Localization and higher branching laws for Harish-Chandra modules

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¿What is ... the Localization?

All groups and varieties are over $\mathbb{C}$. We say X is a G-variety if G acts algebraically on the right of X.

Definition

Let G be an affine group, X be a smooth G-variety. The infinitesimal action induces a homomorphism

$$j : U(\mathfrak{g}) \to D_X := \Gamma(X, \mathcal{D}_X)$$

$U(\mathfrak{g})$: universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)$

$\mathcal{D}_X$: sheaf of algebraic differential operators on X.

Definition

Localization is the right exact functor

$$\mathfrak{g}\text{-Mod} \to \mathcal{D}_X\text{-Mod}, \quad V \mapsto \mathcal{D}_X \otimes_{U(\mathfrak{g})} V.$$

- $\mathfrak{g}$-Mod (resp. $\mathcal{D}_X$-Mod) is the abelian category of left $\mathfrak{g}$-modules (resp. $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules).
- The relevance of localization can be partly explained by its relation to *co-invariants*.

Relatively easy fact

Let $X = H\backslash G$ and $i_x : \mathrm{pt} \to X$ the inclusion of $x = H \cdot 1$. Then

$$\forall V \in \mathfrak{g}\text{-Mod}, \quad i_x^{\bullet}(\mathcal{D}_X \otimes_{U(\mathfrak{g})} V) \simeq V/\mathfrak{h}V.$$

Hence its dual $\simeq \mathrm{Hom}_{\mathfrak{h}}(V, \mathbb{C})$. Note that x (thus H) can vary in X.

Notation

For $f : Y \to X$, denote by $f^{\bullet}$ the inverse image of D-modules: it endows the $\mathcal{O}_Y$-quasi-coherent sheaf f^*M with a $\mathcal{D}_Y$-action.

Slogan: Localization organizes co-invariants into a geometric object.

Examples of localizations

Let G be a connected reductive group.

- Take X to be the flag variety. By allowing twists on $\mathcal{D}_X$, we obtain the *Beilinson–Bernstein localization*.
- Take $G = H \times H$ and $X = H$ (the "group case"). Ben-Zvi–Ganev arXiv:1901.01226 studied the localization and its specialization at infinity in this context, and applied them to study asymptotics of matrix coefficients of *admissible representations*.

Remarks

- For affine X, it suffices to work with D_X.
- One can show that $D_X \otimes_{U(\mathfrak{g})} (\cdot)$ is usually non-exact $\rightsquigarrow$ should look at $D_X \otimes^L_{U(\mathfrak{g})} (\cdot)$.

¿What is … the Branching Law?

In the classical/elementary setting, we consider $H \subset G$: compact Lie groups, fix reasonable (eg. irreducible) representations V of G, and W of H. We look at

$$\mathrm{Hom}_H(V|_H, W) \quad \approx \text{ study how } V|_H \text{ “branches”.}$$

Oftentimes, (eg. in Langlands program), we consider reasonable representations of

- p-adic groups,
- real Lie groups, etc. Simplest version: study

$$\mathrm{Hom}_{\mathfrak{h}}(V|_{\mathfrak{h}}, W), \quad V \in \mathfrak{g}\text{-Mod}, \; W \in \mathfrak{h}\text{-Mod}.$$

It is more useful to consider $(\mathfrak{g}, K)$-Mod instead of $\mathfrak{g}$-Mod.

Let G: connected reductive, $K \subset G$. In representation theory, we are often interested in

$$(\mathfrak{g}, K)\text{-module} := \mathfrak{g}\text{-module} + \text{compatible algebraic } K\text{-action},$$

and particularly in the subcategory of *Harish-Chandra modules*.

Definition

We say a $(\mathfrak{g}, K)$-module V is Harish-Chandra if

- V is finitely generated over $\mathfrak{g}$,
- V is locally $\mathcal{Z}(\mathfrak{g})$-finite, i.e. V is the union of finite-dimensional $\mathcal{Z}(\mathfrak{g})$-submodules.

Here $\mathcal{Z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.

When K is a symmetric subgroup, the second condition can be replaced by admissibility.

Regularity: First take

Consider the localization of Harish-Chandra $(\mathfrak{g}, K)$-modules.
In the work of Ben-Zvi and Ganev (the "group case" $G = H \times H$, $X = H$), one takes

$$K = H \quad (\rightsquigarrow \text{Harish-Chandra bimodules}).$$

They made critical use of the

Theorem (V. Ginzburg, 1989)

In this setting, Harish-Chandra $(\mathfrak{g}, K)$-modules (or Harish-Chandra bimodules) localize to *regular holonomic* D_H-modules.

Natural question

What about more general homogeneous G-spaces? Clue:

$$\text{group case} \subset \text{alg. symm. spaces} \subset \underbrace{\text{spherical varieties}}_{\text{i.e. } \exists \text{ open Borel orbit}}.$$

Deriving the localization functor

Let G be connected reductive, $X = H\backslash G$ and assume X is affine ($\iff$ H is reductive). Left-derive $D_X \otimes_{U(\mathfrak{g})} (\cdot)$ to get

$$D_X \otimes^L_{U(\mathfrak{g})} (\cdot) : \mathrm{D}(\mathfrak{g}\text{-Mod}) \longrightarrow \mathrm{D}(X) := \mathrm{D}(D_X\text{-Mod}).$$

Proposition–Exercise

Let $x = H \cdot 1 \in X$. For every $V \in \mathfrak{g}$-Mod and n,

$$\mathrm{H}_n(\mathfrak{h}; V) \simeq \mathrm{H}_n i_x^\bullet \left(D_X \otimes^L_{U(\mathfrak{g})} V \right), \quad \text{its dual } \simeq \mathrm{Ext}^n_{\mathfrak{h}}(V, \mathbb{C}).$$

Here $i_x^\bullet : \mathrm{D}^-(D_X\text{-Mod}) \to \mathrm{D}^-(D_{\mathrm{pt}}\text{-Mod}) = \mathrm{D}^-(\mathbb{C})$.

This suggests some link to higher branching laws.

Higher/homological/Ext-branching (D. Prasad, ICM 2018)

Take $H \subset G$ with G connected reductive.

- (p-adic case) Study $\mathrm{Ext}^n_H(V|_H, \mathbb{C})$, where V is an adm. rep. of $G(\mathbb{Q}_p)$. If W is an adm. rep. of $H(\mathbb{Q}_p)$ then

$$\mathrm{Ext}^n_H(V|_H, W^\vee) \simeq \mathrm{Ext}^n_H((V \boxtimes W)|_H, \mathbb{C}).$$

  Also interested in $\sum_n (-1)^n \dim \mathrm{Ext}^n_H(V|_H, \mathbb{C})$, better behaved once well-defined.

- (Real case) Study Ext^n  $(V|_H, \mathbb{C})$ where $V \in$ .

Problems in the real case:

1. Sensible choice(s) of these cat's?
2. Relation to localization?
3. Finiteness?

Algebraic formulation of branching

Consider G: connected reductive complex group, and subgroups

$$\begin{array}{ccc} H & \subset & G \\ \cup & & \cup \\ K^H & \subset & K. \end{array}$$

We study

$$(\mathfrak{g}, K)\text{-Mod} \longrightarrow (\mathfrak{h}, K^H)\text{-Mod}, \quad V \mapsto V|_H$$

and $\mathrm{Ext}^n_{\mathfrak{h},K^H}(V|_H, \mathbb{C})$. The case $K^H = \{1\}$ (i.e. $\mathrm{Ext}^n_{\mathfrak{h}}(V|_H, \mathbb{C})$) has been seen to be related to fibers of $\mathcal{D}_{H\backslash G} \otimes^L_{U(\mathfrak{g})} (\cdot)$.

Remark

There are also analytic formulations (with Fréchet representations, continuous Hom...)

Higher localization: equivariant case

- $H, K \subset G$: reductive subgroups,
- $X := H\backslash G$ (affine), $x = H \cdot 1 \in X$.
- (D_X, K)-module := (strongly) K-equivariant D_X-modules.

Goal: Left-derive $D_X \otimes_{U(\mathfrak{g})} (\cdot) : (\mathfrak{g}, K)\text{-Mod} \to (D_X, K)\text{-Mod}$.

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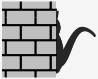

- $\mathrm{D}^b(\mathfrak{g}, K)$: the standard one (ref: Knapp–Vogan).
- $\mathrm{D}^b_K(X)$ = bounded equivariant derived category of D-modules: usually requires various "resolutions" of X (ref: Bernstein–Lunts 94). More generally: theory of D-modules on the stack X/K.

I know of no obvious definition via resolutions (localization does not behave so well under equivariant maps).

Desiderata for higher derivation:

- Commutation with oblivion (i.e. forget equivariance).
- Relation to higher branching.
- Say something about the image of Harish-Chandra $(\mathfrak{g}, K)$-modules. Eg. does it land in $\mathsf{D}^{\mathrm{b}}_{K,\mathrm{rh}}(X)$ when H and K are both spherical subgroups?
    - rh: the subcategory with regular holonomic cohomologies.
    - H is called spherical if $X = H\backslash G$ is.

A proposal

Consider *h-derived categories* on both sides, and take left *h-derived functor* (Beilinson–Ginzburg, Bernstein–Lunts 95) to obtain

$$\mathbf{Loc}_{X,K} : {}^{\mathrm{h}}\mathsf{D}^{\mathrm{b}}(\mathfrak{g}, K) \to {}^{\mathrm{h}}\mathsf{D}^{\mathrm{b}}(D_X, K).$$

Weak and strong modules

Let $A \in \{U(\mathfrak{g}), D_X\}$ on which $K \subset G$ acts algebraically. We have $j : \mathfrak{k} \to A$.

- Weak (A, K)-module = left A-module M with algebraic K-action ρ + compatibility $(ka)(km) = k(am)$.
- (A, K)-module: + require that $d\rho$ = the $\mathfrak{k}$-action via j.

This unifies the notions of $(\mathfrak{g}, K)$-modules and K-equivariant D_X-modules.

Remark

One can allow more general algebras A with K-action. This leads to the notion of Harish-Chandra algebras. See Bernstein–Lunts 1995.

h-complexes

An h-complex over (A, K) is an action-packed object. It comprises:

- a complex (C, d) of weak (A, K)-modules,
- a family of maps $i_\xi \in \mathrm{End}^{-1}_{\mathbb{C}}(C)$, linear in $\xi \in \mathfrak{k}$, such that

$$i_{\mathrm{Ad}(k)(\xi)} = k i_\xi k^{-1},$$
$$i_\xi : A\text{-linear},$$
$$i_\xi i_\eta + i_\eta i_\xi = 0,$$
$$d i_\xi + i_\xi d = (\mathrm{d}\rho - \alpha j)(\xi)$$

  where $\alpha : A \to \mathrm{End}_{\mathbb{C}}(C)$ comes from the first item.

Thus,

- every complex of (A, K)-modules is an h-complex with $i_\xi = 0$;
- the cohomologies of an h-complex are (A, K)-modules.

- Can define Hom-complexes, mapping cones, translation functors for h-complexes.
- The h-homotopy category ${}^{\mathrm{h}}\mathsf{K}(A, K)$ and h-derived category ${}^{\mathrm{h}}\mathsf{D}(A, K)$ are thus defined. Ditto with boundedness conditions.
- Triangulated with t-structure, RHom, Ext; $\heartsuit = (A, K)$-Mod.
- One can also define left and right h-derived functors:
  - as certain Kan extensions of functors between homotopy categories,
  - computed by taking K-projective / K-injective resolutions.

More conceptually:

As the derived category of dg-modules over the Harish-Chandra dg-algebra $(U(\bar{\mathfrak{k}}) \otimes A, K)$, where

$$U(\bar{\mathfrak{k}}) := \underbrace{\bigwedge^{\bullet} \mathfrak{k}}_{\deg \leq 0} \otimes \underbrace{U(\mathfrak{k})}_{\deg = 0} .$$

Once the formalism is set up, it is easy to lift $D_X \otimes_{U(\mathfrak{g})} (\cdot)$ to the level of h-complexes, and obtain left h-derived functor

$$\mathbf{Loc}_X = \mathbf{Loc}_{X,K} : {}^{\mathrm{h}}\mathsf{D}^{\mathrm{b}}(\mathfrak{g}, K) \to {}^{\mathrm{h}}\mathsf{D}^{\mathrm{b}}(D_X, K).$$

For all reductive subgroup $T \subset K$ and $n \in \mathbb{Z}$:

$$\begin{array}{ccccc} {}^{\mathrm{h}}\mathsf{D}(\mathfrak{g}, K) & \xrightarrow{\mathbf{Loc}_{X,K}} & {}^{\mathrm{h}}\mathsf{D}(D_X, K) & \xrightarrow{\mathrm{H}^n} & (D_X, K)\text{-Mod} \\ \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} & & \downarrow{\scriptstyle \mathrm{oblv}} \\ {}^{\mathrm{h}}\mathsf{D}(\mathfrak{g}, T) & \xrightarrow[\mathbf{Loc}_{X,T}]{} & {}^{\mathrm{h}}\mathsf{D}(D_X, T) & \xrightarrow[\mathrm{H}^n]{} & (D_X, T)\text{-Mod} \end{array}$$

commutes. Also, the amplitude of $\mathbf{Loc}_{X,K}$ is in $[-\dim G, 0]$.

It is now *routine* to relate the fibers of $\mathbf{Loc}_X$ to higher branching:

$$\mathrm{Ext}^n_{\mathfrak{h},K^H}(V|_H, \mathbb{C}) \simeq \mathrm{Ext}^n_{D_{\mathrm{pt}},K^H}\left(i_x^\bullet(\mathbf{Loc}_X(V)), \mathbb{C}\right),$$
$$\mathrm{H}_n(\mathfrak{h}, K^H; V|_H) \simeq \mathrm{H}^{-n}\mathrm{L}\left(\mathrm{coInv}^{\mathbb{C},K^H}_{\mathbb{C},\{1\}}\right)\left(i_x^\bullet \mathbf{Loc}_X(V)\right),$$

where

- Ext^n: taken in the h-derived categories (cf. next slide);
- V: any $(\mathfrak{g}, K)$-module;
- $i_x^\bullet : {}^{\mathrm{h}}\mathrm{D}^-(D_X, K^H) \to {}^{\mathrm{h}}\mathrm{D}^-(D_{\mathrm{pt}}, K^H)$ is the left h-derived functor of the inverse image via $i_x : \mathrm{pt} \to X$ (note that $D_{\mathrm{pt}} = \mathbb{C}$);
- $\mathrm{H}_n(\mathfrak{h}, K^H; \cdot)$: relative Lie algebra homologies;
- $\mathrm{L}\left(\mathrm{coInv}^{\mathbb{C},K^H}_{\mathbb{C},\{1\}}\right) : {}^{\mathrm{h}}\mathrm{D}^-(\mathbb{C}, K^H) \to \mathrm{D}^-(\mathbb{C})$ is the left h-derived functor of taking co-invariants of h-complexes.

Coming full circle

So far, we only require K, H, K^H to be reductive subgroups of G.
The following equivalences are all triangulated and t-exact.

Theorem (Bernstein–Lunts 1995, Pandžić 2005)

We have ${}^{\mathrm{h}}\mathrm{D}(\mathfrak{g}, K) \simeq \mathrm{D}(\mathfrak{g}, K)$ and ${}^{\mathrm{h}}\mathrm{D}^{\mathrm{b}}(\mathfrak{g}, K) \simeq \mathrm{D}^{\mathrm{b}}(\mathfrak{g}, K)$, by viewing a complex of $(\mathfrak{g}, K)$-modules as an h-complex with $i_\xi = 0$.

Theorem (Beilinson $\leq$ 1995)

For all smooth affine K-variety X, we have ${}^{\mathrm{h}}\mathrm{D}^{\mathrm{b}}(D_X, K) \simeq \mathrm{D}^{\mathrm{b}}_K(X)$, compatibly with inverse images.

$\implies$ Working in h-derived categories gives the same Ext^n.

Question: What can we say about $\mathbf{Loc}_X(V) : \mathrm{D}^{\mathrm{b}}(\mathfrak{g}, K) \to \mathrm{D}^{\mathrm{b}}_K(X)$?

Theorem of regularity

Let $H, K \subset G$ be reductive subgroups as before. Set $X = H\backslash G$ (affine).

Theorem 1 (L.)

Assume H and K are spherical. For all Harish-Chandra $(\mathfrak{g}, K)$-module V, the cohomologies of

$$\mathbf{Loc}_X(V) \in {}^{\mathrm{h}}\mathsf{D}^{\mathrm{b}}(D_X, K)$$

are regular holonomic. Moreover, their characteristic variety are in $\mu^{-1}(\mathfrak{g}^*_{\mathrm{nil}})$ where $\mu : T^*X \to \mathfrak{g}^*$ is the moment map.

The proof is based on a criterion of L. (2022) arXiv:1905.08135, a variant of Ginzburg's (1989) for the case of symmetric subgroup $H = K$.

Sketch of the proof of Theorem 1

Let $\mathcal{L} := \mathbf{Loc}_{X,\{1\}}(V)$. By loc. cit., it boils down to show that $\forall n$,

(R1) $\mathrm{H}^n\mathcal{L}$ is finitely generated over D_X

(R2) $\mathrm{H}^n\mathcal{L}$ carries a K-equivariant structure

(R3) $\mathrm{H}^n\mathcal{L}$ is locally $\mathcal{Z}(\mathfrak{g})$-finite via $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g}) \xrightarrow{j} D_X$.

- (R1) follows easily since V is finitely generated over $\mathfrak{g}$ + Noetherian properties of $U(\mathfrak{g})$ and D_X.
- (R2) is immediate: $\mathrm{H}^n\mathcal{L}$ comes from $\mathrm{H}^n\mathbf{Loc}_{X,K}(V)$ by oblivion.
- (R3) is the technical part. It reduces to the following

Lemma

Consider X: smooth affine G-variety, V: finitely generated and locally $\mathcal{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-module. Then $\mathrm{Tor}_n^{U(\mathfrak{g})}(D_X, V)$ are locally $\mathcal{Z}(\mathfrak{g})$-finite D_X-modules.

Sketch of the proof of Lemma about $\mathrm{Tor}_n^{U(\mathfrak{g})}(D_X, V)$

After F. Knop (1994), set $\mathcal{Z}(X) := Z(D_X^G)$. When X is spherical, $\mathcal{Z}(X) = D_X^G$.

1. By induction on n, reduce to $V = M_\chi := U(\mathfrak{g})/\ker(\chi)U(\mathfrak{g})$, where χ: an infinitesimal character.
2. Reduce to local $\mathcal{Z}(\mathfrak{g})$-finiteness of

$$D_X \otimes_{\mathcal{Z}(X)} \mathrm{Tor}_n^{\mathcal{Z}(\mathfrak{g})}(\mathcal{Z}(X), N_\chi) \simeq \mathrm{Tor}_n^{U(\mathfrak{g})}(D_X, M_\chi)$$

where $N_\chi := \mathcal{Z}(\mathfrak{g})/\ker(\chi)$, by going through

$$\begin{array}{ccccc} & & D_X & & \\ & \overset{\text{free}}{\nearrow} & & \overset{j}{\nwarrow} & \\ \mathcal{Z}(X) & & & & U(\mathfrak{g}) \\ & \underset{j\text{:finite}}{\nwarrow} & & \underset{\text{free}}{\nearrow} & \\ & & \mathcal{Z}(\mathfrak{g}) & & \end{array} \qquad \begin{aligned} M_\chi &= U(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} N_\chi \\ &\simeq U(\mathfrak{g}) \otimes^L_{\mathcal{Z}(\mathfrak{g})} N_\chi. \end{aligned}$$

Left/right freeness as modules: Knop + Kostant.

3. The problem is now easier since $\mathcal{Z}(X)$ is commutative: $\mathcal{Z}(\mathfrak{g})$ acts on the $\mathcal{Z}(X)$-module $\mathrm{Tor}_n^{\mathcal{Z}(\mathfrak{g})}(\mathcal{Z}(X), N_\chi)$ through χ.
   In fact, $\mathcal{Z}(X)$ also acts locally finitely.
4. $M^\flat := 1 \otimes \mathrm{Tor}_n^{\mathcal{Z}(\mathfrak{g})}(\mathcal{Z}(X), N_\chi) \subset D_X \otimes_{\mathcal{Z}(X)} \mathrm{Tor}_n^{\mathcal{Z}(\mathfrak{g})}(\mathcal{Z}(X), N_\chi)$.
   Let $M^\natural = U(\mathfrak{g}) \cdot M^\flat$: now a $\mathfrak{g}$-submodule, locally $\mathcal{Z}(\mathfrak{g})$-finite.
5. Form $D_X \otimes M^\natural$ where $\mathfrak{g}$ acts on D_X by deriving the natural G-action of "conjugation". There is a $\mathfrak{g}$-linear surjection
$$D_X \otimes M^\natural \twoheadrightarrow D_X \otimes_{\mathcal{Z}(X)} \mathrm{Tor}_n^{\mathcal{Z}(\mathfrak{g})}(\mathcal{Z}(X), N_\chi).$$
6. Since D_X = union of finite-dimensional G-modules, the $\mathfrak{g}$-module $D_X \otimes M^\natural$ is locally $\mathcal{Z}(\mathfrak{g})$-finite ($\because$ a result of Kostant). QED.

Odds and ends

- In fact, Knop obtained an isomorphism à la Harish-Chandra describing the structure $\mathcal{Z}(X)$.
- $\mathcal{Z}(X)$ also affords extra symmetries on $\mathbf{Loc}_X(\cdot)$: every $z \in \mathcal{Z}(X)$ acts on $D_X \otimes_{U(\mathfrak{g})} V$ by

$$D \otimes v \mapsto Dz \otimes v.$$

  Hence $\mathbf{Loc}_X$ "spreads over" Spec $\mathcal{Z}(X)$.
- We expect some applications of Theorem 1 beyond branching laws. Eg. in geometric representation theory.

Some consequences of regularity

Theorem 2

Retain the assumptions on H, K, V and let $K^H \subset H \cap K$. For all n, there are canonical isomorphisms

$$\begin{array}{ccc} \mathrm{Ext}^n_{\mathfrak{h},K^H}(V|_H, \mathbb{C}) & \simeq & \mathrm{Ext}^n_{D_{\mathrm{pt}},K^H}\left(i_x^! \mathbf{Loc}_X(V)[\dim X], \mathbb{C}\right), \\ \updownarrow \text{dual} & & \\ \mathrm{H}_n(\mathfrak{h}, K^H; V|_H) & \simeq & \mathrm{H}^{-n+\dim X} \mathrm{L}\left(\mathrm{coInv}^{\mathbb{C},K^H}_{\mathbb{C},\{1\}}\right)\left(i_x^! \mathbf{Loc}_X(V)\right). \end{array}$$

All the complexes of D-modules above are bounded with *regular holonomic* cohomologies. Consequently, all these vector spaces are finite-dimensional.

Here $i_x^!$ is defined to match the synonymous functor under Riemann–Hilbert.

Remarks

- The finiteness is weaker than the recent results obtained by M. Kitagawa arXiv:2109.05555, who uses B–B localization and Zuckerman functors to bound the dimension of relative Lie algebra cohomologies (related to homologies via Poincaré duality).
- When $K^H = \{1\}$, finiteness is obtained by Aizenbud–Gourevitch–Krötz–Liu (2016) when K is symmetric and in good position relative to K.
- In fact, for $K^H = \{1\}$ one can avoid using h-complexes, although they still appear secretly in proving the Theorem 1 of regularity.

Under the previous assumptions, one can well-define

$$\mathrm{EP}_{\mathfrak{h},K^H}(V|_H,\mathbb{C}) = \sum_n (-1)^n \dim \mathrm{Ext}^n_{\mathfrak{h},K^H}(V|_H,\mathbb{C}).$$

When $K^H = \{1\}$ it points to an interesting topological quantity.

Theorem 3

Retain the previous assumptions and assume $K^H = \{1\}$. Set $\mathcal{L} := \mathbf{Loc}_{X,\{1\}}(V)$. Then $\mathrm{EP}_{\mathfrak{h},\{1\}}(V|_H,\mathbb{C})$ equals the local EP characteristic

$$\chi_x\left(\mathrm{Sol}_X(\mathcal{L})\right)$$

of the solution complex of $\mathcal{L}$ at x, which is expressible in terms of characteristic cycles and Euler obstructions by Kashiwara's local index theorem: $\boxed{\chi_x(\mathrm{Sol}(\mathcal{L})) \approx (\mathrm{Euler}\circ \mathrm{CC}(\mathcal{L}))(x)}$

Nevertheless, I do not know how to access $\mathrm{CC}(\mathcal{L})$.

Twisting by a character

Branching with a twist

- In p-adic branching laws, we are also interested in $\mathrm{Ext}^n_H(V|_H, \chi)$ where $\chi : H(\mathbb{Q}_p) \to \mathbb{C}^\times$.
- In the case of $(\mathfrak{g}, K)$-modules, we take χ: 1-dimensional $(\mathfrak{h}, K^H)$-module ($K^H \subset H \cap K$) and study

$$\mathrm{Ext}^n_{\mathfrak{h},K^H}(V|_H, \chi).$$

Take $\underline{H} \lhd H$ such that $S := H/\underline{H}$ is a torus. Suppose that $\chi : \mathfrak{h} \to \mathbb{C}$ is a character factoring through $\mathfrak{s}$.

Geometric setup — "monodromic structure"

Define $\pi : \tilde{X} = \underline{H}\backslash G \to H\backslash G = X$: it is a (left) S-torsor, (right) G-equivariant. $\rightsquigarrow$ *Twisted Differential Operators* (TDO's) on X.

- Let $\mathfrak{m}_\chi \subset \mathrm{Sym}(\mathfrak{s})$ be the maximal ideal corresponding to χ.
- Set $\tilde{\mathcal{D}} := \pi_*(\mathcal{D}_{\tilde{X}})^S$ and $\mathcal{D}_{X,\chi} := \tilde{\mathcal{D}}/\mathfrak{m}_\chi\tilde{\mathcal{D}}$ ($\Rightarrow$ TDO) and $D_{X,\chi} := \Gamma(X, \mathcal{D}_{X,\chi})$.
- We still have $j : U(\mathfrak{g}) \to D_{X,\chi}$.
- When X (thus $\tilde{X}$) is affine: $D_{X,\chi} = D_{\tilde{X}}^S/\mathfrak{m}_\chi D_{\tilde{X}}^S$; the functors $\mathbf{Loc}_{X,K,\chi}$ can still be defined.

Theorems

All the previous results carry over to this setting, except those concerning solution complexes.

Remark

It seems difficult to extend this formalism to the case when χ is a finite-dimensional module (which does not happen in the p-adic case).

Analytic formulation of branching

In this setting we consider

- a connected reductive real group G and a closed real subgroup H,
- a smooth Fréchet representation E of G, of moderate growth.
- $\mathrm{Hom}_H :=$ the continuous Hom.

Define $E_H = E / \sum_{h \in H} (h - \mathrm{id})E$, equipped with the quotient topology (could be non-Hausdorff). Then

$$\mathrm{Hom}_H(E|_H, \mathbb{C}) \simeq (E_H)^*$$

where $(\cdot)^*$ is the continuous dual.

This formalism is arguably more “realistic” for relative harmonic analysis.

Y. Chen and B. Sun (2021) constructed the *Schwartz homologies* $\mathrm{H}_n^{\mathcal{S}}(H; E)$, which are locally convex topological vector spaces, not necessarily Hausdorff, with $\mathrm{H}_0^{\mathcal{S}}(H; E) \simeq E_H$.

Homological branching in the analytic setting

It amounts to the study of $\mathrm{H}_n^{\mathcal{S}}(H; E)$ for Casselman–Wallach representations E of G.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the complexified Lie algebras. Let $K \subset G$ be a maximal compact subgroup, such that $K^H := K \cap H$ is maximal compact in H.

Theorem (Chen–Sun)

$$\mathrm{H}_n(\mathfrak{h}, K^H; E) \simeq \mathrm{H}_n^{\mathcal{S}}(H; E)$$

as vector spaces. The left hand side is defined using the familiar complex, but without assuming local K^H-finiteness.

Recall that E: Casselman–Wallach $\implies$ $V := E^{K\text{-fini}}$: Harish-Chandra $(\mathfrak{g}, K)$-module.

Comparison map (algebraic vs. analytic branching)

Functoriality for $V \hookrightarrow E$ induces linear maps

$$c_n(E) : \mathrm{H}_n(\mathfrak{h}, K^H; V) \to \mathrm{H}_n^{\mathcal{S}}(H; E).$$

Question: Assuming $H_{\mathbb{C}}$ is reductive and spherical in $G_{\mathbb{C}}$, when is $c_n(E)$ an isomorphism?

Consequences of an affirmative answer:

- $\dim \mathrm{H}_n^{\mathcal{S}}(H; E) < +\infty$ — "homological finiteness" of $E|_H$,
- $\mathrm{H}_n^{\mathcal{S}}(H; E)$ is Hausdorff — "homological separatedness" of $E|_H$,
- *automatic continuity* in the case $n = 0$ (only known for symmetric or maximal unipotent subgroups).

Examples of comparison isomorphisms

1. If $E|_H$ is K^H-admissible (studied by T. Kobayashi and his collaborators), then $c_n(E) :\simeq$.

   $$\text{Reason: it implies } E^{K^H\text{-fini}} = E^{K\text{-fini}}.$$

   This occurs rarely, but includes the interesting case when
   - $K^H\backslash H \hookrightarrow K\backslash G$ is a holomorphic embedding of Hermitian symmetric domains, and
   - E is a unitary highest weight module (eg. holomorphic discrete series).
2. If $G = \mathrm{SL}(2)$ and $H = \begin{pmatrix} * & \\ & * \end{pmatrix}$, $K = \mathrm{SO}(2)$, then $c_n(E) :\simeq$.
   - Step 1: reduce to principal series by a general technique of Hecht–Taylor (1998).
   - Step 2: direct computation (suffices to consider $n = 0, 1$). It does happen that $\mathrm{H}_1 \neq 0$ sometimes.

   Note that H is a symmetric subgroup of G.

Thanks for your attention

arXiv:2207.08994