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New zeta integrals associated with real prehomogeneous vector spaces

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Genesis: Tate's thesis (1950)

We want to study the L-function $L(s, \chi) = \prod_v L(s, \chi_v)$ where $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ is a continuous character, K is a number field, and v ranges over places of v.

Goals

- Represent the local L-factors $L(s, \chi_v)$ as the GCD of **zeta integrals**.
- Interpret the local and global **functional equations** of L as the symmetry of zeta integrals under the **Fourier transform** $\mathscr{F}$ of Schwartz–Bruhat functions.

Local theory

- F: local field with norm $|\cdot|$.
- $\mathscr{S}(F)$: the space of Schwartz–Bruhat functions on F,
- $\chi : F^\times \to \mathbb{C}^\times$: continuous character, $\check{\chi} := \chi^{-1}$.
- Fix Haar measures. For all $\xi \in \mathscr{S}(F)$, define

$$Z(s, \chi, \xi) = Z(\chi|\cdot|^s, \xi) := \int_{F^\times} \chi(t)\xi(t)|t|^s \, \mathrm{d}^\times t.$$

Facts:

- Convergence for $\mathrm{Re}(s) \gg 0$.
- Meromorphic continuation to all $s \in \mathbb{C}$.
- Tate's local functional equation: $\exists\, \gamma(s, \chi)$, meromorphic in s, such that for all ξ

$$Z\left(\check{\chi}|\cdot|^{1-s}, \mathscr{F}\xi\right) = \gamma(s, \chi) Z\left(\chi|\cdot|^s, \xi\right).$$

Generalization 1: Godement-Jacquet (local)

Integral representation for standard L-functions for GL_n.
Idea:

$$\mathrm{GL}_n \hookrightarrow \mathrm{Mat}_{n\times n}, \quad \mathrm{GL}_n \times \mathrm{GL}_n\text{-equivariantly.}$$

Let π be an admissible representation of $\mathrm{GL}_n(F)$, and $\check{v} \otimes v \in \check{\pi} \otimes \pi$. The zeta integral is

$$Z(s, \pi, v \otimes \check{v}, \xi) := \int_{\mathrm{GL}(n,F)} \xi(x) \underbrace{\langle \check{v}, \pi(x)v \rangle}_{\text{matrix coefficient}} |\det(x)|^{s+\frac{n-1}{2}} \, \mathrm{d}^\times x$$

where $\xi \in \mathscr{S}(\mathrm{Mat}_{n\times n}(F))$ and the Haar measures are chosen.

References:

R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Springer LNM 260, 1972.

D. Goldfeld and J. Hundley, *Automorphic representations and L-functions for the general linear group*, Vol 2. CUP, 2011.

- Convergence for $\mathrm{Re}(s) \gg 0$.
- Meromorphic continuation.
- Functional equation with respect to $\mathscr{F} : \mathscr{S}(\mathrm{Mat}_{n\times n}(F)) \to \mathscr{S}(\mathrm{Mat}_{n\times n}(F))$ + gamma factor $\gamma(s, \pi)$.

Further generalizations: the doubling zeta integrals, etc.

Generalization 2: Sato's zeta integrals (local)

Consider the data (G, ρ, X):

- ρ: linear representation of a reductive group G on an F-vector space X.
- $\exists!$ Zariski-open dense orbit X^+ in X (=: prehomogeneity) and $\partial X := X \smallsetminus X^+$ is a hypersurface $f = 0$ where $f \in F[X]$.

In this case,

- (G, ρ, X) is a **regular** PREHOMOGENEOUS VECTOR SPACE (PVS) over F. So is its contragredient $(G, \check{\rho}, \check{X})$;
- f is a **relative invariant**: there exists a unique homomorphism $\chi : G \to \mathbb{G}_m$ such that ${}^g f = \chi(g) f$ for all $g \in G$.

We call χ the eigencharacter of f.

Fix Haar measure on $X(F)$ and define the zeta integral as

$$Z(s,\xi) := \int_{X(F)} \xi|f|^s \, \mathrm{d}x, \quad \xi \in \mathscr{S}(X(F)), \; s \in \mathbb{C}.$$

For simplicity, assume f is irreducible and $X^+(F)$ is a single $G(F)$-orbit.

M. Sato's Fundamental Theorem (1961): local version

- Convergence for $\mathrm{Re}(s) \gg 0$.
- Meromorphic continuation to all s.
- Functional equation with respect to $\mathscr{F} : \mathscr{S}(X(F)) \to \mathscr{S}(\check{X}(F))$ + gamma factor depending only on s.

General case. If f decomposes into n irreducibles, one must replace s by $(s_1, \ldots, s_n)$. The finitely many $G(F)$-orbits in $X^+(F)$ are intertwined in the functional equation. So γ-factor $\rightsquigarrow$ γ-matrix.

Incomplete list of references for Mikio Sato's fundamental theorem:

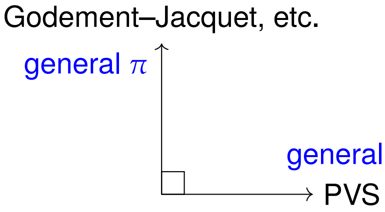
F. Satō, *On functional equations of zeta distributions*. Automorphic forms and geometry of arithmetic varieties, 465–508, Adv. Stud. Pure Math., 15, Academic Press, Boston, MA, 1989.

T. Kimura, *Introduction to prehomogeneous vector spaces*. Translations of Mathematical Monographs, 215. American Mathematical Society, Providence, RI, 2003.

Example (cf. Godement–Jacquet)

Let $G := \mathrm{GL}_n \times \mathrm{GL}_n$ acts linearly on $X := \mathrm{Mat}_{n\times n}$ by left and right translations. Then $X^+ = \mathrm{GL}_n$ and one can take $f = \det$. This PVS is "self-dual".

Observation. The PVS zeta integral in this case = the Godement–Jacquet integral for $\pi = \mathrm{triv}$.



Both rely on embedding a homogeneous G-space in some G-variety on which Schwartz functions, Fourier transforms, etc. are available. Cf. **Braverman–Kazhdan–Ngô theory**.

Remarks

- The L-functions obtained from PVS are usually degenerate ($\because$ it involves very few automorphic representations).
- On the other hand, Sato's Fundamental Theorem is based on simple geometric reasoning, whilst the Godement–Jacquet functional equation requires *ad hoc* arguments over $F = \mathbb{R}$ (reduction to Tate's thesis, etc.)

Best hope

Seek a COMMON GENERALIZATION, and try to establish the expected properties, in the local case at least.

Hereafter, we consider only the case $F = \mathbb{R}$.

The PVS in question

Consider:

- G: connected reductive group over $\mathbb{R}$.
- $\rho : G \to \mathrm{GL}(X)$ a linear representation of G on X.
- $X^+ \subset X$: the Zariski-open G-orbit.
- $\partial X = X \smallsetminus X^+$: a hypersurface. We may take $f \in \mathbb{R}[X]$ such that $\partial X = \{f = 0\}$ and $f \geq 0$ on $X(\mathbb{R})$.

These imply that (G, ρ, X) and its contragredient $(G, \check{\rho}, \check{X})$ are both regular PVS's.

Assumption on sphericity

We assume that the homogeneous G-space X^+ is **absolutely spherical**, i.e. $X^+ \times_{\mathbb{R}} \mathbb{C}$ has an open dense Borel orbit.

If we drop the assumption on ∂X, the resulting data (G, ρ, X) are also known as **multiplicity-free spaces** under G. Over $\mathbb{C}$:

- V. Kac (1980) classified the irreducible multiplicity-free spaces.
- A. Leahy and Benson–Ratcliff achieved the general classification (1996—1997).

Well-known examples:

- $G := D^\times \times D^\times$ acting on $X := D$, where D is a central simple $\mathbb{R}$-algebra (Godement–Jacquet case)
- $G := \mathrm{GL}_n$ acting on $X := \mathrm{Sym}^2\mathbb{R}^n$ or $\bigwedge^2 \mathbb{R}^n$. Ditto for the hermitian version.
- $G := E_6 \times \mathbb{G}_m$ acting on a 27-dimensional X.

Fact. For any π: irreducible Casselman–Wallach representation of $G(\mathbb{R})$, the $\mathbb{C}$-vector space

$$\mathscr{N}_\pi(X^+) := \mathrm{Hom}_{G(\mathbb{R}),\mathrm{cts}}(\pi, C^\infty(X^+))$$

is finite-dimensional.
For any π, a vector v in π and $\eta \in \mathscr{N}_\pi(X^+)$, call

$$\eta(v) \in C^\infty(X^+(\mathbb{R}))$$

a **generalized matrix coefficient** of π.

- They are directly related to relative harmonic analysis / relative Langlands program.
- Those π for which $\mathscr{N}_\pi(X^+) \neq \{0\}$ are said to be **distinguished** by X^+.
- The case when X^+ is a symmetric space is well-known.
- In many cases, $\dim \mathscr{N}_\pi(X^+) > 1$.

A hasty generalization

Fix (G, ρ, X). Assume FOR SIMPLICITY that ∂X and $\partial \check{X}$ are both irreducible, say $\partial X = \{f = 0\}$ with f irreducible.

Definition

For any π, v and $\eta \in \mathcal{N}_\pi(\mathrm{X}^+)$, set

$$Z(s, \pi, v, \eta, \xi) := \int_{\mathrm{X}^+(\mathbb{R})} \eta(v)\ \xi\ |f|^s$$

where $s \in \mathbb{C}$ and $\xi \in \mathscr{S}(\mathrm{X}(\mathbb{R}))$.

Questions in increasing difficulty

1. Convergence for $\mathrm{Re}(s) \gg 0$?
2. Meromorphic continuation?
3. Some sort of functional equation with respect to $\mathscr{F} : \mathscr{S}(\mathrm{X}(\mathbb{R})) \to \mathscr{S}(\check{\mathrm{X}}(\mathbb{R}))$?

Remark

To get a canonical definition, we shall take η and ξ valued in **half-densities**. Advantages:

- The properties of $\mathscr{N}_\pi(X^+)$ are unaltered. In fact, the line bundle of half-densities on $X^+(\mathbb{R})$ can be $G(\mathbb{R})$-equivariantly trivialized by some power of $|f|$.
- Better normalization of zeta integrals (eg. remove the $|\det|^{n/2}$ in Godement–Jacquet).
- $\mathscr{F} : \mathscr{S}(X(\mathbb{R})) \to \mathscr{S}(\check{X}(\mathbb{R}))$ becomes equivariant and truly canonical.

We will BYPASS this issue in this talk.

Related works

- Bopp–Rubenthaler (2005): for a class of symmetric $X^+ \hookrightarrow X$, and π: minimal spherical principal series.
- L., in Springer LNM 2228 (2018): general framework, including a discussion of the non-Archimedean case.
- L., *Towards generalized prehomogeneous zeta integrals*, in Springer LNM 2221 (2018): for symmetric X^+; no discussion of functional equations.

We will NOT discuss the global setting in this talk. Cf.

- H. Saito, *Explicit form of the zeta functions of prehomogeneous vector spaces*. Math. Ann. 315 (1999), no. 4, 587–615.
- T. Ibukiyama, H. Saito, *On zeta functions associated to symmetric matrices I—III*. (The above are mainly for $\pi = \mathrm{triv}$)
- Y. Sakellaridis, *Spherical varieties and integral representations of L-functions*. (2012).

Convergence

The convergence of $Z(s, \pi, v, \eta, \xi)$ for $\mathrm{Re}(s) \gg 0$ (uniformly in η, v) can be reduced to the following

Fact

There exist

- a continuous semi-norm q on the underlying Fréchet space of π,
- a Nash function $p : X^+(\mathbb{R}) \to \mathbb{R}_{\geq 0}$,

such that $|\eta(v)| \leq q(v)p$ for all v.

Fix a maximal compact subgroup $K \subset G(\mathbb{R})$. The fact above follows from either

1. the **moderate growth** of the Casselman–Wallach representation π, or
2. that when v is K-finite, $\eta(v)$ satisfies a **regular holonomic** differential system on X^+ arXiv:1905.08135

Meromorphic continuation

Based on the estimates before, the meromorphic continuation of $Z(s, \pi, v, \eta, \xi)$ to all $s \in \mathbb{C}$ follows by the standard technique via the Bernstein–Sato b-functions.

Steps:

1. Reduce to K-finite v.
2. Use the **holonomicity** of the $D_{X^+_{\mathbb{C}}}$-module generated by $\eta(v)$, which is established in arXiv:1905.08135 or [Aizenbud–Gourevitch–Minchenko, 2016].

The sphericity of X^+ is crucial here.

Towards the functional equation

- Pick $h \in \mathbb{R}[X]$ such that h is a relative invariant, $h \geq 0$ on $X(\mathbb{R})$, $\{h = 0\} = \partial X$ and $\mathrm{d}\log h : X^+ \xrightarrow{\sim} \check{X}^+$. This is possible by the regularity of the PVS.
- Fact: there exists $\check{h} \in \mathbb{R}[\check{X}]$ with the same properties such that $\check{h} \circ \mathrm{d}\log h = 1/h$ and $h \circ \mathrm{d}\log \check{h} = 1/\check{h}$. They are relative invariants with opposite eigencharacters.
- To simplify matters, consider only the integral $Z_X(s, \eta, v, \xi) = \int_{X^+(\mathbb{R})} \eta(v)\xi h^s$: meromorphic family of tempered distributions on $X(\mathbb{R})$. Ditto for $Z_{\check{X}}(s, \check{\eta}, v, \xi)$.

For generic s, the distribution $Z(s, \check{\eta}, v, \mathscr{F}(\cdot))\big|_{X^+(\mathbb{R})}$ is smooth (by elliptic regularity, essentially). Express it as

$$\gamma(s, \check{\eta})(v) \cdot |h|^{-s}, \quad \gamma(s, \check{\eta}) \in \mathcal{N}_\pi(X^+).$$

The γ-**matrix** $\gamma(s, \cdot) : \mathcal{N}_\pi(\check{X}^+) \to \mathcal{N}_\pi(X^+)$ is meromorphic in s.

A comparison with known cases of functional equations leads to the following[1]

Theorem-to-prove

For all $\check{\eta}, v$ and all $\xi \in \mathscr{S}(X(\mathbb{R}))$,

$$Z_{\check{X}}\left(s, \check{\eta}, v, \mathscr{F}\xi\right) = Z_X\left(-s, \gamma(s, \check{\eta}), v, \xi\right)$$

as meromorphic functions in s.

- $\mathscr{F}$ depends on the choice of a $\psi : \mathbb{R} \to \mathbb{C}^\times$.
- Let $\mathfrak{d}(s)$ be the distribution (LHS - RHS), meromorphic in s. The γ-matrix is defined precisely to make

$$\mathfrak{d}(s)|_{X^+(\mathbb{R})} = 0.$$

[1] It might be simpler to start with η rather than $\check{\eta}$.

1. Take v to be a K-finite vector in π.
2. It suffices to show $\mathfrak{d}(s) = 0$ for s in some open ball off the poles.
3. There exists $L \in \mathbb{Z}_{\geq 1}$ such that $h^L \mathfrak{d}(s) = 0$ for all s in any given compact subset.

Definition of Capelli operators

- $C := \check{h}^L \otimes h^L \in \mathbb{R}[\check{X}] \otimes \mathbb{R}[X] \hookrightarrow \mathrm{Diff}_{\check{X}}$. It is G-invariant.
- For all $s \in \mathbb{C}$, let $C_s := \check{h}^{-s} \circ C \circ \check{h}^s$: analytic $G(\mathbb{R})$-invariant differential operator on $X^+(\mathbb{R})$.

$$\underbrace{h^L}_{\text{diff. op.}} \cdot \left(\check{\eta}(v)\check{h}^s\right) = C_s \left(\check{\eta}(v)\right) \check{h}^{s-L} \quad \text{in } C^\infty(\check{X}^+(\mathbb{R})),\ \forall s.$$

Equality still holds when both sides are extended to tempered distributions on $\check{X}(\mathbb{R})$ by zeta integrals.

One can infer from $h^L\mathfrak{d}(s) = 0$ (for s in some ball B) that

$$Z_{\check{X}}(s - L, \chi_s, v, \mathscr{F}\xi) = Z_X\left(L - s, \gamma(s - L, \chi_s), v, \xi\right), \quad s \in B,$$

where $\chi_s := C_s \circ \check{\eta} \in \mathcal{N}_\pi(\check{X}^+)$ varies holomorphically with s.

- It remains to show $(C_s)_* : \mathcal{N}_\pi(\check{X}^+) \to \mathcal{N}_\pi(\check{X}^+)$ is injective ($\implies$ bijective) for generic s.
- As $\det(C_s)_*$ is holomorphic in s, we are reduced to the following

Lemma-to-prove

The function $s \mapsto \det(C_s)_*$ is not identically zero for $s \in \mathbb{Z}$.

Ideas of the proof

1. Set $\mathscr{Z}(\check{X}) = \mathrm{Diff}(\check{X})^G$. Then $C_s \in \mathscr{Z}(\check{X})$ for all $s \in \mathbb{Z}$.
2. Fact: $\mathscr{Z}(\check{X})$ is commutative ($\because$ sphericity). Break $\mathscr{N}_\pi(\check{X}^+)$ into generalized eigenspaces $\mathscr{N}^{(\lambda)}$, where $\lambda : \mathscr{Z}(\check{X}) \to \mathbb{C}$.
3. Goal: Show that the eigenvalue $\lambda(C_s)$ on each $\mathscr{N}^{(\lambda)}$ is nonzero, for general $s \in \mathbb{Z}$.

This turns out to be a question of algebraic geometry or invariant theory.

Strategy

1. Use Knop's **Harish-Chandra isomorphism**

$$p : \mathscr{Z}(\check{X}) \simeq \mathbb{C}\left[\rho + \mathfrak{a}^*_{\check{X}}\right]^{W_{\check{X}}},$$

   so that λ is a point of the variety $(\rho + \mathfrak{a}^*_{\check{X}})/\!\!/W_{\check{X}}$ of "infinitesimal characters".
2. Show that $C \rightsquigarrow C_s$ corresponds to translating λ in some explicit direction determined by the eigencharacter of $\check{h}$.
3. It will ultimately follow from Knop's study of the Capelli operators C that $s \mapsto \lambda(C_s) \neq 0$ for generic s.

The REGULARITY of the PVS (G, ρ, X) plays a pivotal role here.

Reference:

F. Knop, *Some remarks on multiplicity free spaces*. In: Representation theories and algebraic geometry (Montréal, PQ, 1997), 301–317.

Remarks

- The proof also works without the simplifying assumptions.
- In the setting considered by Mikio Sato ($\pi = \mathrm{triv}$, without sphericity assumption on X^+), one can express $C_s \circ \check{\eta}$ in terms of the b-function of the PVS. The γ-matrix has also been determined in numerous cases.
- Very little can be said about $\gamma(s, \check{\eta})$ in our general setting.

Thanks for your attention

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