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On certain zeta integrals

The cover picture is taken from http://lvtu.qunar.com

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Integral representations of L-functions

Common features:

- Use tractable integrals involving automorphic forms + *other stuff* to study L-functions, etc.
- Adélic formulation.
- Meromorphic continuation, bounded in vertical strips, etc.
- Functional equation(s).
- Euler product; theory of local integrals.

Usually called zeta integrals — the core of the L-function machine.

Godement-Jacquet theory (< 1972)

Goal: study the standard L-functions of GL_n, both locally and globally. We begin with the case over a local field F

- $\mathsf{GL}_n \times \mathsf{GL}_n$-equivariant embedding $\mathsf{GL}_n \hookrightarrow \mathsf{Mat}_n$.
- $\xi \in \mathcal{S}(\mathsf{Mat}_{n\times n})$: Schwartz-Bruhat functions.
- For an irrep π of $\mathsf{GL}_n(F)$, integrate against ξ the matrix coefficients $\langle \check{v}, \pi(\cdot)v \rangle$ (= images under equivariant $\pi \boxtimes \check{\pi} \to C^\infty(\mathsf{GL}_n)$, satisfying multiplicity one):

$$Z(s, \pi, v \otimes \check{v}, \xi) := \int_{\mathsf{GL}(n,F)} \xi(x) \langle \check{v}, \pi(x)v \rangle |\det(x)|^{s + \frac{n-1}{2}} \, \mathrm{d}^\times x.$$

Facts: convergence for $\mathrm{Re}(s) \gg 0$, meromorphic/rational continuation.

- Local functional equation: there exists a meromorphic/rational function $\gamma(s, \pi)$ such that

$$Z(1-s, \check{\pi}, \check{v} \otimes v, \mathcal{F}\xi) = \gamma(s, \pi) Z(s, \pi, v \otimes \check{v}, \xi)$$

  where $\mathcal{F}$ is the Fourier transform on $\mathrm{Mat}_{n \times n}(F)$ relative to $(X, Y) \mapsto \psi(\mathrm{tr} XY)$, for a chosen additive character ψ.
- Relation to L-functions: taking greatest common divisor

$$Z(s, \pi, \xi) \rightsquigarrow L(s, \pi).$$

- Shift in s: the $-\frac{1}{2}$ is nice, but where does $n/2$ come from?

Note: $|\det|^n \mathrm{d}^\times x = \mathrm{d}x$

Global case: let $\mathbb{A} = \mathbb{A}_F$, π be cuspidal automorphic, $\phi \in \pi$ and $\check{\phi} \in \check{\pi}$ be cusp forms,

$$\int_{\mathsf{GL}(n,\mathbb{A})} \beta(x)\xi(x)|\det(x)|^{s+\frac{n-1}{2}} \mathrm{d}^\times x$$

where $\Re(s) \gg 0$ and

- $x\phi(\cdot) = \phi(\cdot\, x)$, $\beta(x) = \langle \check{\phi}, x\phi \rangle_{\mathsf{Pet}}$ (the Petersson pairing) is the global matrix coefficient: it factorizes,
- $\xi \in \mathcal{S}(\mathsf{Mat}_{n\times n}(\mathbb{A}))$.

It serves as a prototype for many other integral representations of L-functions.

Exercise: rewrite it as a convergent integral involving ϕ and $\check{\phi}$ on $\mathrm{diag}(\mathfrak{a})\mathsf{GL}(n, F)^2 \backslash \mathsf{GL}(n, \mathbb{A})^2$, for a suitable central connected subgroup $\mathfrak{a} \subset \mathsf{GL}(n, F_\infty)$.

Doubling integrals – a sketch (Piatetski-Shapiro and Rallis)

A typical set-up: let W be a symplectic F-vector space, $\dim W \geq 2$ and $W^\square := (W, \langle , \rangle) \overset{\perp}{\oplus} (W, -\langle , \rangle)$.

- $G = \mathrm{Sp}(W)$, $H = \mathrm{Sp}(W^\square)$;
- $P \subset H$ is the stabilizer of the Lagrangian $x_0 := \mathrm{diag}(W)$ in $W^\square$, $P_{\mathrm{ab}} \simeq \mathbb{G}_m$, and we have a natural algebraic character $\det_P : P_{\mathrm{ab}} \to \mathbb{G}_m$;
- $G \times G \hookrightarrow H$ naturally;
- since $(G \times G) \cap P = \mathrm{diag}(G)$, one obtains a $G \times G$-equivariant embedding $G \hookrightarrow P \backslash H$ with open dense image;
- the boundaries are negligible: every $\gamma \in \partial H$ is stabilized by the unipotent radical of some proper parabolic. Morally, this means ∂H does not interact with cusp forms.

Global integrals. Let F be a number field. Let $\sigma \boxtimes \pi \boxtimes \check{\pi}$ be a cuspidal automorphic representation of $(P_{\mathsf{ab}} \times G \times G)(\mathbb{A}_F)$. Form

$$\int_{[G \times G]} E_f(\sigma, s)(x, x') \phi(x) \phi'(x') \, \mathrm{d}x \, \mathrm{d}x'$$

- $\phi \in \pi$, $\phi' \in \check{\pi}$;
- f a "good section" for $I_P^H(\sigma \otimes \det_P^s)$ (unitary parabolic induction), parametrized by $s \in \mathbb{C}$;
- $E_f(\sigma, s) : H(F) \backslash H(\mathbb{A}_F) \to \mathbb{C}$ the Eisenstein series made from f.

Properties of Eisenstein series (eg. intertwining operators) $\implies$ meromorphic continuation, functional equation...

Local integrals. Let $\sigma \boxtimes \pi \boxtimes \check{\pi}$ be a smooth irreducible representation of $(P_{\mathrm{ab}} \times G \times G)(F)$. Form

$$\int_{G(F)} f(x_0(x,1)) c_\pi(x) \,\mathrm{d}x$$

where

- $f : R_u(P)\backslash H(F) \to \mathbb{C}$ is a "good section" for $I_P^H(\sigma \otimes \det_P^s)$ as before;
- $c_\pi \in C^\infty(G(F))$ is a matrix coefficient of π.

By taking gcd, these integrals represents the Rankin-Selberg L-function $L(s + \frac{1}{2}, \chi \otimes \pi)$.
Furthermore, the global integral factorizes into local ones.

The doubling construction can be adapted to $G = \mathsf{GL}(n)$, $H = \mathsf{GL}(2n)$ to yield the Godement-Jacquet integrals. Piatetski-Shapiro and Rallis claimed that the doubling method is *the correct generalization* of Godement-Jacquet theory.
Things can go the other way around.

Braverman-Kazhdan (2002)

Use the affine embedding $P_{\mathsf{ab}} \times G \hookrightarrow P_{\mathsf{der}} \backslash H =: X$. Replace good section f by suitable test function on $X(F)$. Normalized intertwining operators for $I_H^P(\cdots)$ get replaced by "Fourier transforms". The resulting theory resembles the original Godement-Jacquet.

Furthermore: in the unramified setting, there is a distinguished $P_{\mathsf{ab}} \times H(\mathfrak{o}_F)$-invariant test function ξ° (denoted by $c_{P,0}$ in *loc. cit.*) whose values are closely related to the trace-of-Frobenius of the IC sheaf on Drinfeld's compactification $\overline{\mathsf{Bun}_P}$ of Bun_P, the moduli stack of P-bundles over a smooth projective $\mathbb{F}_q$-curve. Why?

Igusa zeta integrals

One of the interesting integrals in the pre-Langlands era.
It originates from ideas of Gelfand et al. on complex powers. Let F be a local field. We want to study integrals

$$\int_{X(F)} |f|^s \xi, \quad \mathrm{Re}(s) \gg 0$$

for

- appropriate varieties X (often: affine n-space),
- $\xi \in C_c^\infty(X(F))$,
- $f \in F[X]$: "interesting" function.

Seek for meromorphic continuation in s, location of poles, evaluation at special ξ, etc.
It is even more interesting if *symmetries* enter into this picture, by letting a group G act on X so that f is an eigenfunction.

Prehomogeneous zeta integrals

Let X be an F-vector space, on which a reductive group G acts with dense open orbit X^+. Assume that

1. $\partial X = X \smallsetminus X^+$ is a hypersurface $(f = 0)$; we assume $f \in F[X]$ transforms with eigencharacter $\omega : G \to \mathbb{G}_m$;
2. $X^+(F)$ is a single $G(F)$-orbit (for simplicity).

Define

$$Z(s, \xi) := \int_{X(F)} |f|^s \xi, \quad \xi \in \mathcal{S}(X).$$

Local functional equation (M. Sato, T. Shintani, F. Sato, et al)

Assume X "regular" (hence $\check{X}$ is prehomogeneous) and $X^+(F)$ is a single $G(F)$-orbit. $Z(* - s, \mathcal{F}(\cdot))$ and $Z(s, \cdot)$ can be related up to "γ-factor".

Multiple $G(F)$-orbits in $X^+(F)$ leads to functional equation with "γ-matrices", by considering integrals over each orbit separately.

In comparison with the local Godement-Jacquet and doubling constructions, one may consider

- π: irrep of $G(F)$,
- $\varphi \in \mathcal{N}_\pi := \mathrm{Hom}_{G(F)}(\pi, C^\infty(X^+))$, and
- study the zeta integral $Z(s, \varphi, v, \xi) = \int_{X(F)} \varphi(v)|f|^s \xi$.

An attempt in this direction is made by Bopp and Rubenthaler (2005), under various conditions (eg. assuming X^+ is a symmetric space, specific series of π). One indispensable premise seems to be

$$\dim \mathcal{N}_\pi < \infty.$$

Braverman-Kazhdan (1999, 2000)

For split reductive G, they formulated a vast conjectural generalization of Godement-Jacquet integrals, using

- $G \times G$-equivariant embedding of G into some algebraic monoid M with unit group G together with $\det_M : M \to \mathbb{G}_a$, restricting to a homomorphism $G \to \mathbb{G}_m$;
- conjectural Schwartz space $\mathcal{S}$;
- conjectural Fourier transform, satisfying an even more conjectural Poisson formula.

Conjecturally, integration of matrix coefficients against $\xi \in \mathcal{S}$, twisted by $|\det_M|^s$, represents a large family of L-functions attached to $\rho : \widehat{G} \to \mathrm{GL}(N, \mathbb{C})$.

$$\rho \leftrightarrow \text{highest weight} \xrightarrow{\text{Vinberg theory}} M = M_\rho$$

Related works: L. Lafforgue, Bouthier-Ngô-Sakellaridis......

Analysis on homogeneous spaces

- G under $G \times G$ is just a special case of homogeneous spaces — the group case.
- A generally accepted framework for doing harmonic analysis: spherical homogeneous spaces, i.e. $\exists$ open Borel orbit.
- Let X^+ be spherical homogeneous under G and π: irrep. Main issues in harmonic analysis include
  - decomposition of $L^2(X^+)$,
  - intertwining operators $\pi \to C^\infty(X^+)$ yielding the "coefficients" of π,
  - intertwining operators relating different X^+.
- Must also consider spherical embeddings: G-equivariant morphisms $X^+ \hookrightarrow X$ with open dense image and normal X.

Important works are being done by Sakellaridis-Venkatesh.

Let H be a split reductive group. In the global case, Sakellaridis (2012) proposes to generate many integral representations using

- affine spherical embedding $X^+ \hookrightarrow X$,
- $\forall v$, conjectural Schwartz space $\mathcal{S}_v \subset C^\infty(X^+(F_v)))$ from these data.
- for almost all $v \nmid \infty$, there should be a distinguished $G(\mathfrak{o}_v)$-invariant element $\xi_v^\circ \in \mathcal{S}_v$ (the basic function).

The expected behavior of $\xi \in \mathcal{S}_v$ is complicated by the singularities of X.

Example: Bouthier-Ngô-Sakellaridis

In the Braverman-Kazhdan case $X^+ = G \hookrightarrow X = M_\rho$, suppose everything unramified.

They interpreted $\xi_v^\circ \in \mathcal{S}_v$ as the trace of Frobenius of an appropriately defined IC sheaf on L^+X, the formal arc space attached to X^+, by passing to the equal-characteristic case.

Generalizations?

It seems that a *correct* way of viewing zeta integrals is crucial.
We try to refine Sakellaridis' proposal as follows. G: split connected reductive group over a local field F, char$(F) = 0$, common patterns include

1. an affine G-spherical embedding $X^+ \hookrightarrow X$;
2. a Schwartz space $\mathcal{S}$ of "test functions" on a $X^+(F)$;
3. integrate the coefficients of an irrep π (smooth admissible, SAF if $F \supset \mathbb{R}$) against test functions from $\mathcal{S}$;
4. a mechanism to twist coefficients by $|f|^s$ for appropriate $f \in F[X]^{G-\text{eigen}}$;
5. meromorphic/rational continuations of the zeta-integrals/zeta distributions so obtained;
6. functional equation of zeta integrals, which is a manifestation of some "Fourier transform" of Schwartz functions.

Reality check

Any such formalism must give a simple, conceptual explanation of the Godement-Jacquet theory.

Other issues:

- Can we prove anything under such a general framework?
- Clarify the functional-analytic underpinnings.
- Local-global compatibility.
- Try to find manageable examples, for which Schwartz space and Fourier transform are already available.
- It should make zeta integrals appear more natural to an outsider.

Towards a broader framework

G: split connected reductive group

1. $X^+ \hookrightarrow X$: affine spherical embedding.
2. The boundary $X \smallsetminus X^+$ is the union of prime divisors $f_i = 0$ (eigencharacter $=: \omega_i$), $i = 1, \ldots, r$.
3. Assume the eigencharacters $\omega_1, \ldots$ are linearly independent and generate a lattice Λ. Write $|\omega|^\lambda = \prod_i |\omega_i|^{\lambda_i}$, $|f|^\lambda := \prod_i |f_i|^{\lambda_i}$ for $\lambda = \sum_i \lambda_i \omega_i \in \Lambda \otimes \mathbb{C}$.
4. Assume $\mathcal{N}_\pi := \mathrm{Hom}_{G(F)}(\pi, C^\infty(X^+))$ is finite-dimensional. Known for $F = \mathbb{R}$ (Kobayashi-Oshima) or F non-archimedean and X^+ wavefront (Sakellaridis-Venkatesh). Here we should take continuous Hom.
5. Work with a given Schwartz space $C_c^\infty(X^+) \subset \mathcal{S} \subset C^\infty(X^+)$, a continuous smooth $G(F)$-representation.

$$Z_\lambda(\xi, \varphi(v)) := \int_{X^+} \xi\varphi(v)|f|^\lambda, \quad \varphi \in \mathcal{N}_\pi, \xi \in \mathcal{S}, v \in \pi.$$

Digression: What can we integrate?

- Integration applies to any density, locally written as $|\omega|$ (ω: volume form). Density bundle $= \mathscr{L}$.
- The L^2-ness makes sense for a $\frac{1}{2}$-density, locally as $|\omega|^{1/2}$.
- Integration of densities is canonical if we fix a Haar measure on F.
- The L^2-pairing is canonical and invariant under all symmetries of $X^+(F)$, giving rise to unitary representations.

Hence: $L^2(X^+)$ stands for the L^2-sections of the bundle $\mathscr{L}^{1/2}$ of $\frac{1}{2}$-densities. Re-define

- $\mathcal{S} \subset L^2(X^+)$ is $\mathscr{L}^{1/2}$-valued,
- $C^\infty(X^+) := C^\infty(X^+(F), \mathscr{L}^{1/2})$,
- $\varphi \in \mathcal{N}_\pi := \mathrm{Hom}_{G(F)}(\pi, C^\infty(X^+))$.

More generally, we should allow values in some line bundles, eg. the Whittaker-induced case.

What is this good for? Let's return to the Godement-Jacquet scenario

$$\begin{aligned} \mathcal{S} &:= \{\text{Schwartz-Bruhat half-densities on } \mathsf{Mat}_{n\times n}(F)\} \\ &= \{\xi_0 |\mathrm{d}x|^{1/2} : \xi_0 \in \mathcal{S}_{\mathsf{usual}}\}, \end{aligned}$$

where $\mathrm{d}x$ is a translation-invariant volume form. Therefore

$$\xi_0 |\mathrm{d}x|^{1/2} = \xi_0 |\det x|^{n/2} |\det x|^{-n/2} |\mathrm{d}x|^{1/2} = \xi_0 |\det|^{n/2} |\mathrm{d}^\times x|^{1/2}$$

where $|\mathrm{d}^\times x|$ stands for the Haar measure on $\mathsf{GL}(n, F)$.

- This explains the $n/2$-shift.
- Fix ψ, the Fourier transform for $\mathcal{S}$ is canonically defined and $\mathsf{GL}(n, F)$-equivariant (already known to E. Stein ≤ 1967).

Lesson: Don't trivialize $\mathscr{L}$ even though you can.

Partial results

Assume F non-archimedean and $\forall \xi \in \mathcal{S}$, the support of ξ has compact closure in $X(F)$.

1. Using the asymptotics of coefficients $\varphi(v) \in C^\infty(X^+)$ from Sakellaridis-Venkatesh, one can show the convergence for $\mathrm{Re}(\lambda) \gg 0$ when X^+ is wavefront. Idea: use a smooth toroidal compactification of X^+ that is compatible with $X^+ \hookrightarrow X$.
2. In the prehomogeneous case with the usual $\mathcal{S}$, Igusa's theory implies the rationality of zeta integrals.

This framework accommodates:

- the aforementioned generalization of prehomogeneous zeta integrals with X^+ spherical and wavefront,
- Godement-Jacquet $\subset$ Braverman-Kazhdan,
- doubling integrals interpreted via the $P_{\mathrm{ab}} \times G \times G$-equivariant embedding $X^+ := P_{\mathrm{ab}} \times G \hookrightarrow P_{\mathrm{der}} \backslash H \hookrightarrow \overline{P_{\mathrm{der}} \backslash H}^{\mathrm{aff}} = X$.

In the symplectic case at least, the doubling method can be subsumed into Braverman-Kazhdan. The reason is as follows.

Theorem (Rittatore)

View G as a $G \times G$-variety (the "group case"). Its affine spherical embeddings are the same as normal affine algebraic monoids with unit group G.

Moreover, the monoid/embedding $X^+ \hookrightarrow X$ in the doubling method for $G = \mathsf{Sp}(2n)$ matches Ngô's recipe with the expected representation

$$\rho := \mathsf{id} \boxtimes \mathsf{Std} : \mathbb{C}^\times \times \mathsf{SO}(2n+1, \mathbb{C}) \to \mathsf{GL}(2n+1, \mathbb{C})$$

of $(P_{\mathsf{ab}} \times G)^\wedge = \mathbb{C}^\times \times \mathsf{SO}(2n+1, \mathbb{C})$. The verification uses

- a closer look at the stratification of $X = P_{\mathsf{der}} \backslash H$ into $P_{\mathsf{ab}} \times G \times G$-orbits;
- Luna-Vust classification.

Obvious drawbacks

- Currently, the formalism does not include Rankin-Selberg integrals on $\mathsf{GL}(m) \times \mathsf{GL}(n)$ with $n < m$. Doing this will require:
  - allowing homogeneous G-spaces that are "Whittaker-induced" from a Levi,
  - understanding the "unfolding" process.
- The most accessible case: X smooth. Such G-varieties tend to be prehomogeneous (Luna); according to Sakellaridis (2012), the L-functions coming from prehomogeneous zeta integrals have been known by other methods.

Relation to spectral decomposition

Generally, every X^+ has an abstract Plancherel decomposition

$$L^2(X^+) = \int_{\Pi_{\text{unit}}(G)}^{\oplus} \mathcal{H}_\tau \, \mathrm{d}\mu(\tau), \quad \mathcal{H}_\tau = \tau \hat{\otimes} \mathcal{M}_\tau.$$

In the group case: $X^+ = G$ under $G \times G$-action, Harish-Chandra gives a far-reaching refinement of the decomposition above.
In the Godement-Jacquet case $G = \mathsf{GL}(n, F)$, zeta integrals are related to the spectral decomposition of $L^2(\mathsf{GL}(n, F))$:

1. Plancherel formula describes the image of $\xi \mapsto \pi(\xi) \in \mathsf{End}_{\mathbb{C}}(V_\pi)$, $\xi \in C_c^\infty(X^+)$, π in the tempered dual of $G(F)$;
2. for $\xi \in \mathcal{S}(\mathsf{Mat}_{n\times n})$ and $\mathsf{Re}(\lambda) \gg 0$, the image of $Z_\lambda(\xi, \cdots) \leftrightarrow (\pi \otimes |\det|^\lambda)(\xi)$ has a similar description involving $L(\lambda, \pi)$.

Lafforgue turned this approach over to study Braverman-Kazhdan program:

Given local factors $L(s, \pi)$, one defines $\mathcal{S}$ by spectral means.

This leads the notion of functions of L-type, natural formulations of Fourier transforms, etc.

- Heuristically very useful.
- In general: have no L-factor at hand, except when π is in the principal series, etc.
- Putting the cart before the horse?

Issue of spectral decomposition

Clarify the relation between zeta integrals and spectral decomposition of $L^2(X^+)$ for general X^+. Deduce this from properties of $\mathcal{S}$ (not conversely)

Gelfand-Kostyuchenko method

- $L^2(X^+) = \int^{\oplus}_{\Pi_{\mathrm{unit}}(G)} \mathcal{H}_\tau \, \mathrm{d}\mu(\tau)$, assuming the multiplicity space $\mathcal{M}_\tau$ is finite-dimensional.
- In general: the decomposition is not defined by operators $L^2(X^+) \to \mathcal{H}_\tau$ (think of $L^2(\mathbb{R})$ — classical Fourier analysis!)
- Rigged Hilbert spaces: $\mathcal{S} \hookrightarrow L^2(X^+)$ dense continuous, requirement: it is *pointwise defined* by a family $\alpha_\tau : \mathcal{S} \to \mathcal{H}_\tau$ with dense images.
- OK for the minimalist choice $\mathcal{S} = C^\infty_c(X^+) := C^\infty_c(X^+(F), \mathscr{L}^{1/2})$.
- Sufficient condition for the existence of $(\alpha_\tau)_\tau$: $\mathcal{S}$ is nuclear separable; this appeared in *Generalized Functions, Vol. IV*.

Can embed

$$\mathcal{M}_\tau^\vee \simeq \mathrm{Hom}_{G(F)}(\tau \otimes \mathcal{M}_\tau, \tau) = \mathrm{Hom}_{G(F)}(\mathcal{H}_\tau, \tau) \hookrightarrow \mathrm{Hom}_{G(F)}(\mathcal{S}, \tau)$$

since $\mathcal{H}_\tau$ is the completion of $\mathcal{S}$ w.r.t. the continuous semi-norm $\|\alpha_\tau(\cdot)\|_{\mathcal{H}_\tau}$.

When $\mathcal{S} = C_c^\infty(X^+)$, Bernstein (1988) showed

$$\mathrm{Hom}_{G(F)}(\mathcal{S}, \tau) \simeq \mathrm{Hom}_{G(F)}(\pi, C^\infty(X^+)) = \mathcal{N}_\pi$$

where $\pi := \bar{\tau}^\infty$. Summing up, we obtain a relation

Spectral decomposition (L^2-theory) $\leftrightarrow$ distinction (smooth theory).

- $\mathcal{S} = C_c^\infty(X^+)$ is easier to work with.
- Expectation: choice of $\mathcal{S}$ is closely related to affine embeddings $X^+ \hookrightarrow X$.
- Different homogeneous spaces X_1^+, X_2^+ may have isometric L^2 induced from some equivariant $\mathcal{F} : \mathcal{S}_2 \simeq \mathcal{S}_1$, which is difficult to see from C_c^∞ test functions. Again, one may motivate by considering $L^2(\mathbb{R})$ and the usual Schwartz space.

Functional equations: L^2-aspect

Question: why should we expect functional equations?

1. Disintegrate $\mathcal{F} : L^2(X_2^+) \xrightarrow{\sim} L^2(X_1^+)$ using the abstract Plancherel decomposition: get $\eta(\tau) : \mathcal{M}_\tau^{(2)} \to \mathcal{M}_\tau^{(1)}$ on multiplicity spaces. How to handle $\eta(\tau)$?
2. Suppose that $\mathcal{F}$ restricts to $\mathcal{S}_2 \xrightarrow{\sim} \mathcal{S}_1$. By Gelfand-Kostyuchenko, $\mathcal{M}_\tau^{(i),\vee} \subset \mathrm{Hom}_{G(F)}(\mathcal{S}_i, \tau)$ and $\eta(\tau)^\vee$ is induced from a transport of structure via $\mathcal{F}$. How to describe the image of $\mathcal{M}_\tau^{(i),\vee}$?
3. It is relatively easier to identify $\mathcal{M}_\tau^{(i),\vee}$ as a subspace of $\mathcal{N}_\pi^{(i)}$ with $\pi = \bar{\tau}^\infty$. How to compare the relevant embeddings of π into $C^\infty(X^+)$ and $\mathcal{S}_i^\vee$?
4. My proposal: use zeta integrals + meromorphic continuation. This can be justified if for $X^+ = X_1^+, X_2^+$, $C_c^\infty(X^+) \hookrightarrow \mathcal{S}$ induces

$$\mathrm{Hom}_{G(F)}(\pi|\omega|^\lambda, \mathcal{S}^\vee) \hookrightarrow \mathrm{Hom}_{G(F)}(\pi|\omega|^\lambda, C_c^\infty(X^+)^\vee)$$

for $|\omega|^\lambda = \prod_i |\omega_i|^{\lambda_i}$ and $\lambda \in \Lambda_{\mathbb{C}}$ in general position. Plus some technical assumptions...

Assuming the meromorphic continuation, etc. for zeta integrals, we deduce

$$\mathcal{N}_\pi \hookrightarrow \mathcal{L}_\pi := \{\text{nice meromorphic invariant } B_\lambda : \pi_\lambda \otimes \mathcal{S} \to \mathbb{C}\}$$

with $\pi_\lambda := \pi \otimes |\omega|^\lambda$. It maps φ to $B_\lambda(v \otimes \xi) = Z_\lambda(\xi, \varphi(v))$.
Let $\mathcal{K}$ be the field of meromorphic functions on $\mathcal{T} := \{|\omega|^\lambda : \lambda \in \Lambda_\mathbb{C}\}$.
Obtain a $\mathcal{K}$-linear map $\mathcal{N}_\pi \otimes \mathcal{K} \hookrightarrow \mathcal{L}_\pi$.
Now work with $X_i^+ \hookrightarrow X_i$ for $i = 1, 2$. For ease of notations, assume $\Lambda_1 = \Lambda_2$, i.e. allow the same twists by characters $|\omega|^\lambda$.

Definition

The local functional equation for $\mathcal{F} : \mathcal{S}_2 \xrightarrow{\sim} \mathcal{S}_1$ and π holds if:

$$\begin{array}{ccc} \mathcal{L}_\pi^{(1)} & \longrightarrow & \mathcal{L}_\pi^{(2)} \\ \uparrow & & \uparrow \\ \mathcal{N}_\pi^{(1)} \otimes \mathcal{K} & \xrightarrow{\exists \gamma(\pi)} & \mathcal{N}_\pi^{(1)} \otimes \mathcal{K} \end{array}$$

$\gamma(\pi) \leftrightarrow$ $\gamma(\pi, \lambda) : \mathcal{N}_\pi^{(1)} \to \mathcal{N}_\pi^{(2)}$ meromorphic in λ

In comparison of the L^2-aspect, one may infer (under some hypotheses as we have seen) that $\gamma(\pi, 0)$ restricts to $\eta(\tau)^\vee$ for almost all τ and $\pi := \bar{\tau}^\infty$. This is compatible with some known properties of γ-factors: unitarity along the critical line for tempered irreps, etc.

Question

How to obtain local functional equations in this sense?

In the non-archimedean case, one approach is to establish multiplicity-one of invariant bilinear forms $B_\lambda : \pi_\lambda \otimes \mathcal{S} \to \mathbb{C}$ for general λ.

- This is closely related to the geometry of ∂X.
- It is possible to formulate a sufficient condition that implies Godement-Jacquet case, and probably some other prehomogeneous vector spaces.

Speculations on the global case

Let F be a number field, $\mathbb{A} = \mathbb{A}_F$.

- Same conditions on $X^+ \hookrightarrow X$, assuming the Schwartz spaces $\mathcal{S}_v$ defined at each place v.
- Assume that "basic functions" $\xi_v^\circ \in \mathcal{S}_v$ are chosen for almost all $v \nmid \infty$ so that $\mathcal{S} = \bigotimes'_v \mathcal{S}_v$ makes sense.
- Consider a continuous $G(F)$-invariant functional $\vartheta : \mathcal{S} \to \mathbb{C}$, eg.

$$\vartheta(\xi) = \sum_{x \in X^+(F)} \mathrm{ev}_\gamma(\xi)$$

  for appropriate *evaluation maps* ev_γ, with due care on the half-densities, etc.

- Frobenius reciprocity yields $\xi \mapsto \vartheta_\xi \in C^\infty(G(F)\backslash G(\mathbb{A}))$.

It is natural to consider integrals

$$Z_\lambda = \int_{\mathfrak{X}} \phi_\lambda \vartheta_\xi, \quad \Re(\lambda) \gg 0,$$

for ϕ: cusp form, $\phi_\lambda = \phi|\omega|^\lambda$, $\mathfrak{X} = G(F)\backslash G(\mathbb{A})$ divided by some suitable central subgroup of $G(F_\infty)$. Some issues:

- Convergence for $\Re(\lambda) \gg 0$ (look for sufficient conditions).
- Try to prove meromorphic continuation (hard).
- In principle, global functional equations should come from *Poisson formulas* relating Schwartz spaces of different spaces, as in the local case. Idealistically: $\mathcal{F} : \mathcal{S}_2 \xrightarrow{\sim} \mathcal{S}_1$, $\vartheta^{(1)}(\mathcal{F}\xi) = \vartheta^{(2)}(\xi)$ (hard).
- One can relate Z_λ to *period integrals* in its range of convergence; when periods factorize, Z_λ also factorizes into local zeta integrals treated before.

WARNING:
ALL STATEMENTS ABOVE ARE SUBJECT TO REVISION.

Work in progress...