On the Takeda–Wood isomorphism for Hecke algebras and an intertwining relation

19 June 2023 Nankai University
29 June 2023 Hong Kong University, Number Theory Days

Wen-Wei Li
Peking University

Joint with Fei Chen (PKU).

References

C. Bushnell and P. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. LMS, (3) 77 (1998).

W. T. Gan and G. Savin, *Representations of metaplectic groups II: Hecke algebra correspondences*, Represent. Theory, 16 (2012).

S. Takeda and A. Wood. *Hecke algebra correspondences for the metaplectic group*. Transactions of the AMS, 370 (2018).

Setting

- $\varpi \in \mathfrak{o} \subset F$: uniformizer of the ring of integers of a non-Archimedean local field;
- $\mathrm{char}(F) \neq 2$, and p denotes the characteristic of $\mathfrak{o}/\varpi \simeq \mathbb{F}_q$.

We are interested in *smooth representations* over $\mathbb{C}$ of connected reductive groups over F (or their coverings). Specifically:

- $(W, \langle\cdot|\cdot\rangle)$: symplectic F-vector space.
- $G := \mathrm{Sp}(W)$: the symplectic group — $G(F)$ is a locally profinite topological group.
- $\widetilde{\mathrm{Sp}}^{(2)}(W)$: the unique nontrivial twofold cover of $\mathrm{Sp}(W)$ (the METAPLECTIC GROUP).

More conveniently: the eightfold covering

$$1 \to \mu_8 \to \widetilde{\mathrm{Sp}}(W) \to \mathrm{Sp}(W) \to 1, \quad \mu_8 := \{z \in \mathbb{C}^\times : z^8 = 1\}.$$

The metaplectic group $\tilde{G} := \widetilde{\mathrm{Sp}}(W)$ is explicitly described using

- a symplectic basis $e_1, \ldots, e_n, f_n, \ldots, f_1$ of W;
- an unitary additive character $\psi : F \to \mathbb{C}^\times$, trivial on $4 \cdot \mathfrak{o}$ but nontrivial on larger ideals.

It carries the *Weil representation* $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$ realized on the Schwartz space $\mathcal{S}(\bigoplus_{i=1}^n Fe_i)$.

We are interested in *genuine representations*, i.e. smooth representations of $\tilde{G}$ such that μ_8 acts by $\mu_8 \hookrightarrow \mathbb{C}^\times$, such as $\omega_\psi^\pm$.

Our normalization

The ω_ψ is constructed using the Heisenberg group $H(W) := W \times F$ with

$$(x, s)(y, t) = (x + y, s + t + \langle x | y \rangle).$$

Relevance of the covering $\tilde{G}$

To mention a few:

- There are Archimedean and global avatars of $\tilde{G}$.
- Θ-lifts, Fourier–Jacobi models, etc.
- As a key example of *covering groups*, studied by Matsumoto, …, Deligne, Weissman, Fan Gao, Gaitsgory–Lysenko, et al.
- In particular, $\tilde{G}$ serves as a testing ground for *Langlands program* for coverings (Weissman).

Long-term goal

In some sense, $\tilde{G}$ is "close to" being a classical group. Can we achieve an *endoscopic classification* for genuine representations of $\tilde{G}$ à la Arthur, at least when $\mathrm{char}(F) = 0$?

- **Good news # 1:** In the global case with char $= 0$, the *stable trace formula* for $\tilde{G}$ is now available.
- **Bad (?) news:** An extra difficulty for covering groups —

  Things get wild when F is dyadic (i.e. $p = 2$).

  For example, $\tilde{G} \twoheadrightarrow G(F)$ is no longer split over $G(\mathfrak{o})$.

Good news # 2

Some "vector-valued" variants for $\tilde{G}$ of the Iwahori–Hecke algebras can be handled.

- $p \neq 2$: Gan–Savin (2012).
- $p = 2$: Takeda–Wood (2018)

This is achieved by studying suitable TYPES of $\omega_{\psi}^{\pm}$.

Lattices and compact open subgroups

$$\mathcal{L}_i := \bigoplus_{j=1}^{n} \mathfrak{o} e_i \oplus \bigoplus_{1 \leqslant j \leqslant i} \varpi \mathfrak{o} f_j \oplus \bigoplus_{j>i} \mathfrak{o} f_j \subset W,$$

$$K_i := \mathrm{Stab}_{G(F)}(\mathcal{L}_i), \quad i = 0, \dots, n.$$

- $K_i = \mathrm{Stab}_{G(F)}(z_i)$ where $z_0, \dots, z_n$ are the vertices of the standard alcove of the affine building, with preimages $\widetilde{K}_i \subset \tilde{G}$;
- $K_0, \dots, K_n$ represent the maximal open subgroups, up to conjugacy;
- K_0 is the standard hyperspecial subgroup;
- $I := \bigcap_i K_i$ is the standard Iwahori subgroup;
- $J := \bigcap_{i \neq 0} K_i$.

The structures of I and J are accessible via Bruhat–Tits theory.

The types

For $0 \leqslant i \leqslant n$, we have the $\widetilde{K}_i$-submodules of $\omega_\psi^\pm|_{\widetilde{K}_i}$:

$$\tau_i^\pm = \bigotimes_{j<i} \mathcal{S}(\mathfrak{o}f_j/2\varpi\mathfrak{o}f_j)^\pm \otimes \bigotimes_{j\geqslant i} \mathcal{S}(\mathfrak{o}f_j/2\mathfrak{o}f_j),$$

$$\dim \tau_i^\pm = \frac{1}{2}q^{en}(q^i \pm 1), \quad e := \begin{cases} e(F|\mathbb{Q}_2), & p = 2 \\ 0, & p \neq 2. \end{cases}$$

We will mainly focus on

- $\tau_0 = \tau_0^+$: as an $\widetilde{I}$-module.
- τ_1^-: an $\widetilde{J}$-module, noting that $J = I \sqcup Is_0I$, with s_0 being the simple affine reflection.

These representations have a reasonably concrete description via Schrödinger's model for $\omega_\psi^\pm$.

Hecke algebras in general

General reference: Bushnell–Kutzko (1997)

R: locally profinite group, K: compact open subgroup, (τ, V_τ): finite-dimensional K-module.

$\mathcal{H}(R//K;\tau) := \left\{ f : R \to \mathrm{End}_{\mathbb{C}}(V_\tau^\vee),\ C_c^\infty,\ f(k_1 r k_2) = \check{\tau}(k_1) f(r) \check{\tau}(k_2). \right\}.$

It is an algebra under convolution. For each smooth R-representation π, the space

$$\mathbf{M}_\tau(\pi) := \mathrm{Hom}_K(\tau, \pi) \simeq (V_\tau^\vee \otimes V_\pi)^K :$$

is functorially a left $\mathcal{H}(R//K;\tau)$-module.

The goal of Type Theory. Realize each Bernstein component as a category of $\mathcal{H}(R//K;\tau)$-modules, by constructing (K, τ).

Consequence of Takeda–Wood isomorphism — preview

Let $\mathcal{G}_\psi^\pm$ be the Bernstein component (a subcategory of the category of smooth genuine $\tilde{G}$-representations) containing $\omega_\psi^\pm$.

Define

$$H_\psi^+ := \mathcal{H}(\tilde{G}//\tilde{I};\tau_0), \quad H_\psi^- := \mathcal{H}(\tilde{G}//\tilde{J};\tau_1^-).$$

Theorem (Gan–Savin for $p \neq 2$, Takeda–Wood for all p)

The functors $\mathbf{M}_{\tau_0}$ and $\mathbf{M}_{\tau_1^-}$ induce equivalences

$$\mathcal{G}_\psi^\pm \xrightarrow{\sim} H_\psi^\pm\text{—Mod}.$$

In fact, H_ψ^+ is an analogue for $\tilde{G}$ of the Iwahori–Hecke algebra.

Takeda–Wood isomorphism

- Let G^+ be the split $\mathrm{SO}(2n+1)$ and G^- its non-split inner form.
- Let $H^\pm$ be the Iwahori–Hecke algebra of $G^\pm$.

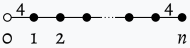
The isomorphism for H_ψ^+ is obtained as follows (modulo technicalities).

1. Construct $T_0, \ldots, T_n \in H_\psi^+$ with $\mathrm{Supp}(T_i) = \tilde{I} s_i \tilde{I}$ where s_i are the generators of the affine Weyl group of G;
2. Prove the braid and quadratic relations for them (see the next page);
3. Compare with the usual presentation of H^+.

The same idea applies to H_ψ^-.

Affine Coxeter diagrams and quadratic relations

- For H_ψ^+:



Generators: $T_0, \ldots, T_n$. Quadratic relations:

$$T_0^2 - 1 = 0 = (T_i - q)(T_i + 1), \quad 0 < i \leqslant n.$$

- For H_ψ^-:

Generators: $T_1, \ldots, T_n$. Quadratic relations:

$$(T_1 + 1)(T_1 - q^2) = 0 = (T_i + 1)(T_i - q), \quad 1 < i \leqslant n.$$

The braid relations can be read off from the diagrams above.

Theorem (Gan–Savin for $p \neq 2$, Takeda–Wood for all p)

This recipe yields isomorphisms

$$\mathrm{TW} : H_\psi^\pm \xrightarrow{\sim} H^\pm$$

between Hilbert $\mathbb{C}$-algebras with involution.

- Hilbert structure: connected to unitary structures in representation theory (eg. Plancherel formula).
- The steps 1 and 2 are largely combinatorial.
- The Weil representations $\omega_\psi^\pm$ are critical analytic ingredients in the proof. In fact, they are needed for defining $H_\psi^\pm$.

Corollaries

1. Equivalence between $\mathcal{G}_\psi^\pm$ and the counterparts for $G^\pm$.
2. Preservation of Harish-Chandra μ-functions (= Plancherel densities), up to a precise constant.

Our task

To study/compare via the Takeda–Wood isomorphism: Jacquet functor, parabolic induction and intertwining operators.

Fact

The (semi-)standard Levi subgroups of $\tilde{G}$ split via Schrödinger's model

$$\tilde{M} = \widetilde{\mathrm{Sp}}(W^\flat) \times \prod_{i \in I} \mathrm{GL}(n_i, F) \quad \text{in the 8-fold covering}$$

where $W^\flat$ is a non-degenerate symplectic subspace of W. The avatars on $\tilde{M}$ of $\tau_i^\pm$ can thus be defined as

$$\tau_i^{\tilde{M},\pm} := \underbrace{\tau_i^{\flat,\pm}}_{\text{for } \widetilde{\mathrm{Sp}}(W^\flat)} \otimes \bigotimes_{i \in I} \mathbf{1},$$

using the standard Iwahori subgroups for each factor.

Difficulties

Let T be the maximal torus of G defined from the chosen symplectic basis. Two Borel subgroups $B^{\rightarrow} \supset T \subset B^{\leftarrow}$:

$$\begin{aligned} B^{\rightarrow}\text{-simple roots} &= \epsilon_1 - \epsilon_2, \dots, 2\epsilon_n, \\ B^{\leftarrow}\text{-simple roots} &= 2\epsilon_1, \dots, \epsilon_n - \epsilon_{n-1}. \end{aligned}$$

- Take parabolic subgroups $P = MU$ with $P \supset B^{\leftarrow}$ and $M \supset T$.
- Define Iwahori subgroups and the $\tau_i^{\pm}$ using $B^{\rightarrow}$-simple roots. Only in this way could $\tau_i^{\tilde{M},\pm}$ be "aligned" with $\tau_i^{\pm}$.
- Caution: $(\tilde{I}, \tau_0)$ is **not a cover** of $(\tilde{I} \cap \tilde{M}, \tau_0^{\tilde{M}})$ if $p = 2$; same for $(\tilde{J}, \tau_1^{-})$.
  - We have to compare $H_{\psi}^{\pm}$ and $H_{\psi}^{\tilde{M},\pm}$.
  - Tools from Bushnell–Kutzko are not directly available here.

Coinvariants

Let $P = MU$ be parabolic with $P \supset B^{\leftarrow}$ and $M \supset T$. Fact: (read Bruhat–Tits!)

$$\text{Iwahori decomposition} \quad I = I_U I_M I_{U^-}$$

holds with $I_M := I \cap M(F)$ (still an Iwahori subgroup), etc.

Definition of coinvariants

For every $\tilde{I}$-module τ, let $\tau_U := \tau / \sum_{u \in I_U} \mathrm{im}(\tau(u) - \mathrm{id})$, as an $\widetilde{I_M}$-module.

Proposition

There is an explicit isomorphism of $\widetilde{I_M}$-modules $\tau_{0,U} \simeq \tau_0^{\tilde{M}}$. Ditto for τ_1^- provided that $\dim W^\flat \geqslant 1$.

The proof is similar to the computation of coinvariants of $\omega_\psi^\pm$.

Let $r_{\tilde{M}}$ be the normalized Jacquet functor from $\tilde{G}$ to $\tilde{M}$. It is the functor of U-coinvariants twisted by $\delta_P^{-1/2}$.

For $\tau \in \{\tau_0, \tau_1^-\}$ and a smooth $\tilde{G}$-representation π, there is a functorial linear map

$$\mathbf{q} : \mathbf{M}_\tau(\pi) \to \mathbf{M}_{\tau^{\tilde{M}}}(r_{\tilde{M}}(\pi)).$$

Define a homomorphism t_{nor} of Hilbert $\mathbb{C}$-algebras with involution

$$H_\psi^{\tilde{M},\pm} \xrightarrow[\sim]{\mathrm{TW}} H^{M^\pm} \xrightarrow{t_{\mathrm{nor}}^\pm} H^\pm \xrightarrow[\sim]{\mathrm{TW}^{-1}} H_\psi^\pm,$$

- $H^{M^\pm}$ are the counterparts of $H_\psi^{\tilde{M},\pm}$;
- TW means the Takeda–Wood isomorphism for $\tilde{M}$ and $\tilde{G}$, identity on GL-factors of $\tilde{M}$;
- $t_{\mathrm{nor}}^\pm$ is from Bushnell–Kutzko (involving $\delta_{P^\pm}^{1/2}$).

Compatibility

Let $\tau \in \{\tau_0, \tau_1^-\}$. Our first goal is the following

Theorem

The linear map $\mathbf{q} : \mathbf{M}_\tau(\pi) \to \mathbf{M}_{\tau^{\tilde{M}}}(r_{\tilde{M}}(\pi))$ is bijective, equivariant with respect to $t_{\mathrm{nor}} : H_\psi^{\tilde{M},\pm} \to H_\psi^\pm$.

- The counterpart for Iwahori–Hecke algebras is in Bushnell–Kutzko.
- Bijectivity follows easily from the work of Takeda–Wood. The point is to show equivariance.
- The assertion is that t_{nor} mirrors normalized Jacquet functor on the level of Hecke modules.
- By adjunction, $\mathrm{Hom}_{H_\psi^{\tilde{M},\pm}}(H_\psi^\pm, \cdot)$ then mirrors normalized parabolic induction.

Sketch of the proof

- In the setting of Iwahori–Hecke algebras for linear groups, the proof goes by "aligning" M-supported and P-positive elements in the Hecke algebras via an avatar of $\mathbf{q}$.
- Based on results from Takeda–Wood, we can obtain similar partial results for metaplectic groups (involving $\delta_P^{1/2}$).
- Must compare the recipe above with t_{nor}.

Strategy

Determine the constants of proportionality by computing the Hecke modules attached to $\omega_\psi^\pm$ and to its Jacquet modules, plus routine manipulations.

- $\omega_\psi^\pm \leftrightarrow$ the trivial $H_\psi^\pm$-module;
- the Jacquet modules of $\omega_\psi^\pm$: well-known.

Unramified principal series

Consider $T \subset B^{\leftarrow}$. Split $\tilde{T}$ using Schrödinger's model in the 8-fold covering. In this way, characters of $T(F)$ and $\tilde{T}$ are identified. Same for subtori of T.

- $I(\chi) := \mathrm{Ind}_{\tilde{B}^{\leftarrow}}^{\tilde{G}}(\chi \otimes \delta_{B^{\leftarrow}}^{1/2})$, where χ: unramified character of $T(F)$.
- $J(\chi) := \mathrm{Ind}_{\tilde{P}}^{\tilde{G}}((\omega_{\psi}^{\flat,-} \boxtimes \chi) \otimes \delta_P^{1/2})$, where $P = M_P U_P \supset B^{\leftarrow}$ with

$$M_P = \mathrm{Sp}(2) \times \mathrm{GL}(1)^{n-1},$$
$$\omega_{\psi}^{\flat} \text{ :Weil representation of } \mathrm{Sp}(2),$$
$$\chi \text{ :unramified character of } \mathrm{GL}(1, F)^{n-1}.$$

$$\mathbf{M}_{\tau_0}(I(\chi)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}[X_*(T)]}(H_{\psi}^{+}, \chi).$$

It maps $f : \tau_0 \to I(\chi)$ to the homomorphism $h \mapsto (hf)(\phi)(1_{\tilde{G}})$ where $\phi \in V_{\tau_0} = S(\bigoplus_j \mathfrak{o}/2\mathfrak{o}f_j)$ satisfies $\phi(0, \ldots, 0) = 1$.

There is a similar description of $\mathbf{M}_{\tau_1^-}(J(\chi))$.

Standard intertwining operators

- Irreducibles in $\mathcal{G}_\psi^+$ (resp. $\mathcal{G}_\psi^-$) are precisely the subquotients of $I(\chi)$ (resp. $J(\chi)$). The case for $G^\pm$ is well-known.
- The inducing characters χ from both sides can be matched.
- Having known that TW respects normalized parabolic induction, one should ask:

  Do the standard intertwining operators match?
- Denote by U the unipotent radical of $B^\leftarrow$ or P.

$$\begin{aligned} A(w,\chi) : I(\chi) &\to I(w\chi), \quad w \in \text{type } \mathrm{B}_n \text{ Weyl group}, \\ J(\chi) &\to J(w\chi), \quad w \in \text{type } \mathrm{B}_{n-1} \text{ Weyl group}. \end{aligned}$$

They are defined by $\int_{U^- \cap U^w \backslash U^w(F)}$ + rational continuation.

In contrast with the case of connected reductive groups, $A(w, \chi)$ requires sensible choice of representatives $\dot{w} \in \tilde{G}$.

- Choose representatives in $\tilde{G}$ for each simple (with respect to $B^{\leftarrow}$) reflection s_i, with corresponding root α_i, as

$$\dot{w}_i := x_{\alpha_i}(1) x_{-\alpha_i}(-1) x_{\alpha_i}(1)$$

  where $x_{\alpha_i} : F \to U_{\alpha_i}(F) \subset G(F)$ lifts canonically to $\tilde{G}$
- The representative of the long reflection is then rescaled to act as Fourier transform via ω_ψ.

Proposition (Gan–L., 2018)

These representatives satisfy the braid relation. We can thus define $\dot{w} \in \tilde{G}$ for general w, using any reduced expression.

Spherical projectors

Recall the maximal compact open subgroups $K_i \subset G(F)$ for $i = 0, \ldots, n$. Define the idempotents

$$H_\psi^+ \ni e_\psi^+(x) := \begin{cases} \frac{1}{\mathrm{vol}(\tilde{K}_0)} \check{\tau}_0(x), & x \in \tilde{K}_0 \\ 0, & x \notin \tilde{K}_0. \end{cases}$$

$$H_\psi^- \ni e_\psi^-(x) := \begin{cases} \frac{1}{\mathrm{vol}(\tilde{K}_1)} \check{\tau}_1^-(x), & x \in \tilde{K}_1 \\ 0, & x \notin \tilde{K}_1. \end{cases}$$

Lemma

They match the spherical idempotents $e^\pm$ in the Iwahori–Hecke algebras $H^\pm$ under TW.

As before, we use properties of $\omega_\psi^\pm$ to check this.

Multiplicity-one results:

Lemma

For all χ, the space $e_\psi^+ \mathbf{M}_{\tau_0}(I(\chi))$ is 1-dimensional with generator

$$\sigma^+_{\psi,\chi} : \phi \mapsto [k \mapsto k\phi(0,\ldots,0)], \quad \phi \in V_{\tau_0},\ k \in \tilde{K}_0.$$

Lemma

For all χ, the space $e_\psi^- \mathbf{M}_{\tau_1^-}(J(\chi))$ is 1-dimensional with generator

$$\sigma^-_{\psi,\chi} : \phi \mapsto [k \mapsto k\phi(\cdot,0,\ldots,0)], \quad \phi \in V_{\tau_1^-},\ k \in \tilde{K}_1.$$

Observe that $e_\psi^+ \mathbf{M}_{\tau_0}(\cdot) = \mathrm{Hom}_{\tilde{K}_0}(\tau_0,\cdot)$ (same for $e_\psi^- \ldots$).

The proof of Lemmas makes use of the computation of coinvariants and Frobenius reciprocity.

Identity of intertwining operators

Suppose χ is an unramified character.

- Let $I^+(\chi)$ (resp. $I^-(\chi)$) denote the principal series of G^+ (resp. G^-) induced from χ. Their spherical parts: $\mathbb{C}\sigma_\chi^\pm$.
- On the level of Hecke modules, they can be identified with $\mathbf{M}_{\tau_0}(I(\chi))$ and $\mathbf{M}_{\tau_1^-}(J(\chi))$ via TW.

Question (recalled)

Do the standard intertwining operators $A(w,\chi)$ match up to an explicit constant? (depends on measures of unipotent radicals)

Strategy

A priori, they differ by a rational function in χ. The function can be determined by comparing the images of $\sigma_{\psi,\chi}^\pm$ and $\sigma_\chi^\pm$.

A vector-valued Gindikin–Karpelevich formula for $\tilde{G}$

Let w be a Weyl group element (type B_n if $+$, type B_{n-1} if $-$).

Theorem

As rational functions in χ, we have

$$\frac{A(w,\chi)\sigma^{\pm}_{\psi,\chi}}{\sigma^{\pm}_{\psi,w\chi}} = |2|_F^{d(w)/2} \frac{A^{G^{\pm}}(w,\chi)\sigma^{\pm}_{\chi}}{\sigma^{\pm}_{w\chi}}$$

where $d(w)$ is the number of long roots flipped by w. (Measures: to be specified)

Corollary

Under the isomorphism TW, $A(w,\chi)$ matches $|2|_F^{d(w)/2} A^{G^{\pm}}(w,\chi)$.

Reduction

In the $+$ case, the multiplicativity of $A(w, \chi)$ reduces the problem to rank 1.

- $G = \mathrm{Sp}(2) \simeq \mathrm{SL}(2)$,
- $G^+ \simeq \mathrm{PGL}(2)$,
- w represented by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,
- χ: unramified character of $F^\times$,
- $z := \chi(\varpi)$, $|z| < 1$ (i.e. in the range of convergence).

Reason: Every reflection using short roots occurs in some copy of GL shared by G and $G^\pm$.

We are led to study an explicit p-adic integral over $\begin{pmatrix} 1 & \\ t & 1 \end{pmatrix}$ with $|t| \geqslant 1$.
Measure: $\mathrm{vol}(\mathfrak{o}) = 1$.

End of the proof of Gindikin–Karpelevich formula

RHS of the Theorem is evaluated by the usual GK formula for PGL(2):

$$A^{G^+}(w, \chi)\sigma_\chi^+ = \frac{1 - q^{-1}z}{1 - z}\sigma_{w\chi}^+.$$

We are ultimately led to the technical calculation

$$\int_{\mathfrak{o}^\times} \gamma_\psi(2\varpi^{-k}a)\, \mathrm{d}a = \begin{cases} (1 - q^{-1})|2|_F^{1/2}, & k : \text{even}, \\ 0, & k : \text{odd} \end{cases}$$

where we put the Haar measure on F with $\mathrm{vol}(\mathfrak{o}) = 1$, and γ_ψ is the *Weil index*:

$$\gamma_\psi(b) = \mathrm{PV}\int_F \psi\left(\frac{bx^2}{2}\right) \mathrm{d}_{\mathrm{selfdual}}x.$$

Remark. There exists a less computational approach via μ-functions.

About the — case

In the — case, we are reduced to the rank 2 case.

- Explicit computations are no longer amenable in $\widetilde{\mathrm{Sp}}(4)$.
- Fortunately, we can avoid (most) p-adic integrals by using the matching of Harish-Chandra μ-functions by Takeda–Wood.
- However, the choice of Haar measures is different in this case.

Prospects

The applications below are most interesting when F is dyadic.

1. Reducibility of normalized parabolic induction is preserved by TW. This implies that TW preserves tempered L-parameters, without using any Θ-lift.
2. Matching of standard intertwining operators from arbitrary $P \supset B^{\leftarrow}$ — reduce to the minimal case we addressed.
3. TW preserves Aubert involution (as a functor).

There are also potential applications to Arthur's *local intertwining relations* for $\tilde{G}$. It serves as the original impetus of this work.

Cf. the Chapter 7 of

J. Arthur, *The endoscopic classification of representations*, AMS. Coll. 61 (2013).

Thanks for your attention