{R}(\mathrm{Hom}(Y, \cdot))$ at X.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): clear.

(iii) $\Rightarrow$ (iv): take an injective resolution $0 \to X \to I^0 \to I^1 \to \cdots$ and put $B^k := \mathrm{im}(d_I^{k-1})$ for $k \geq 1$, with $B^0 := X$, so that there are short exact sequences $0 \to X \to I^0 \to B^1 \to 0$ and $0 \to B^k \to I^k \to B^{k+1} \to 0$. The long exact sequence in the second variable (Proposition 7.6.5) gives, recursively, $\mathrm{Ext}^1(\cdot, B^n) \simeq \mathrm{Ext}^{n+1}(\cdot, X) = 0$, so B^n is injective (Proposition 7.6.5 (iv)), and $0 \to X \to I^0 \to \cdots \to I^{n-1} \to B^n \to 0$ is a resolution of length $\leq n$.

(iv) $\Rightarrow$ (i): a resolution of length $\leq n$ computes $\mathrm{Ext}^{\geq n+1}(Y, X) \simeq \mathrm{H}^{\geq n+1}\,\mathrm{Hom}(Y, I^\bullet)$ for every $Y \in \mathrm{Ob}(\mathcal{A})$, whence $\mathrm{inj.dim}(X) \leq n$ (Definition 8.3.3). $\square$

To justify the remaining equalities in Definition 8.3.4, reading (i) $\Leftrightarrow$ (ii) and its dual for all X at once in Proposition 8.3.5: for $n \in \mathbb{Z}_{\geq 0}$, $n \geq \sup_X \mathrm{inj.dim}(X)$ iff $\mathrm{Ext}^{\geq n+1}(\cdot, \cdot) = 0$ iff $n \geq \sup_X \mathrm{proj.dim}(X)$; taking inf over n yields the desired equalities.

Corollary 8.3.6 Suppose $\mathcal{A}$ has enough injectives and projectives and $\mathrm{gl.dim}(\mathcal{A}) = n < \infty$. Then every left exact (resp. right exact) $F : \mathcal{A} \to \mathcal{A}'$ has $\mathrm{R}\,F$ (resp. $\mathrm{L}\,F$) of dimension $\leq n$.

Proof For $X \in \mathrm{Ob}(\mathcal{A})$, compute $\mathrm{R}\,F(X)$ on an injective resolution of length $\leq n$ supplied by Proposition 8.3.5. The left-derived case is dual. $\square$

Example 8.3.7 Let R be a principal ideal domain. Submodules of free modules are free ([9, Lemma 6.7.4]), so every R-module M has a free resolution $0 \to F_1 \to F_0 \to M \to 0$, whence $\mathrm{gl.dim}(R\text{-}\mathsf{Mod}) \leq 1$ by Proposition 8.3.5. Equality holds as soon as R is not a field: for a nonzero non-unit $t \in R$, the cyclic module R/Rt is not projective because of torsion, hence it cannot have projective resolution of length 0.

Remark 8.3.8 B. L. Osofsky [14] proved the following unexpected result: for a product of countably infinitely many fields, the global dimension is $k+1$ if and only if the cardinal equality $2^{\aleph_0} = \aleph_k$ holds ($k \in \mathbb{Z}_{\geq 0}$); in particular, global dimension $= 2$ is equivalent to the continuum hypothesis $2^{\aleph_0} = \aleph_1$!

Exercise 8.3.9 Show that if $\mathcal{A}$ has enough injectives and the left exact functor $F : \mathcal{A} \to \mathcal{A}'$ has finite-dimensional $\mathrm{R}\,F$, then the induced map $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}')$ is $[A] \mapsto \sum_n (-1)^n [\mathrm{R}^n\,F(A)]$. *Hint.* Since $\mathrm{R}\,F$ has finite dimension, it restricts to $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A}')$ (truncate with Proposition 7.5.9); now transport $[A] \mapsto [\mathrm{R}\,F(A)]$ through the Exercise 7.5.11 (iii).

8.4 Derived bifunctors

We now extend the formalism of Section 8.1 to two variables. Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ be an additive bifunctor, i.e. additive in each variable. Applying F degreewise to complexes $X_i \in \mathrm{Ob}(\mathsf{C}^\star(\mathcal{A}_i))$ produces a double complex $\mathsf{C}^2 F(X_1, X_2)$ (Definition 3.8.1) with (p, q)-entry $F(X_1^p, X_2^q)$; when $\mathcal{B}$ has countable coproducts (resp. products) we may totalize (Remark 3.8.4), obtaining bifunctors

$$\mathsf{C}_\oplus F := \mathrm{tot}^\oplus \circ \mathsf{C}^2 F \quad \text{and} \quad \mathsf{C}_\prod F := \mathrm{tot}^\prod \circ \mathsf{C}^2 F. \tag{8.4.1}$$

Homotopies in either variable induce homotopies on the totalization, hence these descend to bifunctors

$$\mathsf{K}_{\oplus}F,\ \mathsf{K}_{\prod}F : \mathsf{K}^{\star}(\mathcal{A}_1) \times \mathsf{K}^{\star}(\mathcal{A}_2) \to \mathsf{K}^{\star}(\mathcal{B}) \tag{8.4.2}$$

that commute with translation and preserve mapping cones in each variable, i.e. are triangulated in each variable.

Let $Q_i : \mathsf{K}^{\star}(\mathcal{A}_i) \to \mathsf{D}^{\star}(\mathcal{A}_i)$ and $Q : \mathsf{K}^{\star}(\mathcal{B}) \to \mathsf{D}^{\star}(\mathcal{B})$ be the localization functors. The functor (Q_1, Q_2) between product categories is again a localization: it inverts exactly the pairs of quasi-isomorphisms, which form a multiplicative system in the product category, as seen by a direct check from Definition 1.8.5.

Definition 8.4.1 Assume $\mathcal{B}$ has countable products (resp. coproducts). The ***right derived bifunctor*** (resp. ***left derived bifunctor***) of F is

$$\mathrm{R}\,F := \mathrm{Lan}_{(Q_1,Q_2)}\big(Q \cdot \mathsf{K}_{\prod}F\big) \quad (\text{resp.}\quad \mathrm{L}\,F := \mathrm{Ran}_{(Q_1,Q_2)}(Q \cdot \mathsf{K}_{\oplus}F)\ ), \tag{8.4.3}$$

whenever it exists as a bifunctor $\mathsf{D}^{\star}(\mathcal{A}_1) \times \mathsf{D}^{\star}(\mathcal{A}_2) \to \mathsf{D}^{\star}(\mathcal{B})$ that is a **triangulated bifunctor**: a bifunctor R that is triangulated in each variable and such that, writing T_i (resp. T') for the translations $\cdot[1]$ of $\mathsf{D}^{\star}(\mathcal{A}_i)$ (resp. of the target), the square

$$\begin{tikzcd} R(T_1X_1, T_2X_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & T'R(X_1, T_2X_2) \arrow[d] \\ T'R(T_1X_1, X_2) \arrow[r] & (T')^2R(X_1, X_2) \end{tikzcd} \tag{8.4.4}$$

of translation constraints, formed from the triangulated structures in the two variables, commutes up to the sign -1, for all $X_i \in \mathrm{Ob}(\mathsf{D}^{\star}(\mathcal{A}_i))$ (see [10, Definition 4.6.7]).

As in the setting of Section 8.1:

- uniqueness holds up to unique isomorphism by Proposition 1.8.3;
- the derived bifunctor carries comparison morphisms as in Remark 8.1.2;
- we set $\mathrm{R}^n\,F := \mathrm{H}^n \circ \mathrm{R}\,F$ and $\mathrm{L}_n\,F := \mathrm{H}^{-n} \circ \mathrm{L}\,F$.

Existence again goes through resolutions; the following is the two-variable avatar of Proposition 8.2.4.

Proposition 8.4.2 For $i \in \{1,2\}$ let $\mathcal{A}_i^{\flat} \subseteq \mathcal{A}_i$ be an additive full subcategory such that

- for each $X_1 \in \mathrm{Ob}\big(\mathcal{A}_1^{\flat}\big)$, the subcategory $\mathcal{A}_2^{\flat}$ is of I-type relative to $F(X_1,\cdot)$;
- for each $X_2 \in \mathrm{Ob}\big(\mathcal{A}_2^{\flat}\big)$, the subcategory $\mathcal{A}_1^{\flat}$ is of I-type relative to $F(\cdot, X_2)$ (Definition 8.2.3).

Then $\mathrm{R}\,F$ exists and, for $X_i \in \mathrm{Ob}\big(\mathsf{K}^{+}\big(\mathcal{A}_i^{\flat}\big)\big)$,

$$\mathrm{R}\,F(Q_1X_1, Q_2X_2) \simeq Q\mathsf{K}_{\prod}F(X_1, X_2). \tag{8.4.5}$$

Dually, the P-type conditions give

$$\mathrm{L}\,F(Q_1X_1, Q_2X_2) \simeq Q\mathsf{K}_{\oplus}F(X_1, X_2) \tag{8.4.6}$$

for $X_i \in \mathrm{Ob}\big(\mathsf{K}^{-}\big(\mathcal{A}_i^{\flat}\big)\big)$. In particular, if $\mathcal{A}_1$ and $\mathcal{A}_2$ have enough injectives (resp. enough projectives), then $\mathrm{R}\,F$ (resp. $\mathrm{L}\,F$) exists for every additive bifunctor F.

Proof We treat the right-derived case and write $\mathcal{I}_i := \mathsf{K}^+\left(\mathcal{A}_i^\flat\right)$. Resolutions exist in each variable by Theorem 4.1.4. On the other hand, fix $X_1 \in \mathrm{Ob}(\mathcal{I}_1)$ and an acyclic $Y \in \mathrm{Ob}(\mathcal{I}_2)$: for every p, the one-variable statement Proposition 8.2.4, applied to the functor $F(X_1^p, \cdot)$, makes the column $F(X_1^p, Y^\bullet)$ of $\mathsf{C}^2F(X_1, Y)$ acyclic; the double complex has finite support on each total degree, and the acyclic assembly lemma (Theorem 3.8.6, Remark 3.8.4) makes $\mathsf{K}_\prod F(X_1, Y)$ acyclic, and symmetrically in the two variables.

Consequently $Q \cdot \mathsf{K}_\prod F$ inverts pairs of quasi-isomorphisms between objects of $\mathcal{I}_1 \times \mathcal{I}_2$ and descends along the product of equivalences of Proposition 7.4.6, exactly as in the proof of Proposition 8.2.2; the descent is the required left Kan extension. □

Finally, we record the multiplicative structure on derived bifunctors, assuming one-sided exactness of F in each variable.

Proposition 8.4.3 Suppose that F is right exact in each variable (resp. left exact in each variable) and that the P-type (resp. I-type) hypotheses of Proposition 8.4.2 hold. Then there are canonical morphisms, natural in (X_1, X_2) and defined for all $p, q \in \mathbb{Z}$:

$$\begin{gathered} F(\mathrm{H}^p(X_1), \mathrm{H}^q(X_2)) \to \mathrm{H}^{p+q}(\mathrm{L}\, F(X_1, X_2)) \\ (\text{resp.} \quad \mathrm{H}^{p+q}(\mathrm{R}\, F(X_1, X_2)) \to F(\mathrm{H}^p(X_1), \mathrm{H}^q(X_2))\ ). \end{gathered} \tag{8.4.7}$$

Proof (*Sketch*) We only treat the right exact case; the left exact case is dual.

At the level of complexes, writing $Z^n(X) := \ker d_X^n$ for the cocycles, the maps $F(Z^p(X_1), Z^q(X_2)) \to Z^{p+q}(\mathsf{C}_\oplus F(X_1, X_2))$ are tautological, and right exactness in each variable makes them descend to

$$F(\mathrm{H}^p(X_1), \mathrm{H}^q(X_2)) \to \mathrm{H}^{p+q}(\mathsf{C}_\oplus F(X_1, X_2)). \tag{8.4.8}$$

Both sides of the sought morphism depend only on (Q_1X_1, Q_2X_2) up to isomorphism, so by functoriality it suffices to prescribe it when $X_i \in \mathrm{Ob}\left(\mathsf{K}^-\left(\mathcal{A}_i^\natural\right)\right)$ for each i; there the comparison morphism of Remark 8.1.2 is the identity and the construction is the displayed morphism. □

For $F = \otimes_R$ the pairings of Proposition 8.4.3 underlie the graded product on $\bigoplus_n \mathrm{Tor}_n^R$; we will meet them again in Example 8.6.6.

8.5 RHom and derived inverse limits

Throughout this section, $\mathcal{A}$ is a fixed abelian category, and we assume all the required resolutions exist; for $\mathcal{A} = R\text{-}\mathsf{Mod}$ they always do (Example 4.6.6), where R is any ring.

8.5.1 Derived Hom and derived adjunctions

The bifunctor $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$ is left exact in each variable. If there exist additive full subcategories $\mathcal{A}_1^\flat \subset \mathcal{A}^{\mathrm{op}}$ and $\mathcal{A}_2^\flat \subset \mathcal{A}$ as in Proposition 8.4.2, then the right derived bifunctor (Definition 8.4.1) of Hom exists, as a functor

$$\mathsf{D}^+(\mathcal{A}^{\mathrm{op}}) \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Ab}). \tag{8.5.1}$$

Furthermore, we may identify $\mathsf{D}^+(\mathcal{A}^{\mathrm{op}})$ with $\mathsf{D}^-(\mathcal{A})$ by reversing arrows; see [10, Proposition

4.4.16].[4] This justifies the following.

Definition 8.5.1 Under the assumptions above, the ***derived Hom*** is the right derived bifunctor

$$\mathrm{RHom} := \mathrm{RHom}_{\mathcal{A}} : \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Ab}). \tag{8.5.2}$$

If $\mathcal{A}$ is $\Bbbk$-linear, where $\Bbbk$ is a commutative ring, the codomain of $\mathrm{RHom}_{\mathcal{A}}$ lifts to $\mathsf{D}^+(\Bbbk\text{-}\mathsf{Mod})$.

For simplicity, in what follows we identify complexes with their images in the derived category, i.e. we will omit the localization functor Q.

Theorem 8.5.2 Suppose $\mathcal{A}$ has enough injectives or enough projectives. Then:
(i) $\mathrm{RHom} : \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Ab})$ exists;
(ii) if $X_2 \in \mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$ and $X_2 \to I_2$ is an injective resolution, then $\mathrm{RHom}(X_1, X_2) \simeq \mathrm{Hom}^\bullet(X_1, I_2)$ (the Hom complex in Definition 3.4.1);
(iii) if $X_1 \in \mathrm{Ob}(\mathsf{C}^-(\mathcal{A}))$ and $P_1 \to X_1$ is a projective resolution, then $\mathrm{RHom}(X_1, X_2) \simeq \mathrm{Hom}^\bullet(P_1, X_2)$;
(iv) there is a canonical isomorphism

$$\mathrm{H}^n\,\mathrm{RHom}(X, Y) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n]) \tag{8.5.3}$$

for $X \in \mathrm{Ob}(\mathsf{D}^-(\mathcal{A}))$ and $Y \in \mathrm{Ob}(\mathsf{D}^+(\mathcal{A}))$; when $X, Y \in \mathrm{Ob}(\mathcal{A})$, the right-hand side is the $\mathrm{Ext}^n(X, Y)$ of Definition 7.6.4.

Proof (*Sketch*) Suppose $\mathcal{A}$ has enough injectives. For (i) we may take $\mathcal{A}_2^\flat$ to be the subcategory of injectives in $\mathcal{A}$, whereas $\mathcal{A}_1^\flat = \mathcal{A}^{\mathrm{op}}$ in Proposition 8.4.2. Indeed, $\mathrm{Hom}_{\mathcal{A}}(\cdot, X_2)$ is exact for all injective $X_2 \in \mathrm{Ob}(\mathcal{A})$.

For the isomorphism in (ii), the point is to identify $\left(\mathsf{C}_{\prod} \mathrm{Hom}\right)(X_1, X_2)$, modulo switching between $\mathsf{C}(\mathcal{A}^{\mathrm{op}})$ and $\mathsf{C}(\mathcal{A})^{\mathrm{op}}$ by reversing arrows, with $\mathrm{Hom}^\bullet(X_1, X_2)$. This step is routine; see [10, Example 3.5.12]. The reader is invited to try the case where $X_1, X_2 \in \mathrm{Ob}(\mathcal{A})$.

To show (iv), we compute $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$ using Proposition 7.6.1 by choosing an injective resolution $Y \to I_2$ and using (ii).

The case of enough projectives, and item (iii), are proved in the same way. □

Corollary 8.5.3 For $X, Y \in \mathrm{Ob}(\mathcal{A})$, when $\mathcal{A}$ has enough injectives *and* enough projectives, the derived-categorical $\mathrm{Ext}^n(X, Y)$ (Definition 7.6.4) agrees with the classical Ext^n of Proposition 4.4.1.

Proof Interpret the derived-categorical $\mathrm{Ext}^n(X, Y)$ using Theorem 8.5.2 (iv); reconcile the derived-categorical $\mathrm{Ext}^n(X, Y)$ with the classical one by Theorem 8.5.2 (ii), (iii). □

More precisely, one can show that an extension $0 \to Y \to E \to X \to 0$ maps to the image of id_X under the connecting morphism $\mathrm{Hom}(X, X) \to \mathrm{Ext}^1(X, Y)$.

This closes the forward reference planted in Remark 4.4.2.

Theorem 8.5.4 (*derived adjunction*) For an adjunction $F : \mathcal{A} \dashv \mathcal{A}' : G$ of additive functors, with $\mathcal{A}$ having enough projectives and $\mathcal{A}'$ enough injectives, there are canonical isomorphisms

$$\mathrm{RHom}_{\mathcal{A}'}(\mathrm{L}\,F(X), Y) \simeq \mathrm{RHom}_{\mathcal{A}}(X, \mathrm{R}\,G(Y)), \tag{8.5.4}$$

[4] Some annoying signs appear: see [10, Proposition 3.4.4].

for $X \in \mathsf{D}^-(\mathcal{A}), Y \in \mathsf{D}^+(\mathcal{A}')$. Taking H^0 yields $\mathrm{Hom}_{\mathsf{D}(\mathcal{A}')}(\mathrm{L}\,F(X), Y) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, \mathrm{R}\,G(Y))$.

Proof (*Sketch*) Resolve X by projectives P and Y by injectives I; then both sides reduce to $\mathrm{Hom}^\bullet(\mathsf{K}\ F(P), I) \simeq \mathrm{Hom}^\bullet(P, \mathsf{K}\ G(I))$, which is the termwise adjunction $\mathrm{Hom}(F\,\cdot,\cdot) \simeq \mathrm{Hom}(\cdot, G\,\cdot)$ promoted to the complex level; Example 1.6.4 records the central special case. For details, see [10, Theorem 4.9.9]. ☐

Remark 8.5.5 For the unbounded derived categories, the RHom in Theorem 8.5.2 can be extended to

$$\mathrm{RHom} := \mathrm{RHom}_{\mathcal{A}} : \mathsf{D}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathsf{Ab}) \tag{8.5.5}$$

as the derived bifunctor. The key idea is to use K-injective and K-projective resolutions; for details, see [10, Theorem 4.11.7]. Theorem 8.5.4 also extends to the unbounded setting; see [10, Theorem 4.11.8].

8.5.2 Derived inverse limits as homotopy limits

Recall the category $\mathsf{InvSys}(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}_{\geq 0}^{\mathrm{op}}}$ introduced in Definition 4.5.1 and the difference map $\Delta_X = T_X - \mathrm{id} : \prod_k X_k \to \prod_k X_k$, whose kernel is $\lim_k X_k$. The derived version is governed by a homotopy-theoretic construction.

Definition 8.5.6 (*Bökstedt–Neeman*) For a triangulated additive category $\mathcal{D}$ with countable products, the ***homotopy limit*** of an inverse system $X = (X_k, f_k)_{k\geq 0}$ is the object $\mathrm{holim}(X)$ fitting into the exact triangle

$$\mathrm{holim}(X) \to \prod_k X_k \xrightarrow{\Delta_X} \prod_k X_k \xrightarrow{+1} . \tag{8.5.6}$$

Dually define the ***homotopy colimit*** hocolim using $\nabla_X = T_X - \mathrm{id}$ on coproducts, namely

$$\coprod_k X_k \xrightarrow{\nabla_X} \coprod_k X_k \to \mathrm{hocolim}(X) \xrightarrow{+1} . \tag{8.5.7}$$

Remark 8.5.7 By Corollary 7.2.2 and suitable rotation of triangles, $\mathrm{holim}(X)$ and $\mathrm{hocolim}(X)$ are seen to be unique up to non-canonical isomorphism. To restore full canonicity, one needs the ∞-categorical language.

Theorem 8.5.8 If $\mathcal{A}$ has exact countable products and enough injectives, the right derived functor of $\lim : \mathsf{InvSys}(\mathcal{A}) \to \mathcal{A}$ exists as

$$\mathrm{R}\lim : \mathsf{D}^+(\mathsf{InvSys}(\mathcal{A})) \to \mathsf{D}^+(\mathcal{A}), \tag{8.5.8}$$

and for every system X there is a canonical exact triangle in $\mathsf{D}^+(\mathcal{A})$

$$\mathrm{R}\lim(X) \to \prod_k X_k \xrightarrow{\Delta_X} \prod_k X_k \xrightarrow{+1} . \tag{8.5.9}$$

In particular $\mathrm{R}\lim(X) \simeq \mathrm{holim}(X)$, and $\mathrm{R}\lim$ has dimension ≤ 1, with $\mathrm{H}^0(\mathrm{R}\lim(X)) \simeq \lim_k X_k$ and $\mathrm{H}^1(\mathrm{R}\lim(X)) \simeq \lim^1_k X_k$ (Definition 4.5.3), thus recovering Theorem 4.5.4.

Proof (*Sketch*) The injective systems $R = \prod_{m\geq 0} \mathcal{R}_m(I_m)$ of the proof of Theorem 4.5.4 have injective components and epimorphic difference maps, and every system embeds into one of them. By Theorem 4.1.4, each $X \in \mathrm{Ob}(\mathsf{C}^+(\mathsf{InvSys}(\mathcal{A})))$ thus admits a quasi-isomorphism $X \to Y$ into a complex of such systems, with $\mathrm{R}\lim(X) = \lim Y$; in particular $X_k \to Y_k$ is a quasi-isomorphism in $\mathrm{Ob}(\mathsf{C}^+(\mathcal{A}))$, for all $k \geq 0$.

Since each degree Y^n has an epimorphic difference map, we have a degreewise short exact sequence of complexes

$$0 \to \lim Y \to \prod_k Y_k \xrightarrow{\Delta_Y} \prod_k Y_k \to 0, \tag{8.5.10}$$

whose triangle exhibits $\mathrm{R}\lim(X) \simeq \mathrm{Cone}(\Delta_Y)[-1]$. Since each $X_k \to Y_k$ is a quasi-isomorphism, so is $\prod_k X_k \to \prod_k Y_k$ (countable products are exact); Δ being natural, Y may be replaced by X, giving the required triangle and $\mathrm{R}\lim(X) \simeq \mathrm{holim}(X)$.

For a system X of objects, $\mathrm{Cone}(\Delta_X)[-1]$ lives in degrees $[0, 1]$, whence the dimension bound; applying H to the triangle then yields $\mathrm{H}^0 \simeq \lim_k X_k$ and $\mathrm{H}^1 \simeq \lim^1_k X_k$ (Definition 4.5.3). See [10, Theorem 4.10.4]. □

This recovers Theorem 4.5.4 in derived-categorical dress: $\lim^1$ is the sole higher derived functor of lim because $\mathrm{R}\lim$ has dimension ≤ 1.

One can also deduce a version of Theorem 8.5.8 for unbounded derived categories; the key is to use the bound on the dimension of $\mathrm{R}\lim$. See [10, Example 4.11.11].

8.6 K-flat complexes and the derived tensor product

Fix a commutative ring $\Bbbk$. When we talk about an (A, B)-bimodule, where A, B are $\Bbbk$-algebras, the $\Bbbk$-actions through A and B are always assumed to coincide. Hence (A, R)-Mod is nothing but $A \otimes_{\Bbbk} R^{\mathrm{op}}$-$\mathsf{Mod}$.

Let A, R, B be $\Bbbk$-algebras. The tensor product between (A, R)-bimodules and (R, B)-bimodules is right exact in each variable; its left derived bifunctor is the derived tensor product we seek.

For those who only consider tensor products of R-modules over a commutative ring R, take $\Bbbk = A = B = R$ in what follows; all the annoying flatness conditions below will then disappear. See also Example 8.6.6.

Definition 8.6.1 The ***derived tensor product*** is the left derived bifunctor (Definition 8.4.1)

$$\overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} : \mathsf{D}((A, R)\text{-}\mathsf{Mod}) \times \mathsf{D}((R, B)\text{-}\mathsf{Mod}) \to \mathsf{D}((A, B)\text{-}\mathsf{Mod}). \tag{8.6.1}$$

We also write $X \overset{\mathrm{L}}{\otimes} Y$ when R is understood.

To compute it one needs a class of adapted resolutions: K-projective resolutions suffice, but the intrinsic notion is K-flatness.

Definition 8.6.2 (*Spaltenstein*) Let R be a ring.

- A complex X of right R-modules is ***K-flat*** (over R) if Y acyclic implies $X \otimes_R Y$ acyclic.
- A ***K-flat resolution*** of a complex X is a quasi-isomorphism $P \to X$ with P being K-flat over R.

- Suppose A and R are $\Bbbk$-algebras. We say (A,R)-Mod has ***enough K-flats*** (over R) if every complex of (A,R)-bimodules has a resolution that is K-flat over R.

Proposition 8.6.3 Fix a ring R.

(i) A bounded-above complex of flat modules is K-flat.

(ii) Suppose A, R, B are $\Bbbk$-algebras with A (resp. B) flat over $\Bbbk$. Then every K-projective complex of (A,R)-bimodules (resp. (R,B)-bimodules) is K-flat over R.

(iii) For any ring R, Mod-R and R-Mod have enough K-flats; more generally (A,R)-Mod has enough K-flats when A is flat over $\Bbbk$, and symmetrically (R,B)-Mod when B is.

Proof (*Sketch*) (i) Given a complex X of flat right R-modules, let Y be an acyclic complex of left R-modules. If Y is bounded above, then $X^\bullet \otimes_R Y^\bullet$ has finite support on each total degree and acyclic columns, hence $X \otimes_R Y$ is acyclic by the acyclic assembly lemma (Theorem 3.8.6).

In general, write $Y = \operatorname{colim}_m \tau^{\leq m} Y$ (Definition 3.7.1), a filtered colimit of bounded-above acyclic complexes: tensor products and direct sums commute with filtered colimits (Theorem B.4.3), and filtered colimits are exact (Proposition B.2.2), so the formation of $\operatorname{tot}^{\oplus}(X^\bullet \otimes_R \cdot)$ commutes with filtered colimits, and so does taking cohomology; hence $X \otimes_R Y$ is a filtered colimit of acyclic complexes, and therefore acyclic. In particular, K-flatness passes to filtered colimits.

(ii) Let X be a K-projective complex of (A,R)-bimodules with A flat over $\Bbbk$. For an injective left A-module I, the complex-level tensor–Hom adjunction [10, Proposition 4.12.6] gives

$$\operatorname{Hom}^\bullet(X \otimes_R Y, I) \simeq \operatorname{Hom}^\bullet(X, \operatorname{Hom}^\bullet(Y_\Bbbk, I_\Bbbk)). \tag{8.6.2}$$

Suppose Y is acyclic. The forgetful functor A-Mod $\to$ $\Bbbk$-Mod has the exact left adjoint $A \otimes_\Bbbk (\cdot)$, so $I_\Bbbk$ is injective and $\operatorname{Hom}^\bullet(Y_\Bbbk, I_\Bbbk)$ is acyclic, $Y_\Bbbk$ being acyclic as well; by K-projectivity, $\operatorname{Hom}^\bullet(X \otimes_R Y, I)$ is acyclic. As $\operatorname{Hom}(\cdot, I)$ is exact,

$$\mathrm{H}^{-n} \operatorname{Hom}^\bullet(X \otimes_R Y, I) \simeq \operatorname{Hom}(\mathrm{H}^{-n}(X \otimes_R Y), I) = 0. \tag{8.6.3}$$

Since I is arbitrary, and every left A-module embeds into an injective one, $X \otimes_R Y$ is acyclic ([10, Lemma 4.12.7]). The case of B flat over $\Bbbk$ is dual.

(iii) (A,R)-Mod $= A \otimes_\Bbbk R^{\mathrm{op}}$-Mod has enough K-projectives (Example 4.6.6), which are K-flat over R by (ii). □

Theorem 8.6.4 The derived tensor product between (A,R)-bimodules and (R,B)-bimodules exists when A or B is flat over $\Bbbk$, and is computed by K-flat resolutions: if $P \to X$ is a K-flat resolution then $X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y \simeq P \otimes_R Y$ (and symmetrically in Y). Moreover:

(i) $\mathrm{H}^{-n}\left(X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y\right) \simeq \operatorname{Tor}_n^R(X,Y)$ (Proposition 4.4.4, Remark 4.4.5), for X an (A,R)-bimodule and Y an (R,B)-bimodule, viewed as complexes concentrated in degree 0;

(ii) if S is another $\Bbbk$-algebra and A and B are both flat over $\Bbbk$, there is a canonical associativity constraint $\left(X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y\right) \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z \simeq X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} \left(Y \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z\right)$ for complexes of (A,R)-, (R,S)-, resp. (S,B)-bimodules;

(iii) for a map $R \to S$ of $\Bbbk$-algebras, the derived extension-of-scalars

$$\mathrm{L}({}_{R\to S}P) = S \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} (\cdot) \tag{8.6.4}$$

is left adjoint to restriction, and is transitive:

$$\mathrm{L}({}_{S\to T}P) \circ \mathrm{L}({}_{R\to S}P) \simeq \mathrm{L}({}_{R\to T}P). \tag{8.6.5}$$

Proof (*Sketch*) Existence and the computation by K-flat resolutions follow the pattern of Proposition 8.2.2, with the saturated triangulated subcategory $\mathcal{F} \subset \mathsf{K}((A,R)\text{-}\mathsf{Mod})$ of complexes K-flat over R in place of $\mathcal{I}$. The acyclicity condition of Definition 8.2.1, transplanted to bifunctors, splits into two requirements:

(a) $P \in \mathrm{Ob}(\mathcal{F})$ and Y acyclic imply $P \otimes_R Y$ acyclic, which is the very definition of K-flatness;

(b) $X \in \mathrm{Ob}(\mathcal{F})$ acyclic and Y arbitrary imply $X \otimes_R Y$ acyclic. Indeed, upon forgetting B, place Y in a triangle $Q \to Y \to N \xrightarrow{+1}$ with $Q \to Y$ a K-flat resolution and N acyclic (Proposition 8.6.3 (iii)), and apply two-out-of-three (Proposition 7.2.1) to the tensored triangle: $X \otimes_R Q$ is acyclic by K-flatness of Q (applied to the acyclic X), and so is $X \otimes_R N$ by that of X.

When A is flat over $\Bbbk$, the resolution condition is Proposition 8.6.3 (iii), so $Q \cdot \mathsf{K}_{\oplus}(\cdot \otimes_R \cdot)$ descends exactly as in Proposition 8.2.2, whence existence and the computation by K-flat resolutions of either variable; the case of B flat is symmetric. See [10, Proposition 4.12.9].

(i) Reduce to one-sided modules via the forgetful 2-cell of Exercise 8.6.7, then resolve X (or Y) by a bounded-above complex of flat R-modules: it is a K-flat resolution by Proposition 8.6.3 (i), and its H^{-n} is $\mathrm{Tor}_n^R(X,Y)$ by Proposition 4.4.4.

(ii) Fix Y and resolve X (resp. Z) by a complex K-flat over R (resp. S), possible by Proposition 8.6.3 (iii) since A and B are flat over $\Bbbk$; all three derived tensor products are then computed by ordinary tensor products of complexes, and the constraint is the classical associativity isomorphism ([9, Proposition 6.5.12]).

(iii) The adjunction is the derived adjunction Theorem 8.5.4 in its unbounded form, applied to extension-of-scalars $\dashv$ restriction. As for transitivity, the canonical morphism is furnished by the general composite theorem for triangulated functors ([10, Theorem 4.6.5]); it is an isomorphism because extension-of-scalars preserves K-flat complexes: if $P \to Y$ is a K-flat resolution over R, then both sides of the canonical morphism are computed by $T \otimes_R P$, using the associativity $W \otimes_S (S \otimes_R P) \simeq W \otimes_R P$ valid for every complex of right S-modules W. □

For a complex X of (A,R)-bimodules, we write ${}_AX$ (resp. X_R) for the underlying complex of left A-modules (resp. right R-modules). These forgetful functors are exact, hence descend to the derived categories.

Theorem 8.6.5 (*derived tensor-Hom adjunction*) Let A, R, B be $\Bbbk$-algebras. In what follows, X, Y, Z are complexes of (A,R)-, (R,B)-, resp. (A,B)-bimodules.

(i) With A flat over $\Bbbk$ there is a canonical isomorphism

$$\mathrm{RHom}_{(A,B)}\left(X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y, Z\right) \simeq \mathrm{RHom}_{(A,R)}(X, \mathrm{RHom}_B(Y_B, Z_B)); \tag{8.6.6}$$

(ii) when B is flat over $\Bbbk$, there is a canonical isomorphism

$$\mathrm{RHom}_{(A,B)}\left(X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y, Z\right) \simeq \mathrm{RHom}_{(R,B)}(Y, \mathrm{RHom}_A({}_AX, {}_AZ)). \tag{8.6.7}$$

Proof (*Sketch*) We spell out the case (i). Take X K-projective and Z K-injective. Then X_R is K-flat by Proposition 8.6.3 (ii), so $X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y \simeq X \otimes_R Y$, and the left-hand side is computed by $\mathrm{Hom}^\bullet(X \otimes_R Y, Z)$; on the right-hand side, the forgetful functor $(A, B)\text{-}\mathsf{Mod} \to \mathsf{Mod}\text{-}B$ has the exact left adjoint $A \otimes_{\Bbbk} (\cdot)$, hence preserves K-injectives, so $\mathrm{RHom}_B(Y_B, Z_B)$ is computed by the complex of (A, R)-bimodules $\mathrm{Hom}^\bullet(Y_B, Z_B)$, after which K-projectivity of X computes the outer $\mathrm{RHom}_{(A,R)}$ as well, by Remark 8.5.5. The isomorphism is now the complex-level adjunction $\mathrm{Hom}^\bullet(X \otimes_R Y, Z) \simeq \mathrm{Hom}^\bullet(X, \mathrm{Hom}^\bullet(Y_B, Z_B))$, cf. Equation (8.6.2).

The case of (ii) is symmetric, with Y resolved by K-projectives and the forgetful functor $(A, B)\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ in place of the one above. □

Example 8.6.6 Let R be a commutative ring and take $\Bbbk = A = B = R$, so that the flatness hypotheses are automatic. Then $\overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}}$ makes $\mathsf{D}(R\text{-}\mathsf{Mod})$ a symmetric monoidal category (Definition A.6.7) with unit R (flat, hence K-flat): the associativity constraint is that of Theorem 8.6.4 (ii), the commutativity constraint comes from swapping the tensor factors at the complex level with suitable signs, and the other axioms are verified on K-flat representatives. In the same spirit, for a commutative R-algebra X the pairings of Proposition 8.4.3 with $F = \otimes_R$, followed by the multiplication $X \otimes_R X \to X$, make $\bigoplus_n \mathrm{Tor}_n^R(X, X)$ a graded-commutative algebra, the ***Tor-algebra*** :

$$ab = (-1)^{pq} ba, \quad a \in \mathrm{Tor}_p^R(X, X), \quad b \in \mathrm{Tor}_q^R(X, X). \tag{8.6.8}$$

See [10, Examples 4.12.14 – 4.12.15].

Exercise 8.6.7 Let A, B, R be $\Bbbk$-algebras with B flat over $\Bbbk$. Give the canonical 2-cell expressing the compatibility of $\overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}}$ with the forgetful functors, and deduce from it the isomorphism $\mathrm{H}^{-n}\left(X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y\right) \simeq \mathrm{Tor}_n^R(X, Y)$ of Theorem 8.6.4 directly from the classical Tor. *Hint*. Verify the 2-cell on a K-flat representative $P \simeq X$, where both routes are computed by one and the same complex-level tensor product (Proposition 8.6.3 (iii), Theorem 8.6.4); then resolve X by a bounded-above complex of flat modules, which computes both $X \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} Y$ and the classical $\mathrm{Tor}_n^R(X, Y)$ (Proposition 4.4.4).

8.7 Outlook

The shortcomings of $\mathsf{D}(\mathcal{A})$ recorded in the introduction — exact triangles unique only up to non-canonical isomorphism, the lack of kernels and cokernels, and the absence of a clean effaceability characterization of derived functors — all stem from a single source: constructing $\mathsf{D}(\mathcal{A})$ crudely erases the homotopy relation and the higher coherence relations. The remedy is to remember them. An ∞-category is, roughly, a category in which composition is defined and associative only up to coherent homotopy. Combinatorially, these are modeled by simplicial sets: a category is the same as a simplicial set whose inner horns admit *unique* fillers

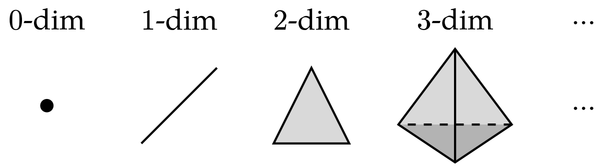
(Proposition 9.3.2), an ∞-category is one where fillers merely exist, and an ∞-groupoid is a Kan complex. The simplicial machinery: horns, nerves, Kan complexes, geometric realization, will be collected in Chapter 9.

A ***stable ∞-category*** is, roughly, a pointed ∞-category with finite (co)limits in which a square is a pushout if and only if it is a pullback: informally, a triangulated category that "remembers its homotopy coherence". Its homotopy category is canonically triangulated, and $\mathsf{D}(\mathcal{A})$ is the prototypical case: it is the homotopy category of the stable ∞-category of complexes over $\mathcal{A}$. In that richer setting the shortcomings above dissolve: mapping cones and homotopy limits such as holim (Definition 8.5.6) become canonical, finite limits and colimits exist, and the total derived functors between stable ∞-categories are governed by the same Kan extensions. Stable ∞-categories are the working language of *derived algebraic geometry* and higher algebra, where the set-theoretic and non-functorial phenomena of triangulated categories disappear; see [12].

Chapter 9

Simplicial Methods (TO REVISE)

Simplicial methods were introduced into algebraic topology by S. Eilenberg and J. A. Zilber around 1950 as a rigorous combinatorial language for decomposing a space into points, segments, triangles, tetrahedra, and their higher-dimensional kin:



The formal object is a *simplicial set*, a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$, where Δ is the category of non-empty finite ordinals. Through the ***geometric realization*** functor, the simplicial sets model topological spaces (Section 9.4), and through the ***nerve*** construction they encode categories (Section 9.3). That objects, arrows, and commutativity of a category correspond respectively to 0-, 1-, and 2-dimensional data is no accident: it is the first hint that category and space can be unified under one combinatorial umbrella.

The bridge to homological algebra proper is the ***Dold–Kan correspondence*** (Section 9.5), which for an abelian category $\mathcal{A}$ gives an equivalence of categories

$$\Gamma : \mathsf{Ch}_{\geq 0}(\mathcal{A}) \simeq \mathsf{s}\mathcal{A} \tag{9.0.1}$$

between connective chain complexes and simplicial objects. Under this equivalence the homotopy theory of chain complexes acquires a purely combinatorial, topologically flavoured counterpart (Section 9.6). Finally, the ***bar construction*** (Section 9.7) manufactures canonical (co)simplicial resolutions from monads and comonads, explaining at a single stroke the standard complexes underlying *Hochschild (co)homology* (Section 3.5) and *group cohomology* (Subsection 6.6.3). We follow [10, Chapter 8]; the heavier topological details are deferred to [5] and [3].

9.1 Simplicial and cosimplicial objects

We begin with the indexing category. The finite ordinals form a category under order-preserving maps; we need only the non-empty ones.

Definition 9.1.1 (*the simplex category*) The ***simplex category*** Δ has as objects the totally ordered sets $[n] := \{0, 1, \cdots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$, and as morphisms the order-preserving (i.e. weakly monotone) maps $[m] \to [n]$.

The notation $[n]$ is not to be confused with $\boldsymbol{n} = \{0, \cdots, n-1\}$. Every morphism $[l] \to [n]$ in Δ admits a unique factorization as an order-preserving surjection followed by an order-preserving injection,

$$[l] \overset{\text{epi}}{\twoheadrightarrow} [m] \overset{\text{mono}}{\hookrightarrow} [n], \quad l \geq m \leq n, \tag{9.1.1}$$

and each injection (resp. surjection) further decomposes into elementary pieces omitting one element (respectively identifying two) at a time. Thus every morphism is a composite of the following two families of generators, the *cofaces* and *codegeneracies*:

coface	map	description
$d^i = d^{n,i}$	$[n-1] \hookrightarrow [n]$	order-preserving injection omitting $i \in [n]$ $(0 \leq i \leq n)$
$s^j = s^{n,j}$	$[n+1] \twoheadrightarrow [n]$	order-preserving surjection taking j twice $(0 \leq j \leq n)$

A direct verification gives the ***cosimplicial identities*** , which present Δ by generators and relations:

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = \mathrm{id} = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases} \tag{9.1.2}$$

Passing to the opposite category Δ^{op}, the cofaces and codegeneracies become the *faces* $d_i = d_i^n : [n] \to [n-1]$ and *degeneracies* $s_j = s_j^n : [n] \to [n+1]$ (morphisms in Δ^{op}). Reversing Equation (9.1.2) yields the ***simplicial identities*** :

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = \mathrm{id} = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases} \tag{9.1.3}$$

Let FinOrd be the category of finite ordinals. It contains Δ as a full subcategory, with exactly one extra object: the empty set.

Definition 9.1.2 Given a category $\mathcal{C}$:

- A ***simplicial object*** in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$; these form the category $\mathsf{s}\mathcal{C} := \mathcal{C}^{\Delta^{\mathrm{op}}}$.
- A ***cosimplicial object*** in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$; these form $\mathsf{cs}\ \mathcal{C} := \mathcal{C}^{\Delta}$.
- A simplicial (or cosimplicial) object admitting an extension to $\mathsf{FinOrd}^{\mathrm{op}}$ (resp. FinOrd) is called ***augmented*** .

Unwinding the definition, a simplicial object X amounts to a family of objects $(X_n)_{n \geq 0}$ of $\mathcal{C}$ together with ***face maps*** $d_i : X_n \to X_{n-1}$ and ***degeneracy maps*** $s_j : X_n \to X_{n+1}$ $(0 \leq i, j \leq n)$ satisfying Equation (9.1.3); we call X_n the *n-th term*. A morphism $X \to Y$ of simplicial objects is a family $f_n : X_n \to Y_n$ commuting with every d_i and s_j.

Dually, a cosimplicial object is a family $(X^n)_{n \geq 0}$ with *cofaces* $d^i : X^{n-1} \to X^n$ and *codegeneracies* $s^j : X^{n+1} \to X^n$ obeying Equation (9.1.2).

We shall write $\varphi^* : X_n \to X_m$ for the map induced by a morphism $\varphi : [m] \to [n]$ in Δ (and $\varphi_* : X^m \to X^n$ in the cosimplicial case).

Remark 9.1.3 The subcategory $\Delta_+ \subset \Delta$ with the same objects and only the order-preserving *injections* indexes ***semi-simplicial objects*** , namely functors $\Delta_+^{\mathrm{op}} \to \mathcal{C}$: these are simplicial objects with the degeneracy maps s_j (and the identities involving them) discarded. Every simplicial object determines a semi-simplicial object by forgetting. Augmented semi-simplicial objects are defined analogously. The cosimplicial case is entirely dual.

Example 9.1.4 For $C \in \mathcal{C}$ the ***constant simplicial object*** $\mathrm{const}(C)$ has $\mathrm{const}(C)_n = C$ and $s_j = \mathrm{id}_C = d_i$ for all n, i, j. A mild exercise verifies that augmentations of X are in bijection with pairs (C, φ) where $\varphi : X \to \mathrm{const}(C)$ is a morphism of simplicial objects: given an augmentation $X_0 \to X_{-1}$, take $C = X_{-1}$ and φ_n the composite $X_n \to X_0 \to X_{-1}$, where the map $X_n \to X_0$ may be any composite of face maps.

Definition 9.1.5 (*monoidal structure*) Any functor $F : \mathcal{C} \to \mathcal{D}$ acts termwise on simplicial objects, $(X_n, d_i, s_j) \mapsto (F(X_n), F(d_i), F(s_j))$. Applying this to the bifunctor $\otimes$ of a monoidal category $\mathcal{C}$ (Section A.6) defines $X \otimes Y \in \mathsf{s}\mathcal{C}$ by $(X \otimes Y)_n = X_n \otimes Y_n$ with faces and degeneracies $d_i \otimes d_i$ and $s_j \otimes s_j$. Thus $\mathsf{s}\mathcal{C}$ inherits a monoidal structure with unit $\mathrm{const}(\mathbf{1})$.

Exercise 9.1.6 Verify the cosimplicial identities Equation (9.1.2) directly from the descriptions of d^i and s^j, and check that they present Δ by generators and relations.

9.2 Simplicial sets

The case $\mathcal{C} = \mathsf{Set}$ is of primary importance.

Definition 9.2.1 A ***simplicial set*** is a simplicial object in Set; a cosimplicial object in Set is a ***cosimplicial set*** . The category of simplicial sets is $\mathsf{sSet} := \mathsf{Set}^{\Delta^{\mathrm{op}}}$. sSet

Definition 9.2.2 For each $n \in \mathbb{Z}_{\geq 0}$, denote by Δ^n the ***standard*** n***-simplex*** : it is the simplicial set

$$\Delta^n := \mathrm{Hom}_\Delta(\cdot, [n]) : \Delta^{\mathrm{op}} \to \mathsf{Set}, \tag{9.2.1}$$

i.e. the functor represented by $[n]$, whose m-simplices are $(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n])$, the order-preserving maps $[m] \to [n]$.

The Yoneda lemma (Theorem 1.4.1) gives the fundamental identification, for every simplicial set X,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X) &\simeq X_n \\ \varphi &\mapsto \varphi([n])\big(\mathrm{id}_{[n]}\big), \end{aligned} \tag{9.2.2}$$

where $\varphi([n]) : \mathrm{End}([n]) = \Delta^{n([n])} \to X([n]) = X_n$. Moreover $[n] \mapsto \Delta^n$ is a fully faithful embedding $\Delta \to \mathsf{sSet}$: an order-preserving $f : [m] \to [n]$ induces $f : \Delta^m \to \Delta^n$, and specifying a subset $\{k_0 < \cdots < k_m\} \subset [n]$ with $m+1$ elements is the same as specifying an injection $[m] \hookrightarrow [n]$, hence a map $\Delta^m \to \Delta^n$.

Example 9.2.3 Two subobjects of the standard simplex are ubiquitous:

- The ***boundary*** $\partial\Delta^n$ has $(\partial\Delta^n)_m$ the order-preserving $f : [m] \to [n]$ whose image is not all of $[n]$.

- For $0 \le k \le n$, the ***horn*** Λ^n_k has $(\Lambda^n_k)_m$ the order-preserving $f : [m] \to [n]$ whose image does not contain $[n] \setminus \{k\}$; we call k its *vertex*.

Clearly $\Delta^n \supset \partial\Delta^n \supset \Lambda^n_k$, and $\partial\Delta^0 = \varnothing$.

Definition 9.2.4 An element of X_n is said to be an ***n-simplex*** of the simplicial set X. It is said to be ***degenerate*** if it lies in the image of some s_j, and ***non-degenerate*** otherwise. Denote the subset of non-degenerate n-simplices as X_n^{nd}.

For instance the non-degenerate m-simplices of Δ^n are the injections $[m] \hookrightarrow [n]$, of which there are $\binom{n+1}{m+1}$. Degenerate simplices will be discarded when we pass to the normalized chain complex (Section 9.5).

Remark 9.2.5 By the density of the Yoneda embedding (Theorem A.4.1) applied to $\mathsf{sSet} = \Delta^\wedge$, for every $X \in \mathrm{Ob}(\mathsf{sSet})$ there is a canonical isomorphism

$$\operatorname*{colim}_{n,x} \Delta^n \simeq X, \tag{9.2.3}$$

indexed by the comma category (h_Δ/X), with objects the pairs (n, x), $x \in X_n$ (i.e. the morphisms $\Delta^n \to X$), and with morphisms $(n, x) \to (m, y)$ the $f : [n] \to [m]$ in Δ such that $f^*(y) = x$. Here the colimit in $\mathsf{sSet} = \Delta^\wedge$ is computed degreewise.

Exercise 9.2.6 Show that the non-degenerate m-simplices of Δ^n are exactly the order-preserving injections $[m] \hookrightarrow [n]$, and count them. Prove that every $x \in X_n$ decomposes uniquely as $x = \varphi^*(y)$ with y non-degenerate and $\varphi : [n] \twoheadrightarrow [m]$ a surjection. *Hint.* An m-simplex $f : [m] \to [n]$ of Δ^n is degenerate iff it factors through some s^j, i.e. iff f is not injective; count the injections by their images. For a general X, peel degeneracies off x one at a time and settle the choices with the simplicial identities Equation (9.1.3), by induction on n.

Exercise 9.2.7 Write Λ^n_i (resp. $\partial\Delta^n$) explicitly as a colimit of n (resp. $n+1$) copies of Δ^{n-1}. *Hint.* Glue the face inclusions $d^j : \Delta^{n-1} \to \Delta^n$ along their pairwise intersections, which are (possibly empty) copies of Δ^{n-2}, omitting the face $j = i$ for the horn.

9.3 The nerve of a category

Simplicial sets need not come from topology. The *nerve* construction turns any small category into a simplicial set and is one of the cornerstones of ∞-*category* theory.

Let Cat be the category of small categories and functors between them.

Definition 9.3.1 For a small category $\mathcal{C}$, the ***nerve*** $\mathrm{N}\,\mathcal{C}$ is the simplicial set whose n-simplices are functors $[n] \to \mathcal{C}$, i.e. composable chains of morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n \tag{9.3.1}$$

in $\mathcal{C}$. Thus $(\mathrm{N}\,\mathcal{C})_0 = \mathrm{Ob}(\mathcal{C})$ and $(\mathrm{N}\,\mathcal{C})_1 = \mathrm{Mor}(\mathcal{C})$. The face maps drop an end morphism or compose adjacent ones, and the degeneracy maps insert identities:

$$d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n) & \text{if } i = 0 \\ (f_1, \cdots, f_{i+1}f_i, \cdots, f_n) & \text{if } 0 < i < n \\ (f_1, \cdots, f_{n-1}) & \text{if } i = n \end{cases} \tag{9.3.2}$$

$$s_j(f_1, \cdots, f_n) = \begin{cases} \left(\mathrm{id}_{C_0}, f_1, \cdots, f_n\right) & \text{if } j = 0 \\ \left(f_1, \cdots, f_j, \mathrm{id}_{C_j}, f_{j+1}, \cdots, f_n\right) & \text{if } 0 < j < n \\ \left(f_1, \cdots, f_n, \mathrm{id}_{C_n}\right) & \text{if } j = n \end{cases} \tag{9.3.3}$$

Every functor $\mathcal{C} \to \mathcal{C}'$ induces $\mathrm{N}\,\mathcal{C} \to \mathrm{N}\,\mathcal{C}'$, giving the ***nerve functor*** $\mathrm{N} : \mathsf{Cat} \to \mathsf{sSet}$. $\mathrm{N}\,\mathcal{C}$

The nerve $\mathrm{N}\,\mathcal{C}$ is no mere shadow of $\mathcal{C}$: it records $\mathcal{C}$ in full. Which simplicial sets arise this way is answered by a beautiful characterization.

Proposition 9.3.2 (*characterization of nerves*) For a simplicial set X the following are equivalent:
(i) $X \simeq \mathrm{N}\,\mathcal{C}$ for some small category $\mathcal{C}$.
(ii) X has the *unique inner horn filling* property: for every $0 < i < n$ and every $\sigma' : \Lambda_i^n \to X$ (an ***inner horn***), there is a unique $\sigma : \Delta^n \to X$ extending σ'.

Proof (*Sketch*) The details can be found in [10, Proposition 8.3.3]. Below we only sketch the reconstruction of $\mathcal{C}$ from $X := \mathrm{N}\,\mathcal{C}$. The objects are exactly the elements of X_0. The morphisms $\mathrm{Hom}_{\mathcal{C}}(C, C')$ are the $f \in X_1$ with $d_1 f = C$, $d_0 f = C'$; define composition via the unique filler of the inner 2-horn

$$\begin{tikzcd} 0 \arrow[rr, dash, "gf"] \arrow[dr, dash, "f"'] & & 2 \\ & 1 \arrow[ur, dash, "g"'] & \end{tikzcd} \tag{9.3.4}$$

and associativity via unique filling of an inner 3-horn. There is then a canonical morphism $X \to \mathrm{N}\,\mathcal{C}$; the proof that it is an isomorphism can be found in *loc. cit.* □

Proposition 9.3.3 The nerve functor $\mathrm{N} : \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful.

Proof In the proof of Proposition 9.3.2, we have seen how the morphisms and their compositions in $\mathcal{C}$ can be reconstructed from $\mathrm{N}\,\mathcal{C}$. This implies that any morphism $\mathrm{N}\,\mathcal{C} \to \mathrm{N}\,\mathcal{C}'$ induces a functor $\mathcal{C} \to \mathcal{C}'$. It is routine to see that this is inverse to the map from functors $\mathcal{C} \to \mathcal{C}'$ to morphisms $\mathrm{N}\,\mathcal{C} \to \mathrm{N}\,\mathcal{C}'$. □

Therefore, Cat identifies with the full subcategory of sSet on the simplicial sets satisfying unique inner horn filling; relaxing "unique" to mere "existence" of fillers for all horns is precisely the definition of a ***Kan complex*** , the combinatorial model for an ∞-groupoid.

Example 9.3.4 For a monoid Γ, define categories $\mathcal{B}\Gamma$ (one object $\star$, endomorphisms Γ) and $\mathcal{E}\Gamma$ (objects Γ, $\mathrm{Hom}_{\mathcal{E}}\,\Gamma(g, g') = \{h \in \Gamma : hg = g'\}$). We obtain the simplicial sets

$$\mathrm{B}\,\Gamma := \mathrm{N}((\mathcal{B}\Gamma)^{\mathrm{op}}), \quad \mathrm{E}\,\Gamma := \mathrm{N}((\mathcal{E}\Gamma)^{\mathrm{op}}), \tag{9.3.5}$$

so that $(\mathrm{B}\,\Gamma)_n = \Gamma^n$ and $(\mathrm{E}\,\Gamma)_n = \Gamma^{n+1}$. We call $\mathrm{B}\,\Gamma$ the ***classifying space*** $\mathrm{B}\,\Gamma$ of Γ; levelwise, the natural projection $\mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma$ is the quotient by the right Γ-action $(g_1, \cdots, g_{n+1}) \cdot g = (g_1, \cdots, g_n, g_{n+1}g)$. When Γ is a group this action is free; from the topological point of view (see Section 9.4), $\mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma$ gives rise to the universal Γ-torsor.

Exercise 9.3.5 For a small category $\mathcal{C}$, show directly that $\mathrm{N}\,\mathcal{C}$ satisfies unique inner horn filling (Proposition 9.3.2) by translating composition and associativity in $\mathcal{C}$ into filler data. Where does uniqueness fail for an arbitrary simplicial set?

Exercise 9.3.6 Check that the nerve functor $\mathrm{N} : \mathsf{Cat} \to \mathsf{sSet}$ admits a left adjoint τ_1: the objects of $\tau_1(X)$ are the elements of X_0, and the morphisms are the words in X_1 (the free category on the graph with objects X_0 and arrows X_1, source d_1 and target d_0), modulo the congruence generated by

$$s_0(x) \sim \mathrm{id}_x, \quad d_1(y) \sim d_0(y) \circ d_2(y) \quad \text{for } x \in X_0, y \in X_2; \tag{9.3.6}$$

cf. Equation (9.3.4). This is called the ***fundamental category*** attached to X.

9.4 Geometric realization and the singular complex

We now give simplicial sets their topological meaning. Let $e_0, \cdots, e_n$ be the standard ordered basis of $\mathbb{R}^{n+1}$ and define the *standard topological n-simplex*

$$|\Delta^n| := \left\{ (x_0, \cdots, x_n) \in (\mathbb{R}_{\geq 0})^{n+1} : \sum_{i=0}^{n} x_i = 1 \right\}, \tag{9.4.1}$$

with vertices labelled by $i \mapsto e_i$ and the orientation in which $e_1 - e_0, \cdots, e_n - e_{n-1}$ is a positively oriented basis. Every order-preserving $f : [m] \to [n]$ induces a linear map $|f| : |\Delta^m| \to |\Delta^n|$, $\sum_i y_i e_i \mapsto \sum_i y_i e_{f(i)}$, giving a functor $|\cdot| : \Delta \to \mathsf{Top}$.

Recall that Top is cocomplete, with colimits given by glueing topological spaces.

Definition 9.4.1 (*geometric realization*) The ***geometric realization*** $|X|$ of a simplicial set X is the topological space

$$|X| := \operatorname*{colim}_{n,x} |\Delta^n|, \tag{9.4.2}$$

where the indexing category (that of Remark 9.2.5) has objects (n, x) with $x \in X_n$ (equivalently $x : \Delta^n \to X$), and a morphism $(n, x) \to (m, y)$ is an $f : [n] \to [m]$ in Δ with $f^*(y) = x$. Equivalently $|X|$ is the quotient

$$|X| = \left(\bigsqcup_{n \geq 0} X_n \times |\Delta^n| \right) / \sim, \tag{9.4.3}$$

where $(f^*(y), t) \sim (y, |f|(t))$ for every morphism $f : [n] \to [m]$ in Δ with $y \in X_m$, $t \in |\Delta^n|$.

The colimit says: glue one copy of $|\Delta^n|$ for each n-simplex x of X, along the maps prescribed by the face and degeneracy operations. A face map d_i specifies the i-th face of the simplex labelled x; a degeneracy map s_j collapses a degenerate $(n+1)$-simplex onto its j-th face. Degenerate simplices may thus be omitted when gluing, though their presence remains essential since a face of a non-degenerate simplex may be degenerate. Taking $X = \Delta^m$ recovers the original $|\Delta^m|$, so the notation is consistent.

Example 9.4.2 To illustrate: upon realization, the $\partial\Delta^n$ of Example 9.2.3 describes the boundary of the topological simplex $|\Delta^n|$ in the intuitive sense, while $|\Lambda^n_k|$ is obtained from $|\partial\Delta^n|$ by deleting the face opposite the vertex k (cf. Exercise 9.2.7).

Definition 9.4.3 For a topological space E, the ***singular set*** $\mathrm{Sing}(E)$ is the simplicial set with n-simplices the singular maps

$$\mathrm{Sing}(E)_n := \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, E), \tag{9.4.4}$$

and face/degeneracy maps given by precomposition with $|d^i|$, $|s^j|$ respectively. This gives the *singular complex functor* $\mathrm{Sing} : \mathsf{Top} \to \mathsf{sSet}$.

Theorem 9.4.4 (*geometric realization* $\dashv$ *singular complex*) Geometric realization is left adjoint to the singular complex: for every simplicial set X and topological space E there is a natural bijection

$$\mathrm{Hom}_{\mathsf{Top}}(|X|, E) \simeq \mathrm{Hom}_{\mathsf{sSet}}(X, \mathrm{Sing}(E)). \tag{9.4.5}$$

Proof For each pair (n, x), write $i_{n,x} : |\Delta^n| \to |X|$ for the structure map into the colimit of Definition 9.4.1; the family $\left(i_{n,x}\right)_{n,x}$ is compatible, i.e. $i_{n,x} = i_{m,y}|f|$ whenever $f : (n, x) \to (m, y)$ is a morphism of the indexing category. Consequently, a continuous map $\theta : |X| \to E$ is the same thing as a family of continuous maps $\theta_{n,x} : |\Delta^n| \to E$, compatible in the same sense; explicitly $\theta_{n,x} = \theta i_{n,x}$.

Likewise, since $X \simeq \operatorname*{colim}_{n,x} \Delta^n$ (Remark 9.2.5), a morphism $\psi : X \to \mathrm{Sing}(E)$ is the same thing as a family of morphisms $\psi_{n,x} : \Delta^n \to \mathrm{Sing}(E)$ satisfying $\psi_{n,x} = \psi_{m,y} f$ for every morphism $f : (n, x) \to (m, y)$ of the indexing category, where $f : \Delta^n \to \Delta^m$ also denotes the morphism induced by the underlying $f : [n] \to [m]$.

By the Yoneda lemma (Theorem 1.4.1), a morphism $\Delta^n \to \mathrm{Sing}(E)$ is an element of $\mathrm{Sing}(E)_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, E)$, that is, a continuous map $|\Delta^n| \to E$; under this identification the two compatibility conditions agree, since the action of f^* on $\mathrm{Sing}(E)$ is precomposition with $|f|$.

Assembling, we obtain mutually inverse bijections: θ corresponds to the morphism ψ with components $\psi_n(x) := \theta i_{n,x}$, and conversely ψ determines the unique θ with $\theta i_{n,x} = \psi_n(x)$ for all (n, x). Naturality in X and E is clear. □

Note that sSet admits products: they are formed degreewise, as $\left(\prod_{i\in I} X_i\right)_n = \prod_{i\in I} (X_i)_n$, with faces and degeneracies defined componentwise; this is the monoidal structure of Definition 9.1.5 for $(\mathsf{Set}, \times)$.

This adjunction (a *Quillen adjunction* for the standard model structures) is the cornerstone of combinatorial homotopy theory. A second fundamental result, whose proof requires the compactly generated refinement of Top, is the following.

Theorem 9.4.5 (*realization of products*) In the category CGHaus of compactly generated Hausdorff spaces ([13, Chapter 5]; the gluing of Definition 9.4.1 may be performed there, since each $|\Delta^n|$ is compactly generated Hausdorff) there is a natural isomorphism $|X \times Y| \simeq |X| \times |Y|$ for all simplicial sets X, Y. If one of X, Y has only finitely many non-degenerate simplices, the isomorphism already holds in Top.

The proof, which we omit, is a non-trivial subdivision argument; see [3, Ch. III] or [5, Proposition I.2.4].

Guided by Theorem 9.4.5, we define a *simplicial homotopy* from g to f, where $f, g : X \to Y$, to be a morphism $H : X \times \Delta^1 \to Y$ with $H \circ (\mathrm{id}_X \times d^0) = f$ and $H \circ (\mathrm{id}_X \times d^1) = g$, i.e. the restrictions of H along the two endpoints $d^0, d^1 : \Delta^0 \to \Delta^1$, under the canonical identification $X \times \Delta^0 \simeq X$. For general Y this is not an equivalence relation; one restores good behaviour by requiring Y to be a Kan complex. We shall not pursue the homotopy theory of simplicial sets further, and return instead to the algebraic core.

9.5 The Dold–Kan correspondence

Henceforth $\mathcal{A}$ denotes an additive category. Following the topological convention we work with chain complexes $A = (A_n, \partial_n)_{n \in \mathbb{Z}}$ with

- $A_n \in \mathrm{Ob}(\mathcal{A})$, $A_n = 0$ for $n < 0$, and
- $\partial_n : A_n \to A_{n-1}$ with $\partial_{n-1}\partial_n = 0$ for all n.

These form the category $\mathsf{Ch}_{\geq 0}(\mathcal{A})$, also known as the category of *connective chain complexes* in topology.

We shall relate $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ to the category of simplicial objects $\mathsf{s}\mathcal{A}$ in $\mathcal{A}$ by three functors,

$$\Gamma : \mathsf{Ch}_{\geq 0}(\mathcal{A}) \to \mathsf{s}\mathcal{A}, \quad \mathrm{N}, \mathrm{C} : \mathsf{s}\mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A}), \tag{9.5.1}$$

of which N is defined only when $\mathcal{A}$ is abelian, while both N and C work already for semi-simplicial objects (Remark 9.1.3). Our treatment follows [10, §8.5].

Definition 9.5.1 For a semi-simplicial object X in $\mathcal{A}$, set

$$\partial_n := \sum_{i=0}^{n} (-1)^i d_i : X_n \to X_{n-1} \quad (n \geq 1), \quad \partial_0 := 0. \tag{9.5.2}$$

Then $(X_n, \partial_n)_{n\geq 0}$ is an object of $\mathsf{Ch}_{\geq 0}(\mathcal{A})$, called the ***unnormalized chain complex*** (or ***Moore complex***). We denote it by $\mathrm{C}\,X$, with $(\mathrm{C}\,X)_n = X_n$.

One must check $\partial_{n-1}\partial_n = 0$ for $n \geq 1$: the simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$ makes the terms cancel in pairs,

$$\partial_{n-1}\partial_n = \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_{j-1} d_i + \sum_{n-1 \geq i \geq j \geq 0} (-1)^{i+j} d_i d_j = 0. \tag{9.5.3}$$

The construction is functorial, giving a functor $\mathrm{C} : \mathsf{s}\mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$.

Definition 9.5.2 For $A = (A_n, \partial_n)_{n \geq 0} \in \mathsf{Ch}_{\geq 0}(\mathcal{A})$ define a simplicial object $\Gamma(A)$ by

$$(\Gamma A)_n := \bigoplus_{t : [n] \twoheadrightarrow [k]} A_k, \tag{9.5.4}$$

the direct sum over order-preserving surjections $t : [n] \twoheadrightarrow [k]$. For a morphism $f : [m] \to [n]$ in Δ, the induced

$$f^* = \left(\Phi_{u,t} : A_k \to A_l\right)_{\substack{u:[m]\twoheadrightarrow[l] \\ t:[n]\twoheadrightarrow[k]}} : (\Gamma A)_n \to (\Gamma A)_m \tag{9.5.5}$$

is specified as follows. Given $t : [n] \twoheadrightarrow [k]$, factor the composite $tf : [m] \to [n] \to [k]$ as $[m] \twoheadrightarrow [l] \overset{v}{\hookrightarrow} [k]$; set

- $\Phi_{u,t} = \mathrm{id}$ if $l = k$,
- $\Phi_{u,t} = \partial_k$ if $l = k - 1$ and $v = d^0$ (the injection omitting 0),
- $\Phi_{u,t} = 0$ otherwise.

If $u : [m] \twoheadrightarrow [l]$ does not arise from the epi-mono factorization, we set $\Phi_{u,t} = 0$. For fixed f and t, there exists at most one u such that $\Phi_{u,t} \neq 0$.

It is routine to verify that ΓA is indeed a simplicial object and that $A \mapsto \Gamma A$ is functorial. We omit the details.

Definition 9.5.3 Assume $\mathcal{A}$ abelian. For a semi-simplicial object X set

$$\begin{aligned}(\mathrm{N}\,X)_n &:= \bigcap_{i=1}^{n} \ker[d_i : X_n \to X_{n-1}] \quad (n \geq 1),\\ (\mathrm{N}\,X)_0 &:= X_0,\end{aligned} \tag{9.5.6}$$

with differential $\partial_n : (\mathrm{N}\,X)_n \to (\mathrm{N}\,X)_{n-1}$ induced by d_0. It is called the ***normalized chain complex*** $\mathrm{N}\,X$ of X.

Again, one readily verifies using Equation (9.1.3) that $\mathrm{N}\,X$ is an object of $\mathsf{Ch}_{\geq 0}(\mathcal{A})$, and the construction is functorial in X.

Proposition 9.5.4 The inclusions $u_n : (\mathrm{N}\,X)_n \hookrightarrow X_n$ assemble into a monomorphism $u : \mathrm{N}\,X \hookrightarrow \mathrm{C}\,X$ of chain complexes, natural in X.

Proof Immediate from the definitions. □

Although $\mathrm{N}\,X$ was defined as a subobject of $\mathrm{C}\,X$, it is equally a quotient: taking the quotient of X_n by all degenerate simplices (the image of $\Psi_n : \bigoplus_{i<n} X_{n-1} \to X_n$ induced by the s_j) reproduces $(\mathrm{N}\,X)_n$, as follows.

Proposition 9.5.5 Assume $\mathcal{A}$ abelian. The cokernel of Ψ_n is naturally isomorphic to $(\mathrm{N}\,X)_n$. The resulting quotient map $v_n : X_n \twoheadrightarrow (\mathrm{N}\,X)_n$ assembles into $v : \mathrm{C}\,X \twoheadrightarrow \mathrm{N}\,X$ with $vu = \mathrm{id}_{\mathrm{N}\,X}$, natural in X.

Proof See [10, Proposition 8.5.11]. □

The next two lemmas are the heart of the correspondence.

Lemma 9.5.6 Assume $\mathcal{A}$ abelian. For every $A \in \mathsf{Ch}_{\geq 0}(\mathcal{A})$ and $n \geq 0$ one has $\mathrm{N}\,(\Gamma A)_n = A_n$. This gives a natural isomorphism $\eta : \mathrm{id} \simeq \mathrm{N}\,\Gamma$.

Proof Fix $n \geq 0$. For $1 \leq i \leq n$ the composite $\mathrm{id}_{[n]}\, d^i = d^i$ has monic part $d^i \neq d^0$, so Definition 9.5.2 gives $\Phi_{u,\,\mathrm{id}_{[n]}} = 0$ for every u: the face map $d_i = (d^i)^*$ annihilates the direct summand A_n indexed by $\mathrm{id}_{[n]}$, which therefore lies in $(\mathrm{N}\,\Gamma A)_n$ (Definition 9.5.3).

Suppose on the contrary that there exists $x \in (\mathrm{N}\,\Gamma A)_n$ with $x \neq 0$ and $x_{\mathrm{id}_{[n]}} = 0$ (let us pretend $\mathcal{A} = \mathsf{Ab}$, just for ease of exposition); write $x = (x_t)_t$ for the decomposition into the summands of $(\Gamma A)_n$. Take $t : [n] \twoheadrightarrow [k]$ subject to the following properties, in order: $x_t \neq 0$, k is maximal, and $\sum_{j=0}^{n} t(j)$ is maximal. Then $k < n$. Take $1 \leq i \leq n$ such that $t(i-1) = t(i)$; then $td^i : [n-1] \twoheadrightarrow [k]$. From $d_i x = 0$ and the conditions above, we get

$$\sum_{\substack{t':[n]\twoheadrightarrow[k]\\ t'd^i=td^i}} x_{t'} = 0 \quad \text{in } A_k \tag{9.5.7}$$

by inspecting Definition 9.5.2. To specify $t'd^i$ is to insert a value w with $t(i-1) \le w \le t(i+1)$ (put $t(n+1) := k$); every solution $t' \ne t$ then corresponds to $w = t'(i) > t(i)$, so that $\sum_{j=0}^n t'(j) > \sum_{j=0}^n t(j)$ and hence $x_{t'} = 0$ by the maximal choice of t. Thus $x_t = 0$, a contradiction.

Finally, the factorization $\mathrm{id}_{[n+1]}\, d^0 = d^0\, \mathrm{id}_{[n]}$ shows that the differential $\mathrm{N}\,(\Gamma A)_{n+1} \to \mathrm{N}\,(\Gamma A)_n$ is ∂_{n+1}; the identifications are functorial in A and assemble into η. □

Lemma 9.5.7 Assume $\mathcal{A}$ abelian. The functors Γ and N form an adjoint pair

$$\Gamma : \mathsf{Ch}_{\ge 0}(\mathcal{A}) \rightleftarrows \mathsf{s}\mathcal{A} : \mathrm{N}, \tag{9.5.8}$$

with unit η from Lemma 9.5.6 and counit $\varepsilon : \Gamma\,\mathrm{N} \to \mathrm{id}$ given as follows: on the direct summand of $\Gamma(\mathrm{N}\,X)_n$ indexed by $t : [n] \twoheadrightarrow [k]$, the component $\varepsilon_{X,n}$ of ε_X is the composite $(\mathrm{N}\,X)_k \hookrightarrow X_k \xrightarrow{t^*} X_n$.

Proof (*Sketch*) One shows the composite

$$\mathrm{Hom}_{\mathsf{s}\mathcal{A}}(\Gamma A, X) \xrightarrow{\mathrm{N}} \mathrm{Hom}_{\mathsf{Ch}_{\ge 0}}(\mathrm{N}\,\Gamma A, \mathrm{N}\,X) \xrightarrow{\eta_A^*} \mathrm{Hom}_{\mathsf{Ch}_{\ge 0}}(A, \mathrm{N}\,X) \tag{9.5.9}$$

is a bijection by writing its inverse explicitly: a chain map $\varphi = (\varphi_n) : A \to \mathrm{N}\,X$ is sent to the simplicial map $\Phi_n : (\Gamma A)_n \to X_n$ whose component on the summand indexed by $t : [n] \twoheadrightarrow [k]$ is $A_k \xrightarrow{\varphi_k} (\mathrm{N}\,X)_k \hookrightarrow X_k \xrightarrow{t^*} X_n$; commutativity with the structure morphisms for simplicial objects reduces to the recipe in Definition 9.5.2. Taking $\varphi = \mathrm{id}_{\mathrm{N}\,X}$ recovers ε. For details, see [10, Lemma 8.5.7]. □

The key computation is the case $\mathcal{A} = \mathsf{Ab}$, carried out inside the proof below.

Theorem 9.5.8 (*A. Dold, D. Kan*) For every additive category $\mathcal{A}$ the functor $\Gamma : \mathsf{Ch}_{\ge 0}(\mathcal{A}) \to \mathsf{s}\mathcal{A}$ is fully faithful.

(i) If $\mathcal{A}$ is a Karoubian category (i.e. all idempotent endomorphisms have kernels, see [10, §2.5]), then Γ is an equivalence.

(ii) If moreover $\mathcal{A}$ is abelian, then N is a quasi-inverse and

$$\Gamma : \mathsf{Ch}_{\ge 0}(\mathcal{A}) \simeq \mathsf{s}\mathcal{A} : \mathrm{N} \tag{9.5.10}$$

is an adjoint equivalence with unit and counit as in Lemma 9.5.7.

Proof (*Sketch*) The following sketch is extremely condensed; more details are available in [10, Lemma 8.5.8, Theorem 8.5.9]. Let us begin with the special case $\mathcal{A} = \mathsf{Ab}$. Since the unit η is an isomorphism by Lemma 9.5.6, it suffices to show the counit ε is an isomorphism as well. Indeed, the $\varepsilon_{X,n}$ given in Lemma 9.5.7 is mono by a lowest-degree argument and epi by induction on n using the simplicial identities, for all X.

Now consider general $\mathcal{A}$. The functor $\mathcal{A} \to \tilde{\mathcal{A}}^\wedge := \mathrm{Fct}(\mathcal{A}^{\mathrm{op}}, \mathsf{Ab})$ given by $\tilde{h}_{\mathcal{A}} : Y \mapsto \mathrm{Hom}_{\mathcal{A}}(\cdot, Y)$ is fully faithful by an Ab-enriched version of the Yoneda lemma (i.e. replacing Set and maps by Ab and group homomorphisms everywhere). The square

$$
\begin{tikzcd}
\mathsf{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\Gamma"] \arrow[d, "\tilde{h}"'] & \mathsf{s}\mathcal{A} \arrow[d, "\tilde{h}"] \\
\mathsf{Ch}_{\geq 0}\left(\tilde{\mathcal{A}}^{\wedge}\right) \arrow[r, "\Gamma"'] & \mathsf{s}\tilde{\mathcal{A}}^{\wedge}
\end{tikzcd}
\tag{9.5.11}
$$

commutes up to natural isomorphism. The vertical functors are fully faithful, and the bottom row is an equivalence by reducing to the case $\mathcal{A} = \mathsf{Ab}$: for each $Y \in \mathcal{A}$, the evaluation functor $\tilde{\mathcal{A}}^{\wedge} \to \mathsf{Ab}$, $F \mapsto F(Y)$, is exact and preserves direct sums, hence commutes with Γ, N and with the unit and counit; as the latter are isomorphisms in the case $\mathcal{A} = \mathsf{Ab}$, they remain so after evaluation at every Y, and a morphism of functors is an isomorphism precisely when it is so objectwise. Hence the top row is fully faithful. When $\mathcal{A}$ is Karoubian, by noting that $\tilde{h}_{\mathcal{A}}$ preserves biproducts, the essential image of $\tilde{h}$ (Definition 1.2.2) is closed under retracts: a retract of $\tilde{h}(Y)$ determines an idempotent endomorphism of Y, via the full faithfulness and additivity of $\tilde{h}$, and Karoubianness splits it inside $\mathcal{A}$. Since $\left(\mathrm{N}\,\tilde{h}X\right)_n$ is a retract of $\left(\Gamma\,\mathrm{N}\,\tilde{h}X\right)_n \simeq \tilde{h}(X)_n$ (degreewise the section is the unit η, i.e. the inclusion of the summand indexed by the identity, as in the proof of Lemma 9.5.6), each $\left(\mathrm{N}\,\tilde{h}X\right)_n$ lies in that essential image, so Γ is essentially surjective. □

When $\mathcal{A}$ is abelian, $\mathrm{N}\,X$ and $\mathrm{C}\,X$ compute the same homology, as explained below.

Theorem 9.5.9 (*N and C are quasi-isomorphic*) For $\mathcal{A}$ abelian and $X \in \mathrm{Ob}(\mathsf{s}\mathcal{A})$, both $u : \mathrm{N}\,X \to \mathrm{C}\,X$ (Proposition 9.5.4) and $v : \mathrm{C}\,X \to \mathrm{N}\,X$ (Proposition 9.5.5) are quasi-isomorphisms.

Proof (*Sketch*) Below is again extremely condensed; see [10, Theorem 8.5.12] for details.

Since $vu = \mathrm{id}$, it suffices to show v a quasi-isomorphism; as v is split epi, the long exact sequence (Proposition 3.3.1) reduces us to showing $\ker(v)$ acyclic. Reduce to $X = \Gamma A$ by Theorem 9.5.8. One filters $\mathrm{C}\,X$ by subcomplexes $\mathrm{C}^{\leq i}\,X$ recording degeneracies in suitable degrees and proves, via suitable congruence relations modulo $\mathrm{C}^{\leq i-1}\,X$, that each subquotient $D = \mathrm{C}^{\leq i}\,X / \mathrm{C}^{\leq i-1}\,X$ is acyclic by producing an explicit contracting homotopy $h_n = (-1)^{n-i-1} s_{n-i-1}$. □

Remark 9.5.10 A cosimplicial object in $\mathcal{A}$ is the same datum as a simplicial object in $\mathcal{A}^{\mathrm{op}}$, with the directions of morphisms reversed: $\mathsf{cs}\,\mathcal{A} \simeq (\mathsf{s}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$. The Dold–Kan correspondence therefore transports to a chain of equivalences

$$\mathsf{cs}\,\mathcal{A} \simeq \left(\mathsf{Ch}_{\geq 0}(\mathcal{A}^{\mathrm{op}})\right)^{\mathrm{op}} \simeq \mathsf{C}_{\geq 0}(\mathcal{A}) \tag{9.5.12}$$

with cochain complexes concentrated in degrees ≥ 0: a chain complex in $\mathcal{A}^{\mathrm{op}}$ is precisely such a cochain complex in $\mathcal{A}$, and the last step merely reverses arrows. See [10, Remark 8.5.13] for details.

Exercise 9.5.11 For the Dold–Kan functor Γ, compute $\Gamma(A)$ explicitly for the chain complex $A = \left[\cdots \to 0 \to A_1 \xrightarrow{\partial_1} A_0 \to 0\right]$, and verify $\mathrm{N}\,\Gamma(A) = A$ (Lemma 9.5.6) in degrees $0, 1, 2$. *Hint.* Routine bookkeeping with Definition 9.5.2: the two surjections $[2] \twoheadrightarrow [1]$ give $(\Gamma A)_2 = A_0 \oplus A_1 \oplus A_1$, and in degrees ≤ 2 the entry $\Phi_{u,t} = \partial_1$ occurs among the cofaces only for $f = d^0$.

9.6 Homology of simplicial objects

The functors C and N define the homology of a simplicial set and, via Dold–Kan (Section 9.5), relate simplicial homotopy to chain homotopy. Our treatment follows [10, §8.6].

Definition 9.6.1 For $K \in \mathrm{Ob}(\mathsf{sSet})$ let $\mathbb{Z}K \in \mathrm{Ob}(\mathsf{sAb})$ have $(\mathbb{Z}K)_n$ the free abelian group on K_n, with d_i, s_j extended linearly. This gives a functor $\mathbb{Z}(\cdot) : \mathsf{sSet} \to \mathsf{sAb}$, left adjoint to the forgetful functor. Moreover, for simplicial sets X, Y, with respect to the termwise tensor product of Definition 9.1.5 one has $\mathbb{Z}(X \times Y) \simeq \mathbb{Z}X \otimes \mathbb{Z}Y$: degreewise this is the natural isomorphism $\mathbb{Z}(X_n \times Y_n) \simeq \mathbb{Z}\, X_n \otimes_{\mathbb{Z}} \mathbb{Z}\, Y_n$.

Definition 9.6.2 For a simplicial set K and $n \in \mathbb{Z}_{\geq 0}$, the ***n-th homology of** K* with $\mathbb{Z}$-coefficients is

$$\mathrm{H}_n(K; \mathbb{Z}) := \mathrm{H}_n(\mathrm{C}(\mathbb{Z}K)). \tag{9.6.1}$$

More generally $\mathrm{H}_n(K; M) := \mathrm{H}_n(\mathrm{C}(\mathbb{Z}K) \otimes M)$ and $\mathrm{H}^n(K; M) := \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}(\mathbb{Z}K), M))$ for a $\mathbb{Z}$-module M.

By Theorem 9.5.9 one may replace $\mathrm{C}(\mathbb{Z}K)$ by $\mathrm{N}(\mathbb{Z}K)$, which is often smaller: from Proposition 9.5.5,

$$\mathrm{N}(\mathbb{Z}K)_n = \bigoplus_{x \in K_n^{\mathrm{nd}}} \mathbb{Z}\, x, \tag{9.6.2}$$

the free abelian group on the *non-degenerate* n-simplices. For instance

$$\begin{aligned} \mathrm{N}(\mathbb{Z}\Delta^0) &= \mathbb{Z} \text{ concentrated in degree } 0, \\ \mathrm{C}(\mathbb{Z}\Delta^0) &= \Big[\cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}\Big]. \end{aligned} \tag{9.6.3}$$

Example 9.6.3 Let $(Q, \leq)$ be a non-empty poset with a least element e. Viewing Q as a category and applying the nerve gives a simplicial set $S := \mathrm{N}(Q)$. The maps $[0] \xrightarrow{0 \mapsto e} Q \to [0]$ yield $\Delta^0 \xrightarrow{i} S \xrightarrow{q} \Delta^0$ with $qi = \mathrm{id}$, hence $\mathbb{Z} \simeq \mathrm{N}(\mathbb{Z}\Delta^0) \xrightarrow{\mathrm{N}\, i} \mathrm{N}(\mathbb{Z}S) \xrightarrow{\mathrm{N}\, q} \mathbb{Z}$ are mutually inverse in the homotopy category of chain complexes. Thus $\mathrm{H}_0(Q; M) \simeq M$ and $\mathrm{H}_n(Q; M) = 0$ for $n \neq 0$. The homotopy $\mathrm{N}\, i \circ \mathrm{N}\, q \sim \mathrm{id}$ is given on a non-degenerate chain $q_0 < \cdots < q_n$ by prepending e when $q_0 \neq e$, and by 0 when $q_0 = e$.

Homology is invariant under *simplicial homotopy*, whose combinatorial definition we now record.

Definition 9.6.4 (*simplicial homotopy*) Let $X, Y \in \mathsf{s}\mathcal{C}$ and $f, g : X \to Y$. A ***simplicial homotopy*** from g to f is a family $h_i = h_i^n : X_n \to Y_{n+1}$ $(0 \leq i \leq n)$ with

$$
\begin{aligned}
d_0 h_0 &= f_n, \\
d_{n+1} h_n &= g_n, \\
d_i h_j &= h_{j-1} d_i \quad (i < j), \\
d_j h_j &= d_j h_{j-1} \quad (j \geq 1), \\
d_i h_j &= h_j d_{i-1} \quad (i > j+1), \\
s_i h_j &= h_{j+1} s_i \quad (i \leq j), \\
s_i h_j &= h_j s_{i-1} \quad (i > j).
\end{aligned} \tag{9.6.4}
$$

In the case $\mathcal{C} = \mathsf{Set}$ this agrees with the topological notion $H : X \times \Delta^1 \to Y$ of Section 9.4: writing $(\Delta^1)_n = \{\alpha_{-1}, \cdots, \alpha_n\}$, where $\alpha_i : [n] \to [1]$ maps $k \leq i$ to 0 and $k > i$ to 1, the dictionary is

$$
H^n_{-1} = f_n, \quad H^n_n = g_n, \quad H^n_i = d_{i+1} h^n_i \quad (0 \leq i \leq n-1), \tag{9.6.5}
$$

with inverse $h^n_i = H^{n+1}_i s_i$. The translation passes termwise through $\mathcal{C} = \mathsf{Ab}$ using $X \otimes \mathbb{Z}\Delta^1$.

Proposition 9.6.5 If h is a simplicial homotopy from g to f, then $\left(\sum_{j=0}^{n} (-1)^j h^n_j\right)_{n \geq 0}$ is a chain homotopy between $\mathrm{C}\, f, \mathrm{C}\, g : \mathrm{C}\, X \to \mathrm{C}\, Y$.

Proof (*Sketch*) A direct computation with the simplicial identities: after expanding $\partial h_j + h_j \partial$, the terms with $i = j$ cancel pairwise ($d_i h_i = d_i h_{i-1}$), and the cross terms telescope to $f_n - g_n$. $\square$

Exercise 9.6.6 Show that a simplicial homotopy (Definition 9.6.4) between $f, g : X \to Y$ in sSet is the same datum as a map $H : X \times \Delta^1 \to Y$ restricting to f and g at the two endpoints of Δ^1. Then prove Proposition 9.6.5 in detail.

9.7 The bar construction

The bar construction produces canonical simplicial resolutions from monads and comonads, unifying the standard complexes of Hochschild and group (co)homology. We need two preliminary notions on *augmented* semi-simplicial objects (Remark 9.1.3); write the augmentation as $\varepsilon : X_0 \to X_{-1}$.

Definition 9.7.1 (*augmented chain complex*) For an augmented semi-simplicial object X in an additive category, $\mathrm{C}\, X$ extends to the chain complex

$$
\mathrm{C}^{\mathrm{aug}}\, X := \left[\cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} X_{-1} \to 0 \to \cdots \right]. \tag{9.7.1}
$$

The identity $\varepsilon d_0 = \varepsilon d_1$ ensures $\varepsilon \circ \partial_1 = 0$. We denote this augmented complex by $\mathrm{C}^{\mathrm{aug}}\, X$.

Definition 9.7.2 (*contractible augmented object*) An augmented semi-simplicial object X is ***right contractible*** if there are maps $k_n : X_n \to X_{n+1}$ $(n \geq -1)$ with $\varepsilon k_{-1} = \mathrm{id}$, $d_{n+1} k_n = \mathrm{id}$, and $d_i k_n = k_{n-1} d_i$ $(0 \leq i \leq n,\ n \geq 1)$, and $d_0 k_0 = k_{-1} \varepsilon$. Dually one defines ***left contractible*** . Contractibility is *absolute*: any functor $F : \mathcal{C} \to \mathcal{D}$ preserves it.

Proposition 9.7.3 (*contractible $\Longrightarrow$ null-homotopic*) If X is left or right contractible, then $\mathrm{id}_{\mathrm{C}^{\mathrm{aug}}\, X}$ is null-homotopic (hence $\mathrm{C}^{\mathrm{aug}}\, X$ is acyclic).

Proof (*Sketch*) For right contractibility, the maps $((-1)^{n+1}k_n)_n$ witness a null-homotopy: $\partial_{n+1}k_n - k_{n-1}\partial_n = (-1)^{n+1}\operatorname{id}_{X_n}$, verified separately for $n = -1, 0$, and $n \geq 1$ using the simplicial identities. □

Example 9.7.4 (*EΓ is contractible*) For any monoid Γ, the augmented simplicial set $\mathrm{E}\Gamma$ (with $(\mathrm{E}\Gamma)_{-1} = \{1\}$ and constant augmentation) is right contractible: $k_n : (g_1, \cdots, g_{n+1}) \mapsto (g_1, \cdots, g_{n+1}, 1)$ satisfies the identities of Definition 9.7.2 by inspection.

We now specialise to a comonad (L, δ, ε) on a category $\mathcal{D}$: an endofunctor L with a comultiplication $\delta : L \to L^2$ and a counit $\varepsilon : L \to \mathrm{id}$ obeying the coassociativity and counitality axioms; this is the dual of a monad (T, μ, η) on $\mathcal{C}$, an endofunctor T with a multiplication $\mu : T^2 \to T$ and a unit $\eta : \mathrm{id} \to T$. As in [10, §8.5], the comonad yields an augmented simplicial object in the endofunctor category $\mathcal{D}^{\mathcal{D}}$, with n-th term L^{n+1}, faces $d_i = L^i \varepsilon L^{n-i}$ and degeneracies $s_j = L^j \delta L^{n-j}$, and augmentation $\varepsilon : L \to \mathrm{id}$. The most important source of comonads is an adjunction.

Example 9.7.5 (*the bar construction of an adjunction*) For an adjunction $F : \mathcal{C} \dashv \mathcal{D} : G$ with unit η and counit ε, the comonad $L = FG$ (with $\delta = F\eta G$) gives the ***bar construction*** $\mathrm{Bar}(F, G) : \mathsf{FinOrd}^{\mathrm{op}} \to \mathcal{D}^{\mathcal{D}}$, with n-th term L^{n+1}. Composing with G on the left yields the augmented simplicial object

$$G\mathrm{Bar}(F, G)_n = GL^{n+1} = (GF)^{n+1}G = T^{n+1}G \quad (n \geq -1), \tag{9.7.2}$$

where $T = GF$ is the monad on $\mathcal{C}$. Its faces and degeneracies, written in terms of the multiplication $\mu = G\varepsilon F$ and unit η, are

$$d_i = \begin{cases} T^i \mu T^{n-i-1} G & \text{if } 0 \leq i < n \\ T^n G \varepsilon & \text{if } i = n \end{cases}, \quad s_j = T^{j+1} \eta T^{n-j} G. \tag{9.7.3}$$

Evaluating at $Y \in \mathcal{D}$ gives an augmented simplicial object $\mathrm{Bar}(Y)$ in $\mathcal{D}$ with $\mathrm{Bar}(Y)_{-1} = Y$. When $\mathcal{C}, \mathcal{D}$ are additive, applying $\mathrm{C}^{\mathrm{aug}}$ yields a canonical morphism in $\mathsf{Ch}_{\geq 0}(\mathcal{D})$,

$$\varepsilon_Y : \mathrm{C}(\mathrm{Bar}(Y)) \to Y, \tag{9.7.4}$$

whose degree-0 part is the augmentation $\varepsilon_Y : FG(Y) \to Y$.

Theorem 9.7.6 (*the bar resolution is null-homotopic*) For an adjunction $F : \mathcal{A} \dashv \mathcal{B} : G$ between additive categories and $Y \in \mathcal{B}$, the augmented chain complex $\mathrm{C}^{\mathrm{aug}(G\,\mathrm{Bar}(Y))}$ is null-homotopic (hence acyclic).

Proof (*Sketch*) $G\,\mathrm{Bar}(F, G)$ is left contractible, with contracting homotopy $k_n = \eta T^{n+1} G : T^{n+1}G \to T^{n+2}G$: the unit identities $\mu \circ (\eta T) = \mathrm{id}$ and $G\varepsilon \circ (\eta G) = \mathrm{id}$, together with the functoriality of η, verify the conditions of Definition 9.7.2. Evaluation at Y preserves contractibility, and Proposition 9.7.3 concludes. □

Corollary 9.7.7 (*the bar resolution is a resolution*) For an adjunction between abelian categories and $Y \in \mathcal{B}$, the morphism $G\varepsilon_Y : \mathrm{C}(G\,\mathrm{Bar}(Y)) \to GY$ is a quasi-isomorphism.

The bar construction thus supplies, for free, a canonical resolution of Y; and because contractibility is absolute, this resolution is functorial and preserved by any additive functor. We conclude with the two headline applications.

Example 9.7.8 (*Hochschild (co)homology from the bar resolution*) Let $\Bbbk$ be a commutative ring and R a $\Bbbk$-algebra. The free–forgetful adjunction $R \otimes (\cdot) : \Bbbk\text{-Mod} \dashv R\text{-Mod}$ has comonad $L(M) = R \otimes_{\Bbbk} M$ on R-Mod. Applying the bar construction to a left R-module M gives

$$\mathrm{C}(\mathrm{Bar}(M))_n = L^{n+1}(M) = R^{\otimes(n+1)} \otimes M, \tag{9.7.5}$$

with differential the familiar bar (Hochschild) differential. By Corollary 9.7.7 the augmentation $\mathrm{C}^{\mathrm{aug}(\mathrm{Bar}(R))} \to R$ is a quasi-isomorphism over $\Bbbk$. Specialising to $M = R$ recovers the bar complex of Section 3.5:

$$\mathrm{C}(\mathrm{Bar}(R))_n = R^{\otimes(n+2)} = R \otimes R^{\otimes(n)} \otimes R, \tag{9.7.6}$$

which, as an (R, R)-bimodule complex, is precisely the bar resolution $\mathsf{B}R$ defined there. Its Hochschild (co)homology is thus

$$\mathrm{HH}_n(M) = \mathrm{H}_n(M \otimes_{R^e} \mathsf{B}R), \quad \mathrm{HH}^n(M) = \mathrm{H}^n(\mathrm{Hom}_{R^e}(\mathsf{B}R, M)), \tag{9.7.7}$$

reconciling the differential definition of Section 3.5 with the Ext/Tor description of Example 4.4.7.

Example 9.7.9 (*group homology from the bar resolution*) For a group Γ and the group algebra $R = \Bbbk[\Gamma]$, continuing Example 9.7.8, the augmentation $\varepsilon_M : \mathrm{C}(\mathrm{Bar}(M)) \to M$ is a $(\cdot)_\Gamma$-acyclic resolution of the Γ-module M: each $\mathrm{Bar}(M)_n = \Bbbk[\Gamma]^{\otimes(n+1)} \otimes M$ is induced (hence acyclic by Shapiro's lemma). By Corollary 4.2.14,

$$\mathrm{H}_n(\Gamma, M) \simeq \mathrm{H}_n(\mathrm{C}((\mathrm{Bar}(M))_\Gamma)). \tag{9.7.8}$$

Taking Γ-coinvariants contracts the first tensor factor via $\Bbbk[\Gamma] \twoheadrightarrow \Bbbk$, giving the standard chain complex computing group homology, with differential

$$\begin{aligned} g_1 \otimes \cdots \otimes g_n \otimes x \mapsto{}& g_2 \otimes \cdots \otimes x \\ &+ \sum_{k=1}^{n-1} (-1)^k \cdots \otimes g_k g_{k+1} \otimes \cdots \\ &+ (-1)^n g_1 \otimes \cdots \otimes g_{n-1} \otimes g_n x. \end{aligned} \tag{9.7.9}$$

This is the complex underlying the Lyndon–Hochschild–Serre spectral sequence of Subsection 6.6.3.

Exercise 9.7.10 Verify that $\mathsf{E}\,\Gamma$ (Example 9.3.4) is right contractible via the homotopy $k_{n(g_1,\cdots,g_{n+1})} = (g_1, \cdots, g_{n+1}, 1)$, and deduce that $\mathsf{B}\,\Gamma$ has $\mathrm{H}_0(\mathsf{B}\,\Gamma; \mathbb{Z}) \simeq \mathbb{Z}$ when Γ is a group.

Exercise 9.7.11 Let R be a $\Bbbk$-algebra. Starting from the bar resolution Equation (9.7.4) of a left R -module M, write down the chain complex $\mathrm{C}(\mathrm{Bar}(M))$ and its differential explicitly, and check that the homology of $N \otimes_R \mathrm{C}(\mathrm{Bar}(M))$ is $\mathrm{Tor}_n^R(N, M)$ for any right R-module N (compare Example 4.4.7). For $R = \Bbbk[t]$ and the resolution of Example 4.4.7, verify $\mathrm{HH}^n(R) = 0$ for $n > 1$.

9.8 Bisimplicial objects and the Eilenberg–Zilber theorem

We close with a brief look at bisimplicial objects, needed to make the Dold–Kan functor N monoidal. A ***bisimplicial object*** in $\mathcal{C}$ is a functor $(\Delta^{\mathrm{op}})^2 \to \mathcal{C}$; these form $\mathsf{s}^2\mathcal{C}$. Unwound, it is a family of objects $\left(X_{p,q}\right)_{p,q\geq 0}$ with face maps ${}^k d_i$ and degeneracies ${}^k s_j$ ($k = 1, 2$, acting on the p

- resp. q-variable) subject to Equation (9.1.3) in each variable, the maps from the two variables commuting. The ***diagonal*** $d(X)_n = X_{n,n}$ defines a functor $d : \mathsf{s}^2\mathcal{C} \to \mathsf{s}\mathcal{C}$.

For $\mathcal{A}$ additive there is a Moore *double complex* $\mathrm{C}^2\, X$ with horizontal and vertical differentials $\partial^{\mathrm{h}}_{p,q} = \sum_{i=0}^{p} (-1)^{i1} d_i$ and $\partial^{\mathrm{v}}_{p,q} = \sum_{i=0}^{q} (-1)^{i2} d_i$, and a total complex $\mathrm{tot}(\mathrm{C}^2(X))$ (formed as in Definition 3.8.3). The two composite functors $\mathrm{tot} \circ \mathrm{C}^2$ and $\mathrm{C} \circ d$ from $\mathsf{s}^2\mathcal{A}$ to $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ are related by the classical maps of Eilenberg–Zilber and Alexander–Whitney.

Theorem 9.8.1 (*Eilenberg–Zilber*) For $\mathcal{A}$ abelian and $X \in \mathsf{s}^2\mathcal{A}$ there are natural chain maps, the ***shuffle map*** $\overline{\mathrm{EZ}} : \mathrm{tot}(\mathrm{C}^2(X)) \to \mathrm{C}(d(X))$ and the ***Alexander–Whitney map*** $\overline{\mathrm{AW}} : \mathrm{C}(d(X)) \to \mathrm{tot}(\mathrm{C}^2(X))$, that are mutually inverse in homology. In particular both are quasi-isomorphisms.

This is the *generalized Eilenberg–Zilber theorem* of [10, Theorem 8.8.9], attributed to S. Eilenberg, J. A. Zilber and P. Cartier; the normalized descendants EZ, AW are constructed in the same section.

The shuffle map is a signed sum over (p,q)-shuffles (order-preserving injections $\sigma : [p+q] \hookrightarrow [p] \times [q]$, equivalently lattice paths from $(0,0)$ to (p,q)), weighted by the sign $\mathrm{sgn}(\sigma)$, defined as the sign of the permutation reordering the word of p horizontal and q vertical steps of σ into horizontal steps first.

We omit the proof of the theorem, which by Dold–Kan reduces to the case $\mathcal{A} = \mathsf{Ab}$ and then, by filtration and the contractibility of posets with a lower bound (Example 9.6.3), to $X = \mathbb{Z}(\Delta^p \times \Delta^q)$; see [10, §8.8] or [5, IV]. Two structural consequences of these maps are worth recording.

- ▷ **Monoidal structures** Assuming $\mathcal{A}$ monoidal (Definition A.6.1), with the Koszul signs of Section 3.1 on the tensor product of chain complexes, the normalized descent $\mathrm{EZ} : \mathrm{N}\,A \otimes \mathrm{N}\,B \to \mathrm{N}(A \otimes B)$ of the shuffle map (and its unnormalized counterpart for C) gives N and C the structure of *lax monoidal* functors (Definition A.6.6), while the Alexander–Whitney maps give oplax monoidal structures.
- ▷ **Symmetric case** When $\mathcal{A}$ is moreover symmetric monoidal (Definition A.6.7), the shuffle map is compatible with the Koszul braiding, so N is a symmetric lax monoidal functor; this offers a justification of the $(-1)^{pq}$ signs in the Koszul sign rule.

Appendix A

Categorical Foundations and the Embedding Theorem

Throughout these notes we have repeatedly availed ourselves of the freedom to chase diagrams elementwise in a category of modules, even when a result was stated for an abstract abelian category. The license for this switch is a deep theorem: every small abelian category embeds, fully faithfully and exactly, into a module category. This appendix states that theorem and sketches its proof, which passes through the ***Ind-completion*** of a category, a construction of independent interest. We also record the ***adjoint functor theorem*** , the standard existence criterion for adjoints. We follow [10, Appendix B] for the embedding theorem. A closing section, drawn from [9, §§3.1 and 3.3], develops the basics of monoidal categories: the structure borne by the tensor products of modules (Section B.4) and of simplicial objects (Definition 9.1.5), the accompanying functoriality, and symmetry.

A.1 The embedding theorem

Recall (Definition 2.2.9) that a functor between abelian categories is exact if it preserves short exact sequences, and fully faithful (Definition 1.2.2) if it induces bijections on Hom-sets.

Theorem A.1.1 (*Freyd–Mitchell*) Every small abelian category $\mathcal{A}$ admits a fully faithful, exact functor

$$\mathcal{A} \to R\text{-}\mathsf{Mod} \tag{A.1.1}$$

for some ring R.

A few words on the hypotheses and consequences.

▷ **Smallness** The hypothesis that $\mathcal{A}$ be small (a set of objects, a set of morphisms) is a set-theoretic precaution. In practice one shrinks a large abelian category to the small subcategory generated by the finitely many objects appearing in a given diagram, embeds that, and chases elements there; any conclusion expressible by a finite commutative diagram transfers back. By passing to a larger Grothendieck universe one may always arrange smallness.

▷ **Element-chasing** Since a fully faithful exact functor $\mathcal{A} \to R\text{-}\mathsf{Mod}$ preserves kernels, cokernels, images, exactness and commutativity, and reflects them in the sense that any cone (resp. cocone) as per Remark 1.5.3 is a limit (resp. colimit) in $\mathcal{A}$ if so is its image in $R\text{-}\mathsf{Mod}$, any diagram lemma (snake, five) of Section 2.5, once established in $R\text{-}\mathsf{Mod}$ (where objects have elements), is valid in $\mathcal{A}$. This is precisely the move announced in Section 1.7.

A.2 Ind-objects and the Ind-completion

The cleanest route to Theorem A.1.1, due to Grothendieck, enlarges $\mathcal{A}$ to a category large enough to admit a projective generator. The enlargement is the ***Ind-completion*** .

Fix a category $\mathcal{C}$. Via the Yoneda embedding (Theorem 1.4.1) we view $\mathcal{C}$ inside its presheaf category $\mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, which has all small colimits. We suppress the symbol $h_{\mathcal{C}}$ for the Yoneda embedding.

Definition A.2.1 An ***ind-object*** of $\mathcal{C}$ is a presheaf $X \in \mathcal{C}^\wedge$ that can be written as a filtered colimit

$$X = \text{“colim”}\ X_i \quad (i \in \mathrm{Ob}(I), X_i \in \mathrm{Ob}(\mathcal{C})) \tag{A.2.1}$$

of representables, indexed by a small filtered category I. Dually, a ***pro-object*** of $\mathcal{C}$ is a formal filtered limit of objects of $\mathcal{C}$, taken in $\mathcal{C}^\vee$. The full subcategories of ind- and pro-objects are denoted

$$\mathrm{Ind}(\mathcal{C}) \subset \mathcal{C}^\wedge, \quad \mathrm{Pro}(\mathcal{C}) \subset \mathcal{C}^\vee; \tag{A.2.2}$$

in other words, $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Informally, an ind-object is a “formal filtered colimit” of objects of $\mathcal{C}$ taken in the presheaf category $\mathcal{C}^\wedge$; this should not be conflated with the colimits taken in $\mathcal{C}$, and for this reason we wrote “colim” X_i.

A word on sizes: the Hom formula of Proposition A.2.2 below shows that when $\mathcal{C}$ is locally small as required in Section 1.1, so are $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$, since each $\mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$ is a set, and the limit and the colimit in the formula are indexed by small categories. The construction of Ind and Pro also depends on the choice of the Grothendieck universe; for the precise relationship, see the discussions preceding [6, Proposition 6.1.21].

The Yoneda embedding lands in $\mathrm{Ind}(\mathcal{C})$, and one shows as in [10, §B]:

Proposition A.2.2 The Ind-completion $\mathrm{Ind}(\mathcal{C})$ is a category containing $\mathcal{C}$ as a full subcategory, and it admits all small filtered colimits. A morphism between ind-objects $\text{“colim”}_i X_i \to \text{“colim”}_j Y_j$ is computed by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\big(\text{“colim”}_i X_i, \text{“colim”}_j Y_j\big) &\simeq \lim_i \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\big(X_i, \text{“colim”}_j Y_j\big) \\ &\simeq \lim_i \operatorname*{colim}_j \mathrm{Hom}_{\mathcal{C}}\big(X_i, Y_j\big) \end{aligned} \tag{A.2.3}$$

The defining identity of Proposition A.2.2 is the key: morphisms between filtered colimits are computed object-wise, with a limit over the source index and a colimit over the target. Dually, morphisms between pro-objects are computed object-wise, with a colimit over the source index and a limit over the target.

Example A.2.3 $\mathrm{Ind}(\mathsf{FinSet}) \simeq \mathsf{Set}$: every set is the filtered colimit of its finite subsets. Similarly $\mathrm{Ind}\big(\mathsf{Vect}_f(\Bbbk)\big) \simeq \mathsf{Vect}(\Bbbk)$ for vector spaces over a field $\Bbbk$. Ind-completion is thus a small-to-large construction controlled by filtered colimits.

Exercise A.2.4 Let $X = \text{“colim”}_i X_i$ be an ind-object of $\mathcal{C}$, with I small filtered. Show that X sends finite colimits in $\mathcal{C}$ to finite limits of sets: for every finite diagram $\left(Y_j\right)_{j \in J}$ in $\mathcal{C}$ admitting a colimit, the canonical map $X\left(\operatorname{colim}_j Y_j\right) \to \lim_j X\left(Y_j\right)$ is a bijection. Now assume $\mathcal{C}$ abelian, and deduce that finite limits in $\operatorname{Ind}(\mathcal{C})$ exist and are computed object-wise, as in $\mathcal{C}^\wedge$ (cf. Theorem A.3.1).

Hint. Filtered colimits commute with finite limits in Set.

A.3 Sketch of the embedding theorem

The proof of Theorem A.1.1 combines three facts, each of independent importance; we state them and indicate how they fit together.

Theorem A.3.1 (*Ind-completion of an abelian category*) If $\mathcal{A}$ is a small abelian category, then $\operatorname{Ind}(\mathcal{A})$ is a Grothendieck category (Section 2.8), and the inclusion $\mathcal{A} \to \operatorname{Ind}(\mathcal{A})$ is fully faithful and exact.

This is non-trivial: one must show that $\operatorname{Ind}(\mathcal{A})$ is abelian, has a generator, and that small filtered colimits in it are exact; one also has to show that finite (co)limits in $\operatorname{Ind}(\mathcal{A})$ are computed object-wise, so the inclusion is exact.

Since every Grothendieck category has a generator and enough injectives (Section 2.8), $\operatorname{Ind}(\mathcal{A}^{\mathrm{op}})$ has an injective cogenerator (take the coproduct of the quotients of a generator, a small set, and embed it into an injective); by duality $\operatorname{Pro}(\mathcal{A})$ therefore has a *projective generator* S.

Lemma A.3.2 Let S be a projective generator of an abelian category $\mathcal{C}$, and set $R = \operatorname{End}_{\mathcal{C}}(S)$, so that

$$G := \operatorname{Hom}_{\mathcal{C}}(S, \cdot) : \mathcal{C} \to \mathsf{Mod}\text{-}R \tag{A.3.1}$$

is an exact and faithful functor. If $X \in \operatorname{Ob}(\mathcal{C})$ is a quotient of a finite direct sum of copies of S, then the map induced by G,

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_R(G(X), G(Y)), \tag{A.3.2}$$

is a bijection for every $Y \in \operatorname{Ob}(\mathcal{C})$. In particular, G is fully faithful on the full subcategory of objects that are quotients of finite direct sums of copies of S.

Proof Exactness is the projectivity of S; faithfulness holds because S generates. Take $m \in \mathbb{Z}_{\geq 0}$ and a short exact sequence

$$0 \to X' \to S^{\oplus m} \to X \to 0 \tag{A.3.3}$$

in $\mathcal{C}$. Since G is exact, applying it yields a short exact sequence in $\mathsf{Mod}\text{-}R$, and every $Y \in \operatorname{Ob}(\mathcal{C})$ then gives a commutative diagram with exact rows in Ab:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(S^{\oplus m}, Y) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(X', Y) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_R(G(X), G(Y)) \arrow[r] & \operatorname{Hom}_R(G(S^{\oplus m}), G(Y)) \arrow[r] & \operatorname{Hom}_R(G(X'), G(Y))
\end{tikzcd} \tag{A.3.4}$$

The vertical arrows are induced by the faithful functor G, hence are injective. For the middle one, observe that

$$\mathrm{Hom}_{\mathcal{C}}(S,Y) \simeq G(Y) \simeq \mathrm{Hom}_R(R, G(Y)) = \mathrm{Hom}_R(G(S), G(Y)). \tag{A.3.5}$$

Indeed, the first identification is the identity, and the second evaluates $\psi \mapsto \psi(1_R)$. The composite is precisely the homomorphism induced by G on Hom's. Since G is additive, the middle vertical arrow is therefore an isomorphism. A simple chase in the diagram shows that the left-hand arrow is an isomorphism too. □

Proof (*Sketch, for Theorem A.1.1*) View the small abelian category $\mathcal{A}$ as a full subcategory of $\mathrm{Pro}(\mathcal{A})$: by Theorem A.3.1 applied to $\mathcal{A}^{\mathrm{op}}$ and duality, the inclusion is exact and fully faithful. Choose a projective generator S of $\mathrm{Pro}(\mathcal{A})$, so large that every object of $\mathcal{A}$ is a quotient of a (finite) direct sum of copies of S; this is possible because $\mathcal{A}$ is small. Set $R = \mathrm{End}_{\mathrm{Pro}(\mathcal{A})}(S)^{\mathrm{op}}$; since a right $\mathrm{End}_{\mathrm{Pro}(\mathcal{A})}(S)$-module is the same thing as a left R-module, the target $\mathsf{Mod}\text{-}R$ of Lemma A.3.2 becomes $R\text{-}\mathsf{Mod}$ here. By Lemma A.3.2 the functor

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(S, \cdot) : \mathrm{Pro}(\mathcal{A}) \to R\text{-}\mathsf{Mod} \tag{A.3.6}$$

is fully faithful on $\mathcal{A}$, and it is exact (projectivity of S). Its restriction to $\mathcal{A}$ is the desired embedding.

$$\mathcal{A} \xrightarrow{\text{exact, fully faithful}} \mathrm{Pro}(\mathcal{A}) \xrightarrow{\mathrm{Hom}(S,\cdot)} R\text{-}\mathsf{Mod} \tag{A.3.7}$$

□

This is the embedding that licenses element chasing.

A.4 Density of the Yoneda embedding

In Section A.2, the filteredness requirement of the indexing category I is what singles out the Ind-objects inside $\mathcal{C}^\wedge$: in fact, *every* presheaf is canonically a colimit of representables. The precise statement is the ***density*** theorem, for which we will

- reinstate the symbol $h_{\mathcal{C}}$ for the Yoneda embedding (Theorem 1.4.1), and
- write colim for colimits in $\mathcal{C}^\wedge$ instead of the previous "colim", since there will be no confusion here.

For $A \in \mathrm{Ob}(\mathcal{C}^\wedge)$, view A as a functor $\mathbf{1} \to \mathcal{C}^\wedge$ and form the comma category $(h_{\mathcal{C}}/A)$ (Definition 1.5.2): its objects are the pairs

$$(S, \varphi : h_{\mathcal{C}}(S) \to A) \quad (S \in \mathrm{Ob}(\mathcal{C})), \tag{A.4.1}$$

a morphism from (S,φ) to (S',φ') being an $f \in \mathrm{Hom}_{\mathcal{C}}(S,S')$ with $\varphi = \varphi' h_{\mathcal{C}}(f)$. Dually, for $B \in \mathrm{Ob}(\mathcal{C}^\vee)$ the comma category $(B/k_{\mathcal{C}})$ has objects $(S, \psi : B \to k_{\mathcal{C}}(S))$, the morphism ψ being regarded as one of $\mathcal{C}^\vee$, and the evident morphisms. When $\mathcal{C}$ is small these categories are small (cf. Section 1.1); we will only invoke the density theorem for small $\mathcal{C}$.

Theorem A.4.1 For every $A \in \mathrm{Ob}(\mathcal{C}^\wedge)$ and $B \in \mathrm{Ob}(\mathcal{C}^\vee)$, the structure morphisms φ and ψ induce canonical isomorphisms in $\mathcal{C}^\wedge$ and $\mathcal{C}^\vee$, respectively:

$$\operatorname*{colim}_{(S,\varphi)} h_{\mathcal{C}}(S) \simeq A, \quad B \simeq \lim_{(S,\psi)} k_{\mathcal{C}}(S). \tag{A.4.2}$$

In other words, the Yoneda embedding is ***dense*** in the sense that every presheaf is a canonical colimit of representables.

Proof We establish only the first isomorphism; the second follows by duality. For any $A' \in \mathrm{Ob}(\mathcal{C}^\wedge)$, composition with the φ's yields a map into the set of *compatible families* of morphisms $a_{S,\varphi} : h_{\mathcal{C}}(S) \to A'$, indexed by the objects (S,φ) of $(h_{\mathcal{C}}/A)$, a family being *compatible* when $a_{S,\varphi} = a_{S',\varphi'} h_{\mathcal{C}}(f)$ for every morphism $f : (S,\varphi) \to (S',\varphi')$ of $(h_{\mathcal{C}}/A)$; explicitly,

$$\begin{gathered} \mathrm{Hom}_{\mathcal{C}^\wedge}(A, A') \to \left\{ \text{compatible families } \left(a_{S,\varphi}\right)_{S,\varphi} \right\} \\ \theta \mapsto \left(a_{S,\varphi} := \theta\varphi\right)_{S,\varphi} \end{gathered} \tag{A.4.3}$$

Define the map in the reverse direction as follows. Given such a family $\left(a_{S,\varphi}\right)_{S,\varphi}$, take any $S \in \mathrm{Ob}(\mathcal{C})$ and $u \in A(S)$. By Theorem 1.4.1, u corresponds to a morphism $\varphi : h_{\mathcal{C}}(S) \to A$ with $\varphi_S(\mathrm{id}_S) = u$, thereby determining the object (S,φ) of $(h_{\mathcal{C}}/A)$, and $u \mapsto \left(a_{S,\varphi}\right)_S(\mathrm{id}_S)$ (i.e. sending u to the element of $A'(S)$ corresponding to $a_{S,\varphi}$ per Yoneda) defines a map $A(S) \to A'(S)$. We claim:

- as S varies, the maps $A(S) \to A'(S)$ so obtained assemble into a morphism $\theta : A \to A'$ of $\mathcal{C}^\wedge$;
- the assignment $\left(a_{S,\varphi}\right)_{S,\varphi} \mapsto \theta$ and the map of Equation (A.4.3) are mutually inverse.

As with many results concerning the Yoneda embedding, the detailed verifications are nearly tautological and are omitted; the reader may also consult the treatment in [8, pp. 76–77].

Consequently, the map Equation (A.4.3) is a bijection, and in the particular case $A' = A$ it sends id_A to the family $\left(a_{S,\varphi} = \varphi\right)_{S,\varphi}$. This verifies that A, together with the morphisms $\varphi : h_{\mathcal{C}}(S) \to A$, has the universal property of the colimit (Definition 1.5.4). Note that the argument uses no smallness: the universal property holds for an arbitrary $\mathcal{C}$, the indexing category $(h_{\mathcal{C}}/A)$ being possibly large. Cf. [6, Proposition 2.6.3]. □

A.5 The adjoint functor theorem

We collect, for reference, the standard criterion ensuring that a functor *has* an adjoint. Throughout, let $G : \mathcal{D} \to \mathcal{C}$ be a functor between locally small categories with $\mathcal{D}$ complete (all small limits).

Theorem A.5.1 (*general adjoint functor theorem*) The functor G has a left adjoint if and only if:

- G preserves all small limits; and
- for every object $c \in \mathcal{C}$, the coslice category (c/G) has a ***solution set*** , i.e. a (small) set of morphisms $f_i : c \to G(d_i)$ such that every morphism $f : c \to G(d)$ factors through some f_i.

The hard direction constructs the adjoint value $F(c)$ as the limit of a large diagram built from the solution set; the solution-set condition is precisely what keeps this limit small. The dual statement gives a criterion for a right adjoint (cocompleteness, colimit preservation, a co-solution set).

Remark A.5.2 If $\mathcal{D}$ has a cogenerating set (Remark 2.8.5), the solution-set condition of Theorem A.5.1 is automatic. This special case is *Freyd's adjoint functor theorem*. For full proofs see [8, ch.5].

Exercise A.5.3 Using Theorem A.1.1, give a one-line justification for chasing elements in the snake lemma (Theorem 2.5.2) for an arbitrary abelian category.

Hint. Embed the finite diagram into R-Mod and apply the module-theoretic snake lemma.

A.6 Monoidal categories

The tensor product of modules over a commutative ring is associative and unital only up to canonical isomorphism. The abstract home for such structure is the notion of a ***monoidal category*** (called a ***tensor category*** in parts of the literature): a category equipped with a multiplication and a unit object, whose associativity and unit laws hold up to specified natural isomorphisms, subject to coherence conditions. We transcribe the basic definitions from [9, §3.1] and add a concise account of symmetric monoidal categories from [9, §3.3]; the further theory (the braid category, the Yang–Baxter equation, enriched categories) lies beyond the needs of these notes.

Definition A.6.1 A ***monoidal category*** is the data $(\mathcal{V}, \otimes, a, \mathbf{1}, \iota)$ consisting of

- a category $\mathcal{V}$;
- a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, whose values on objects and morphisms are written $(X, Y) \mapsto X \otimes Y$ and $(f, g) \mapsto f \otimes g$;
- an isomorphism a in the functor category $\mathrm{Fct}(\mathcal{V} \times \mathcal{V} \times \mathcal{V}, \mathcal{V})$,

$$a : (\cdot \otimes \cdot) \otimes \cdot \xrightarrow{\sim} \cdot \otimes (\cdot \otimes \cdot), \tag{A.6.1}$$

  called the ***associativity constraint*** , such that for all objects X, Y, Z, W of $\mathcal{V}$ the following pentagon commutes:

$$\begin{tikzcd}[column sep=small]
 & & ((X \otimes Y) \otimes Z) \otimes W \arrow[dll, "a(X,Y,Z) \otimes \mathrm{id}_W"'] \arrow[drr, "a(X \otimes Y, Z, W)"] & & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[dr, "a(X, Y \otimes Z, W)"'] & & & & (X \otimes Y) \otimes (Z \otimes W) \arrow[dl, "a(X, Y, Z \otimes W)"] \\
 & X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\mathrm{id}_X \otimes a(Y, Z, W)"'] & & X \otimes (Y \otimes (Z \otimes W)) &
\end{tikzcd} \tag{A.6.2}$$

- a ***unit object*** $\mathbf{1} \in \mathrm{Ob}(\mathcal{V})$, such that the functors $\mathbf{1} \otimes \cdot$ and $\cdot \otimes \mathbf{1}$ are equivalences (Definition 1.2.4) from $\mathcal{V}$ to itself;
- an isomorphism $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$.

The commutative pentagon in Definition A.6.1 is **Mac Lane's *pentagon axiom*** : its vertices are the five ways of multiplying X, Y, Z, W in order, two vertices being joined when a single application of a connects them. In real life $\otimes$ is rarely strictly associative, i.e. $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are in general different objects, so the associativity law is relaxed to the family of natural isomorphisms

$$a(X, Y, Z) : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \tag{A.6.3}$$

which can be iterated; the pentagon axiom is the coherence condition that keeps the iterations under control. Likewise the unit is subject not to the strict equality $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$ but to the isomorphism ι. As is customary, we speak of "the monoidal category $\mathcal{V}$" and suppress the rest of the structure from the notation.

A subcategory $\mathcal{V}' \subset \mathcal{V}$ is a ***monoidal subcategory*** if it contains $\mathbf{1}$ and $X, Y \in \mathrm{Ob}(\mathcal{V}')$ implies $X \otimes Y \in \mathrm{Ob}(\mathcal{V}')$.

Definition A.6.2 Let $(\mathcal{V}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. Since the functors $\mathbf{1} \otimes \cdot$ and $\cdot \otimes \mathbf{1}$ are equivalences, in particular fully faithful, there is for each object X a unique morphism $\lambda_X : \mathbf{1} \otimes X \to X$, necessarily an isomorphism, such that $\mathrm{id}_{\mathbf{1}} \otimes \lambda_X$ is the composite

$$\mathbf{1} \otimes (\mathbf{1} \otimes X) \xrightarrow{a(\mathbf{1}, \mathbf{1}, X)^{-1}} (\mathbf{1} \otimes \mathbf{1}) \otimes X \xrightarrow{\iota \otimes \mathrm{id}_X} \mathbf{1} \otimes X \tag{A.6.4}$$

and dually a unique morphism $\rho_X : X \otimes \mathbf{1} \to X$, necessarily an isomorphism, such that $\rho_X \otimes \mathrm{id}_{\mathbf{1}}$ is the composite

$$(X \otimes \mathbf{1}) \otimes \mathbf{1} \xrightarrow{a(X, \mathbf{1}, \mathbf{1})} X \otimes (\mathbf{1} \otimes \mathbf{1}) \xrightarrow{\mathrm{id}_X \otimes \iota} X \otimes \mathbf{1} \tag{A.6.5}$$

The λ_X and ρ_X assemble into isomorphisms of functors

$$\lambda : \mathbf{1} \otimes \cdot \xrightarrow{\sim} \cdot, \quad \rho : \cdot \otimes \mathbf{1} \xrightarrow{\sim} \cdot, \tag{A.6.6}$$

called the left and right ***unit constraints*** respectively; their role is analogous to that of a.

Example A.6.3 The following are monoidal categories.

- If $\mathcal{C}$ has finite products, take $\otimes := \times$ and for a the canonical isomorphisms of Exercise 1.5.10; the unit is a terminal object $*$, the empty product (Example 1.5.5). Dually, finite coproducts make $\mathcal{C}$ monoidal under $\otimes := \sqcup$, with unit an initial object.
- For any small category $\mathcal{C}$, the endofunctor category $\mathrm{Fct}(\mathcal{C}, \mathcal{C})$ is monoidal under composition: $F \otimes G := F \circ G$; for morphisms (natural transformations) $\theta : F \Rightarrow G$ and $\psi : F' \Rightarrow G'$, the induced morphism is the horizontal composite $\psi\theta : F'F \Rightarrow G'G$; the unit is $\mathrm{id}_{\mathcal{C}}$. Here a is the identity: composition is strictly associative.
- For a commutative ring R, the category R-Mod is monoidal under $\otimes := \otimes_R$, with unit R itself; the required natural isomorphisms are recorded in Section B.4.

At first sight, Definition A.6.1 and Definition A.6.2 do not exhaust the properties expected of a unit object; for instance, one would like $\lambda_{\mathbf{1}}$, $\rho_{\mathbf{1}}$ and ι to agree. The remaining properties follow from the pentagon axiom and functoriality; the definitive vindication of the definitions is the coherence theorem, for which we refer to [9, §3.2]. The first step is the following lemma, which we state without proof.

Lemma A.6.4 (*Kelly*) Let $(\mathcal{V}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. For every object X,

$$\begin{aligned} \lambda_{\mathbf{1}\otimes X} &= \mathrm{id}_{\mathbf{1}} \otimes \lambda_X : \mathbf{1} \otimes (\mathbf{1} \otimes X) \to \mathbf{1} \otimes X, \\ \rho_{X\otimes \mathbf{1}} &= \rho_X \otimes \mathrm{id}_{\mathbf{1}} : (X \otimes \mathbf{1}) \otimes \mathbf{1} \to X \otimes \mathbf{1}, \\ \lambda_{\mathbf{1}} &= \rho_{\mathbf{1}} = \iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}, \end{aligned} \tag{A.6.7}$$

and for all objects X, Y the following diagrams commute:

$$\begin{tikzcd} (X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a(X,\mathbf{1},Y)"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \lambda_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{A.6.8}$$

$$\begin{tikzcd} (X \otimes Y) \otimes \mathbf{1} \arrow[rr, "a(X,Y,\mathbf{1})"] \arrow[dr, "\rho_{X\otimes Y}"'] & & X \otimes (Y \otimes \mathbf{1}) \arrow[dl, "\mathrm{id}_X \otimes \rho_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{A.6.9}$$

$$\begin{tikzcd} (\mathbf{1} \otimes X) \otimes Y \arrow[rr, "a(\mathbf{1},X,Y)"] \arrow[dr, "\lambda_X \otimes \mathrm{id}_Y"'] & & \mathbf{1} \otimes (X \otimes Y) \arrow[dl, "\lambda_{X\otimes Y}"] \\ & X \otimes Y & \end{tikzcd} \tag{A.6.10}$$

$$\begin{tikzcd} (\mathbf{1} \otimes X) \otimes \mathbf{1} \arrow[rr, "a(\mathbf{1},X,\mathbf{1})"] \arrow[d, "\rho_{\mathbf{1}\otimes X}"'] & & \mathbf{1} \otimes (X \otimes \mathbf{1}) \arrow[d, "\lambda_{X\otimes \mathbf{1}}"] \\ \mathbf{1} \otimes X \arrow[r, "\lambda_X"'] & X & X \otimes \mathbf{1} \arrow[l, "\rho_X"] \end{tikzcd} \tag{A.6.11}$$

For the proof, see [9, Lemma 3.1.5].

Remark A.6.5 Some references, e.g. [8], make the unit constraints λ, ρ and the triangle axiom part of the definition of a monoidal category; Lemma A.6.4 recovers them from the leaner formulation of Definition A.6.1.

Definition A.6.6 Let $(\mathcal{V}_1, \otimes, a_1, \mathbf{1}_1, \iota_1)$ and $(\mathcal{V}_2, \otimes, a_2, \mathbf{1}_2, \iota_2)$ be monoidal categories. A ***monoidal functor*** from $\mathcal{V}_1$ to $\mathcal{V}_2$ is a pair (F, ξ) consisting of a functor $F : \mathcal{V}_1 \to \mathcal{V}_2$ and an isomorphism of bifunctors

$$\xi(X, Y) : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y), \tag{A.6.12}$$

together with an isomorphism $\varphi : \mathbf{1}_2 \xrightarrow{\sim} F(\mathbf{1}_1)$, such that for all objects X, Y, Z of $\mathcal{V}_1$ the following diagram commutes:

$$\begin{tikzcd}
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_1(X,Y,Z))"] & & F(X \otimes (Y \otimes Z)) \\
F(X \otimes Y) \otimes F(Z) \arrow[u, "\xi(X \otimes Y, Z)"] & & F(X) \otimes F(Y \otimes Z) \arrow[u, "\xi(X, Y \otimes Z)"'] \\
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[u, "\xi(X,Y) \otimes \mathrm{id}"] \arrow[rr, "a_2(F(X), F(Y), F(Z))"'] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[u, "\mathrm{id} \otimes \xi(Y,Z)"']
\end{tikzcd} \tag{A.6.13}$$

The isomorphism φ then admits a standard choice, compatible with the unit constraints of Definition A.6.2; we routinely suppress ξ and φ from the notation. In much of the literature, what we have defined is called a **strong** monoidal functor.

Keeping the direction of the arrows but no longer requiring ξ and φ to be invertible, and imposing the unit conditions

$$\begin{aligned}
F(\lambda_1(X)) \circ \xi(\mathbf{1}_1, X) \circ \left(\varphi \otimes \mathrm{id}_{F(X)}\right) &= \lambda_2(F(X)), \\
F(\rho_1(X)) \circ \xi(X, \mathbf{1}_1) \circ \left(\mathrm{id}_{F(X)} \otimes \varphi\right) &= \rho_2(F(X)),
\end{aligned} \tag{A.6.14}$$

yields a ***lax monoidal functor*** reversing both arrows, i.e. taking morphisms $\eta(X,Y) : F(X \otimes Y) \to F(X) \otimes F(Y)$ and $\psi : F(\mathbf{1}_1) \to \mathbf{1}_2$ subject to the mirrored conditions, gives an ***oplax monoidal functor*** . These are called right-lax and left-lax monoidal functors, respectively, in [9, Remark 3.1.8].

Definition A.6.7 Let $(\mathcal{V}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. A ***braiding*** (or ***commutativity constraint***) on $\mathcal{V}$ is an isomorphism of bifunctors

$$c(X,Y) : X \otimes Y \xrightarrow{\sim} Y \otimes X, \tag{A.6.15}$$

natural in both variables, compatible with the structure of $\mathcal{V}$ as follows; the commutative diagrams can be found in [9, §3.3]:

- braiding X past $Y \otimes Z$ in one stroke agrees with braiding it past Y first and then past Z, and dually braiding $X \otimes Y$ past Z agrees with braiding Y past Z first and then X; in the informal picture where the associativity constraint is suppressed, this reads

$$\begin{aligned}
c(X, Y \otimes Z) &= (\mathrm{id}_Y \otimes c(X,Z)) \circ (c(X,Y) \otimes \mathrm{id}_Z), \\
c(X \otimes Y, Z) &= (c(X,Z) \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c(Y,Z)),
\end{aligned} \tag{A.6.16}$$

- $c(\mathbf{1}, X) = \rho_X^{-1} \lambda_X$ and $c(X, \mathbf{1}) = \lambda_X^{-1} \rho_X$.

A monoidal category endowed with a braiding is said to be ***braided*** . If moreover the symmetry condition

$$c(Y,X) \circ c(X,Y) = \mathrm{id}_{X \otimes Y} \tag{A.6.17}$$

holds for all objects X, Y, then $(\mathcal{V}, \otimes, a, \mathbf{1}, \iota, c)$ is called a ***symmetric monoidal category*** .

The monoidal categories of Example A.6.3 built from products, from coproducts, and from $\otimes_R$ are symmetric, the braiding being the evident "swap"; for R-Mod with R commutative it is $m \otimes n \mapsto n \otimes m$, the symmetry used throughout Section B.4. The endofunctor category of Example A.6.3, in contrast, is in general not braided: functors seldom commute up to isomorphism. Genuine non-symmetric braidings arise in the braid category, which lends the notion its name and connects it to the Yang–Baxter equation; see [9, §3.3].

Appendix B

Background on Modules

The category of modules over a ring is our running example: it is the prototypical abelian category (Section 2.3), the setting in which we chase diagrams (Theorem A.1.1), and the source of the derived functors Ext and Tor (Section 4.4). This appendix collects, in one place, the module-theoretic facts used throughout: the explicit construction of filtered colimits, tensor products and the *Hom–tensor adjunction*, base change, and the characterizations of projective, injective, and flat modules. We recall the bare definition of a (left) module; full details are in [9, Chapter 6].

B.1 Modules and homomorphisms

Fix a ring R (associative, unital, $1 := 1_R \neq 0_R =: 0$). A ***left R-module*** is an abelian group M together with a scalar multiplication $R \times M \to M$, written as $(r, m) \mapsto r \cdot m$, satisfying the usual axioms

$$\begin{gathered} r(m_1 + m_2) = rm_1 + rm_2, \quad (r_1 + r_2)m = r_1 m + r_2 m, \\ (r_1 r_2)m = r_1(r_2 m), \quad 1_R m = m. \end{gathered} \tag{B.1.1}$$

Right modules are defined analogously.

If $\Bbbk$ is a commutative ring, a ***$\Bbbk$-algebra*** is a ring R together with a ring homomorphism $\Bbbk \to R$ whose image commutes with every element of R; equivalently, R is then a $\Bbbk$-module and the multiplication $R \times R \to R$ is $\Bbbk$-bilinear. A homomorphism of $\Bbbk$-algebras is a $\Bbbk$-linear ring homomorphism.

Every ring R has an ***opposite ring*** R^{op}: the same underlying abelian group, the same zero and identity, and the product of elements r, r' in R^{op} is their product $r'r$ in R, in reverse order; clearly $(R^{\mathrm{op}})^{\mathrm{op}} = R$.

A right R-module is the same thing as a left R^{op}-module. An ***(S, R)-bimodule*** M is an abelian group carrying both a left S-module structure and a right R-module structure, whose actions commute: $(sm)r = s(mr)$.

A module homomorphism or R-linear map $f : M \to N$ is a map of the underlying abelian groups with $f(rm) = rf(m)$. The R-linear maps form an abelian group $\mathrm{Hom}_R(M, N)$; composition makes R-Mod into a preadditive category. Kernels, cokernels, images, and quotients are formed on the underlying abelian groups and inherit an R-action, so:

Proposition B.1.1 For every ring R, the category R-Mod of left R-modules is abelian. A sequence $M' \to M \to M''$ is exact in R-Mod if and only if it is exact on the underlying abelian groups.

The forgetful functor R-Mod $\to$ Ab is thus faithful and exact. In particular, elementwise diagram-chasing in R-Mod is just diagram-chasing in Ab.

B.2 Filtered colimits of modules

Colimits in R-Mod admit a completely explicit description, as explained in [9, §6.2]: for a functor $M : I \to R$-Mod with I small, set

$$\operatorname*{colim}_{i \in \mathrm{Ob}(I)} M_i := \left(\bigoplus_{i \in \mathrm{Ob}(I)} M_i \right) / N, \tag{B.2.1}$$

where N is the submodule generated by the elements $x - f(x)$, for every arrow $f : i \to j$ of I and every $x \in M_i$, with $f(x)$ sitting in the summand M_j. A family of homomorphisms $M_i \to P$ compatible with the arrows of I is the same as a homomorphism $\bigoplus_{i \in \mathrm{Ob}(I)} M_i \to P$ vanishing on N, so the quotient satisfies the universal property of the colimit.

Proposition B.2.1 Suppose furthermore that I is filtered (Definition 1.1.8). Then

- every element of $\operatorname*{colim}_{i \in \mathrm{Ob}(I)} M_i$ is the class of some $x \in M_i$,
- $x \in M_i$ and $y \in M_j$ define the same class if and only if they acquire equal images in some M_k receiving arrows $i \to k \leftarrow j$.

In particular, for a filtered poset P, the elements of $\operatorname*{colim}_{p \in P} M_p$ are classes of elements $x \in M_p$, and such $x \in M_p$ and $y \in M_q$ are identified precisely when their images in some M_r with $p, q \leq r$ coincide.

Proof Filteredness implies that finitely many objects of I admit arrows into a common one. Hence every finite sum $\sum_l x_l$ with $x_l \in M_{i_l}$ is equivalent, modulo N, to the single element $\sum_l f_l(x_l) \in M_k$ determined by the transition maps $f_l : M_{i_l} \to M_k$: this proves the first claim, and the "if" direction of the second is the defining relation of N. For the "only if" direction, write $x - y$ as a finite combination of the generators $z - f(z)$, and choose k receiving arrows from i, j, and every index occurring therein; the homomorphism $\bigoplus_l M_{i_l} \to M_k$ given by the transition maps kills those generators, hence also $x - y$, which is to say that x and y have the same image in M_k. □

Proposition B.2.2 For every ring R, filtered colimits are exact in R-Mod in the following sense. Consider three functors $I \to R$-Mod from a small category I, written as $i \mapsto M_i', M_i, M_i''$, fitting into a short exact sequence in R-Mod^I:

$$0 \to (M_i')_i \to (M_i)_i \to (M_i'')_i \to 0; \tag{B.2.2}$$

exactness amounts to the condition that $0 \to M_i' \to M_i \to M_i'' \to 0$ is exact for all $i \in \mathrm{Ob}(I)$. If I is filtered, then the induced sequence

$$0 \to \operatorname*{colim}_{i \in \mathrm{Ob}(I)} M_i' \to \operatorname*{colim}_{i \in \mathrm{Ob}(I)} M_i \to \operatorname*{colim}_{i \in \mathrm{Ob}(I)} M_i'' \to 0 \tag{B.2.3}$$

is exact in R-Mod.

Proof Everything is an element chase with Proposition B.2.1. If $x \in M_i'$ maps to zero in colim M_i, then x dies in some M_k; injectivity of $M_k' \to M_k$ then kills x already in M_k', so its class in colim M_i' is zero. If $m \in M_i$ maps to zero in colim M_i'', choose $i \to k$ under which the image of m in M_k'' is zero; the image of m in M_k then lies in the image of M_k', so the class of m comes from colim M_i'. Finally, every class in colim M_i'' is represented by some $z \in M_i''$, which lifts to M_i, giving surjectivity. □

This is the only non-trivial point in verifying that R-Mod is a Grothendieck category (Definition 2.8.2), and it is the input to Proposition B.5.5 (ii). For instance, $R[U^{-1}]$ is a filtered colimit of copies of R, indexed by the divisibility poset of U.

B.3 Free modules

The ***free module*** on a set I is $R^{\oplus I} := \bigoplus_{i \in \mathrm{Ob}(I)} Re_i$, the direct sum of copies of R indexed by I; its elements are finite formal sums $\sum_i r_i e_i$. It satisfies the universal property

$$\mathrm{Hom}_R(R^{\oplus I}, M) \simeq M^I, \quad f \mapsto (f(e_i))_{i \in \mathrm{Ob}(I)}, \tag{B.3.1}$$

so $R^{\oplus I}$ is the value of the left adjoint to the forgetful functor R-Mod $\to$ Set. When R is commutative, the rank of $R^{\oplus I}$ is well-defined and equals $|I|$ (idea: reduce a basis modulo a maximal ideal and count dimensions).

Definition B.3.1 An R-module M is ***free*** if there exists a set X and an isomorphism of R-modules $\varphi : R^{\oplus X} \xrightarrow{\sim} M$. In this case we call $\{\varphi(e_x) : x \in X\} \subset M$ a basis of M.

This is the "internal" version of freeness: for $M = R^{\oplus I}$ with $\varphi = \mathrm{id}$, the basis is $\{e_i : i \in \mathrm{Ob}(I)\}$, and it is convenient to identify I with this subset of $R^{\oplus I}$ via $i \mapsto e_i$. It is routine to check that an R-module M is free with basis $X \subset M$ if and only if

- X generates M;
- X is R-linearly independent, i.e. $\sum_{x \in X} r_x x = 0$ (finite sum) implies $r_x = 0$ for all $x \in X$.

Free $\mathbb{Z}$-modules are nothing but free abelian groups.

Proposition B.3.2 Every module is a quotient of a free module.

This is immediate: send $R^{\oplus M}$ onto M by $e_m \mapsto m$.

B.4 The tensor product

The tensor product linearizes *bilinear* maps.

Let M be a right R-module and N a left R-module. A map $\varphi : M \times N \to P$ (of sets, into an abelian group P) is ***R-balanced*** if

$$\begin{gathered}\varphi(mr, n) = \varphi(m, rn), \quad \varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n), \\ \varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2).\end{gathered} \tag{B.4.1}$$

Definition B.4.1 The ***tensor product*** $M \otimes_R N$ is the abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$, written as $(m, n) \mapsto m \otimes n$, universal among R-balanced maps: every R-balanced $\varphi : M \times N \to P$ factors uniquely through a group homomorphism $\tilde{\varphi} : M \otimes_R N \to P$.

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dashed, "\exists! \tilde{\varphi}"] \\ & P \end{tikzcd} \tag{B.4.2}$$

Concretely, $M \otimes_R N = F/K$ where F is the free abelian group with basis $M \times N$ and K is the subgroup generated by the evident bilinearity and balance relations; then $m \otimes n$ is the class of (m, n).

If M is an (S, R)-bimodule, then $M \otimes_R N$ is a left S-module via $s \cdot (m \otimes n) = (sm) \otimes n$; likewise, if N is an (R, S)-bimodule then $M \otimes_R N$ is a right S-module via $(m \otimes n) \cdot s = m \otimes (ns)$.

The tensor product is functorial in each variable, and (for a commutative R) it makes R-Mod a *symmetric monoidal category*: there are natural, coherent isomorphisms

$$\begin{gathered} (M \otimes_R N) \otimes_R P \simeq M \otimes_R (N \otimes_R P), \\ M \otimes_R N \simeq N \otimes_R M, \quad R \otimes_R M \simeq M. \end{gathered} \tag{B.4.3}$$

For non-commutative R one keeps associativity but loses symmetry (the tensor product is only defined for appropriately matched left/right modules).

Remark B.4.2 An (S, R)-bimodule is the same thing as a left module over the ring $S \otimes_{\mathbb{Z}} R^{\mathrm{op}}$, the action being $(s \otimes r)m := (sm)r$ (with r acting on the right), and the multiplication on $S \otimes_{\mathbb{Z}} R^{\mathrm{op}}$ given by $(s_1 \otimes r_1)(s_2 \otimes r_2) = s_1 s_2 \otimes r_2 r_1$. When R and S are algebras over a commutative ring $\Bbbk$, an (S, R)-bimodule on which the two $\Bbbk$-actions induced via S and via R agree is the same thing as a left $S \otimes_{\Bbbk} R^{\mathrm{op}}$-module; this is the envelope used in Hochschild theory (Section 3.5, with $S = R$).

Theorem B.4.3 (*Hom–tensor adjunction*) Let M be an (S, R)-bimodule. For every left R-module N and every left S-module P, there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_S(M \otimes_R N, P) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, P)). \tag{B.4.4}$$

In particular $M \otimes_R (\cdot) : R\text{-Mod} \to S\text{-Mod}$ is left adjoint to $\mathrm{Hom}_S(M, \cdot) : S\text{-Mod} \to R\text{-Mod}$.

Proof Here $\mathrm{Hom}_S(M, P)$ carries the left R-action $(r \cdot g)(m) = g(mr)$. Given an S-linear map $f : M \otimes_R N \to P$, define $\tilde{f} : N \to \mathrm{Hom}_S(M, P)$ by $\tilde{f}(n)(m) = f(m \otimes n)$; this is R-linear in n because $\tilde{f}(rn)(m) = f(m \otimes rn) = f(mr \otimes n) = \tilde{f}(n)(mr) = \big(r \cdot \tilde{f}(n)\big)(m)$. Conversely, an R-linear $g : N \to \mathrm{Hom}_S(M, P)$ yields an R-balanced map $M \times N \to P$, $(m, n) \mapsto g(n)(m)$, hence by Definition B.4.1 a unique group homomorphism $\hat{g} : M \otimes_R N \to P$, which is in fact S-linear. The two constructions $f \mapsto \tilde{f}$ and $g \mapsto \hat{g}$ are mutually inverse and natural in N and P. $\square$

This is the adjunction promised in Example 1.6.4. As an immediate corollary of Theorem 1.6.6, the functor $M \otimes_R (\cdot)$ preserves colimits, hence is *right exact*; dually $\mathrm{Hom}_S(M, \cdot)$ is left exact. The failure of $M \otimes_R (\cdot)$ to be left exact (i.e. to preserve monomorphisms) is precisely measured by Tor (Section 4.4).

Remark B.4.4 (*base change*) Two special cases of the adjunction underlie *change of rings* (cf. Example 2.3.2). Let $\rho : R \to S$ be a ring homomorphism.

- ***Extension of scalars*** $S \otimes_R (\cdot) : R\text{-Mod} \to S\text{-Mod}$ is left adjoint to the ***restriction of scalars*** $S\text{-Mod} \to R\text{-Mod}$ (view an S-module as an R-module via ρ). This is Theorem B.4.3 with S an (S, R)-bimodule. Extension of scalars is exact when S is flat as an R-module (see Definition B.5.3, e.g. a localization $R[U^{-1}]$).
- Restriction of scalars has a further right adjoint, ***coextension of scalars*** $\mathrm{Hom}_R(S, \cdot) : R\text{-Mod} \to S\text{-Mod}$, again by Theorem B.4.3. Restriction of scalars is *exact* (kernels and cokernels are computed on the underlying abelian groups) and admits adjoints on both sides.

These two adjunctions are the engine behind the change-of-rings spectral sequence of Remark 6.5.5.

B.5 Projective, injective, and flat modules

We collect the module-theoretic characterizations of the projective and injective objects of Section 2.7 and of the flat modules (Definition B.5.3); for projective and injective modules this expands Example 2.7.4.

Proposition B.5.1 Let R be a ring and P a left R-module. The following are equivalent.

(i) P is projective, i.e. $\mathrm{Hom}_R(P, \cdot)$ is exact;
(ii) every epimorphism $M \twoheadrightarrow P$ has a section (i.e. right inverse);
(iii) P is a direct summand of a free R-module;
(iv) the lifting property of Equation (2.7.1) holds.

Proof (i) $\Rightarrow$ (ii): an epimorphism $\pi : M \twoheadrightarrow P$ induces a surjection $\mathrm{Hom}_R(P, M) \twoheadrightarrow \mathrm{Hom}_R(P, P)$, and a preimage s of id_P is a section.

(ii) $\Rightarrow$ (iii): by Proposition B.3.2 there is an epimorphism $\pi : F \twoheadrightarrow P$ with F free; the section provided by (ii) splits the sequence $0 \to \ker(\pi) \to F \to P \to 0$, so P is a direct summand of F (Proposition 2.6.2).

(iii) $\Rightarrow$ (i): a free module $R^{\oplus I}$ is projective, since lifting a map $f : R^{\oplus I} \to Y$ along an epimorphism $X \twoheadrightarrow Y$ merely requires choosing, for each basis vector e_i, a preimage of $f(e_i)$ in X; and $\mathrm{Hom}_R(P, \cdot)$ is a direct summand of the exact functor $\mathrm{Hom}_R(F, \cdot)$, hence exact.

Finally, the equivalence of (i) and (iv) is Equation (2.7.1). □

In particular, free modules are projective, and Proposition B.3.2 then shows that R-Mod has enough projectives (Section 2.7).

Proposition B.5.2 (*Baer's criterion*) A left R-module I is injective if and only if every R-linear map $\mathfrak{a} \to I$ from a left ideal $\mathfrak{a} \subset R$ extends to $R \to I$.

Baer's criterion reduces the injectivity of I to a test on the left ideals $\mathfrak{a} \subset R$, which is what makes it practical. Moreover, R-Mod has enough injectives: every module embeds into an injective one; see [9, Theorem 6.9.14]. For $R = \mathbb{Z}$, the injective abelian groups are exactly the ***divisible*** groups ([9, Lemma 6.9.11]), so $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of Ab (Exercise 2.7.7).

Definition B.5.3 A right R-module M is ***flat*** if the functor $M \otimes_R (\cdot)$ on left R-modules is exact; a left R-module N is flat if the functor $(\cdot) \otimes_R N$ on right R-modules is exact.

Proposition B.5.4 For a right R-module M:

(i) if M is free (hence if it is projective), then M is flat;

(ii) M is flat if and only if $\mathrm{Tor}_n^R(M, N) = 0$ for every left R-module N and every $n \geq 1$ (Section 4.4);

(iii) (Govorov–Lazard) M is flat if and only if it is a filtered colimit of finitely generated free modules.

The characterization (ii) above is the bridge to homological algebra: flatness is detected by the vanishing of the derived functors of the tensor product. Govorov–Lazard (iii) gives a purely module-theoretic description; see [11, §5.10]. Localizations $R[U^{-1}]$ are flat R-modules: take this as an exercise, or see [11, Lemma 1.6.3].

Proposition B.5.5 Let R be a ring, and work with right R-modules.

(i) A direct sum $\bigoplus_{i\in \mathrm{Ob}(I)} M_i$ is flat if and only if every M_i is flat; the same holds for projectivity.

(ii) A filtered colimit of flat modules is flat.

(iii) Over a commutative R, the tensor product of flat R-modules is flat.

(iv) If $\rho : R \to S$ is a ring homomorphism and M is flat (resp. projective), then $M \otimes_R S$ is flat (resp. projective) as a right S-module.

Proof (i) Tensor products preserve colimits (Theorem B.4.3), so $\left(\bigoplus_{i\in \mathrm{Ob}(I)} M_i\right) \otimes_R (\cdot) \simeq \bigoplus_{i\in \mathrm{Ob}(I)}(M_i \otimes_R (\cdot))$, and a direct sum of sequences of modules is exact if and only if each summand sequence is (Proposition B.1.1); in particular direct summands of flat modules are flat. The projective case follows from Proposition B.5.1: writing each $M_i \oplus M_i' \simeq F_i$ with F_i free, the sum $\bigoplus_{i\in \mathrm{Ob}(I)} M_i$ is a direct summand of the free module $\bigoplus_{i\in \mathrm{Ob}(I)} F_i$; conversely, each M_j is a direct summand of $\bigoplus_{i\in \mathrm{Ob}(I)} M_i$, and direct summands of projectives are projective.

(ii) Likewise $\left(\operatorname{colim}_i M_i\right) \otimes_R (\cdot) \simeq \operatorname{colim}_i(M_i \otimes_R (\cdot))$, and filtered colimits of sequences are exact in R-Mod (Proposition B.2.2).

(iii) By associativity, $(M \otimes_R N) \otimes_R (\cdot) \simeq M \otimes_R (N \otimes_R (\cdot))$ is a composite of exact functors.

(iv) For flatness, note that for every left S-module P there is a canonical isomorphism $(M \otimes_R S) \otimes_S P \simeq M \otimes_R P$, where P is regarded as a left R-module via ρ; since restriction of scalars is exact (Remark B.4.4), $M \otimes_R (\cdot)$ remains exact on exact sequences of S-modules. For projectivity, write $M \oplus M' \simeq \bigoplus_{i\in \mathrm{Ob}(I)} R$ (Proposition B.5.1); tensoring with S and using additivity and preservation of coproducts gives $(M \otimes_R S) \oplus (M' \otimes_R S) \simeq \bigoplus_{i\in \mathrm{Ob}(I)} S$, and we conclude by Proposition B.5.1 again. $\square$

Exercise B.5.6 Show directly from Definition B.4.1 that $R \otimes_R M \simeq M$ (via $r \otimes m \mapsto rm$) and that $M \otimes_R (N \otimes_R P) \simeq (M \otimes_R N) \otimes_R P$ for a commutative R. Verify the coherence with the associator.

Hint. Use the universal property; do not choose a presentation.

Exercise B.5.7 Deduce from Theorem B.4.3 that, for a commutative ring R and an R-module M, the functor $M \otimes_R (\cdot)$ is right exact. Show that for a prime number p and $M = \mathbb{Z}/p\,\mathbb{Z}$, the functor fails to be left exact, therefore $\mathbb{Z}/p\,\mathbb{Z}$ is not flat over $\mathbb{Z}$.

Exercise B.5.8 Prove the nontrivial direction of Baer's criterion (Proposition B.5.2): assuming the left-ideal test, show that injectivity of I follows by Zorn's lemma, extending a map from a submodule of an arbitrary M step by step.

Exercise B.5.9 Show that over a PID, a module is flat if and only if it is torsion-free. Deduce that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, and give a second proof that $\mathbb{Z}/p\,\mathbb{Z}$ is not flat for every prime number p.

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