ICCM International Consortium of Chinese Mathematicians

Contragredient representations over function fields

李文威 | Wén-Wēi Lǐ

中國科學院 | Chinese Academy of Sciences

丁酉年冬月初十・廣州 | Dec 27, 2017; Guǎngzhōu

Desiderata

Let G be a connected reductive group over a field F, such as GL_n, Sp_{2n} or E_8. Let $Z \subset G$ be the maximal F-split central torus.

- F local: classify the irreducible smooth representations of $G(F)$, temporarily with coefficients in $\mathbf{C}$.
- F global: decompose the unitary representation $L^2(G(F)\backslash G(\mathbf{A}_F))$ as explicitly as possible, on which $\forall g \in G(\mathbf{A}_F)$ acts by $f(x) \mapsto f(xg)$.

Here $\mathbf{A}_F = \prod'_{v:\text{places}} F_v$. A closely related problem is to decompose

$$L^2(G(F)\backslash G(\mathbf{A}_F)/\Xi), \quad F : \text{global}$$

for an appropriate subgroup $\Xi \subset Z(F)\backslash Z(\mathbf{A}_F)$, such that $G(F)\backslash G(\mathbf{A}_F)/\Xi$ has finite volume ("reduction theory").

TODAY: Some tiny aspect (in progress) of this vast terrain.

Langlands parameterization

- Denote by W_F and WD_F the Weil and Weil–Deligne groups associated to F.
- The L-group ${}^L G = \hat{G} \rtimes \mathrm{Gal}(\tilde{F}|F)$ (say over $\mathbf{C}$) is defined combinatorially by reversing the root datum of G, where $\tilde{F}|F$ is the splitting field of G.
- For F local, Langlands proposes a conjectural arrow

$$\Pi(G) := \{\text{irreps of } G(F)\}/\simeq\; \to \Phi(G)$$

  where $\Phi(G)$ is the set of L-parameters $\phi : \mathrm{WD}_F \to {}^L G$ up to $\hat{G}$-conjugacy. For any $\phi \in \Phi(G)$, let $\Pi_\phi \subset \Pi(G)$ be its fiber, also known as the L**-packet**.
- Fundamental issues, for tempered L-packets at least: surjectivity of this arrow, internal structures of Π_ϕ, "stability", relation to inner twists, etc.

- For global F, Langlands and Arthur conjectured a decomposition of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$ indexed by global parameters $\psi : L_F \times \mathrm{SL}_2 \to {}^L G$ (here: Arthur's SL_2), where L_F is the hypothetical **Langlands group**.
- At any rate, L_F should admit homomorphisms $L_F \twoheadrightarrow \mathrm{W}_F$ and $\mathrm{WD}_{F_v} \hookrightarrow L_F$ for each place v of F.
- The local and global conjectures are inextricably linked.

We will review some recent progresses in due course.

Let F be local and G be quasi-split. An enhanced version of the **local Langlands correspondence** (LLC) predicates on the internal structure of packets Π_ϕ extended à la Vogan across pure inner twists, when ϕ is tempered.

- Fix a **Whittaker datum** $\mathfrak{w} = (U, \chi)$ where $U \subset G$ is a maximal unipotent subgroup and χ is a generic character on $U(F)$. Eg. by fixing an F-pinning and an additive character ψ of F......
- Set $\mathscr{S}_\phi := \pi_0(\mathrm{Z}_{\hat{G}}(\mathrm{im}\ \phi))$.
- Conjecturally, Π_ϕ is in bijection with $\mathrm{Irr}(\mathscr{S}_\phi)$. The trivial representation of $\mathscr{S}_\phi$ should match the unique $\mathfrak{w}$-generic member in Π_ϕ (cf. the tempered L-packet conjecture by Shahidi.)

The case of equal characteristics

Assume $\mathrm{char}(F) = p > 0$ and fix a prime $\ell \neq p$. Consider global fields $F = \mathbf{F}_q(X)$ for a geom. irred. smooth proper curve $X_{/\mathbf{F}_q}$.

- We can and do replace $\mathbf{C}$ by $\overline{\mathbf{Q}_\ell}$.
- The representation theory is now of an **algebraic** nature. Algebro-geometric tools are directly available.
- For example, every irrep π (local) or every cusp form (global) can be defined over some finite extension $E|\mathbf{Q}_\ell$.

The L-parameters in question are

$$\phi : \mathrm{W}_F \to {}^L G := \hat{\mathrm{G}}(\overline{\mathbf{Q}_\ell}) \rtimes \mathrm{Gal}(\tilde{F}|F),$$

continuous, Frobenius-semisimple and commuting with projections to $\mathrm{Gal}(\tilde{F}|F)$.
NOTE: We will disregard Arthur's SL_2 and the parameters will always emit from Gal_F or W_F.

- The global equal-characteristic case with $G = \mathrm{GL}_n$ is accomplished by L. Lafforgue (2002), following the ideas of Drinfeld *et al.*
- In the equal-characteristic case, V. Lafforgue arXiv:1209.5352 and Genestier–Lafforgue arXiv:1709.00978 gave such a parameterization $\pi \mapsto \phi$ for general G, which we will review later.
- The case of local $F \supset \mathbf{Q}_p$: the Fargues–Scholze program. arXiv:1602.00999

Contragredients

Let G: reductive group over a local field F. The LLC is expected to "respect" natural operations on representations, such as parabolic inductions.

The contragredient

If π is an irrep of $G(F)$, then $\check{\pi}$ = the smooth dual endowed with $\langle \check{\pi}(g)\lambda, v \rangle = \langle \lambda, \pi(g^{-1})v \rangle$: still irreducible.

Natural questions

1. What is the contragredient in terms of Langlands parameters?
2. How about its effect on the members of the packet?

Surprisingly, this has not been discussed in the literature until 2012.

Conjecture (Adams–Vogan, D. Prasad)

If π has L-parameter ϕ, then $\check{\pi}$ has L-parameter ${}^L\theta \circ \phi$, where ${}^L\theta$ is the **Chevalley involution** on ${}^LG$ (see below).
If $\pi \in \Pi_\phi$ corresponds to ρ: an irreducible character of $\mathscr{S}_\phi$, then $\check{\pi}$ corresponds to $(\rho \circ {}^L\theta)^\vee$ tensored with an explicit character ι_{-1} of $\mathscr{S}_\phi$.

Recall that $\hat{G}$ has a Gal_F-stable pinning $(\hat{B}, \hat{T}, \ldots)$. The Chevalley involution θ of $\hat{G}$ is characterized by

- θ preserves that pinning;
- θ acts as $t \mapsto w_0(t^{-1})$ on $\hat{T}$, where w_0 = the longest Weyl element;
- θ extends canonically to an L-automorphism ${}^L\theta$ of ${}^LG$ (in the obvious manner).

For every semisimple $g \in \hat{G}$, we have $\theta(g) \overset{\text{conj}}{\sim} g^{-1}$.

Construction of ι_{-1} (cf. [Kaletha 2013, §4]) There are canonical homomorphisms

$$\begin{array}{ccc} \pi_0\left(Z_{\hat{G}}(\mathrm{im}\,\phi)/Z_{\hat{G}}^{\mathrm{Gal}}\right) & \longrightarrow & \ker\left[H^1(\mathrm{W}_F, Z_{\hat{G}^{sc}}) \to H^1(\mathrm{W}_F, Z_{\hat{G}})\right] \\ \Big\uparrow & & \Big\downarrow\!\!\!\!\downarrow \\ \mathscr{S}_\phi & & \left(G^{\mathrm{ad}}(F)/G(F)\right)^{\text{Pontryagin dual}} \end{array}$$

We take the $g_1 \in T^{\mathrm{ad}}(F)$ acting as -1 on each $\mathfrak{g}_\alpha$ where α is any B-simple root. It yields a character of $\mathscr{S}_\phi$.

Known cases:

- $F = \mathbf{R}$: Adams and Vogan (2016).
- char$(F) = 0$ and G: quasisplit orthogonal or symplectic group: Kaletha (2013). In this case, the LLC comes from Arthur's endoscopic classification.
- F is non-Archimedean, depth-zero or epipelagic parameters: Kaletha (2013).

A precondition is to have the Langlands parameterization $\pi \rightsquigarrow \phi$, or some approximation thereof.

The works of Lafforgue (global)

Let $F = \mathbf{F}_q(X)$ be a global field. Fix a finite closed $N \subset X$ (level structure) $\rightsquigarrow K_N \subset G(\mathbf{A}_F)$: compact open subgroup.

$$\underbrace{\mathrm{Bun}_{G,N}(\mathbf{F}_q)}_{\text{as a set}} = \bigsqcup_{\alpha:\text{some inner twists}} G_\alpha(F)\backslash G(\mathbf{A}_F)/K_N,$$

$$\forall \alpha,\ G(\mathbf{A}_F) = G_\alpha(\mathbf{A}_F).$$

Let $E \supset \mathbf{Q}_\ell$ be sufficiently large. V. Lafforgue (2012) obtained a decomposition

$$H_{\varnothing,1} := C_c^{\mathrm{cusp}}\left(\mathrm{Bun}_{G,N}(\mathbf{F}_q)/\Xi; E\right) = \bigoplus_{\sigma:\mathrm{Gal}_F \to {}^L G} \mathfrak{H}_\sigma.$$

Roughly speaking, this is done in two steps.

1. One uses the geometry of the moduli space of штука to define **excursion operators** $S_{I,W,x,\xi,\vec{\gamma}}$ where
   - I: finite set, $\vec{\gamma} = (\gamma_i)_{i\in I} \in \mathrm{Gal}_F^I$;
   - $W \in \mathrm{Rep}_E({}^L G^I)$, and $x \in W$, $\xi \in W^\vee$ are $\hat{G}$-invariant.

   Re-encoded as $S_{I,f,\vec{\gamma}}$ where $f \in \mathscr{O}(\hat{G}\backslash {}^L G^I /\!\!/ \hat{G}; E)$, they generate a **commutative** subalgebra $\mathscr{B}$ of $\mathrm{End}_E(H_{\varnothing,1})$, hence decompose $H_{\varnothing,1} = \bigoplus_\nu \mathfrak{H}_\nu$ into generalized eigenspaces.
2. From ν to L-parameters $\sigma : \mathrm{Gal}_F \to {}^L G(\overline{\mathbf{Q}_\ell})$ up to $\hat{G}$-conjugacy: an invariant-theoretic construction, via the so-called ${}^L G$-pseudo-characters.

NOTE. σ is semisimple.

This furnishes the automorphic-to-Galois direction of Langlands' conjecture, for general G.

Local case

In arXiv:1709.00978, Genestier and Lafforgue obtained a Langlands parameterization over local fields $F \supset \mathbf{F}_p$.

- This is done by constructing elements $\mathfrak{z}_{I,f,\vec{\gamma}}$ in Bernstein's center (over $\mathfrak{o}_E$) of G, where $f \in \mathscr{O}(\hat{G}\backslash {}^L G^I /\!\!/ \hat{G}; \mathfrak{o}_E)$ and $\vec{\gamma} \in \mathrm{W}_F^I$.
- Compatible with normalized parabolic induction. Moreover: local-global compatibility **up to semi-simplification**.
- The apparatus of pseudo-characters attaches to π a **semisimple** L-parameter ϕ.

Note

We expect that ϕ is the semi-simplification of the "real" L-parameter of π.

Sketch of the ideas

Let $I = I_1 \sqcup \cdots \sqcup I_k$ (finite sets) and $W \in \mathrm{Rep}_E({}^L G^I)$.

- **Geometric Satake** $\rightsquigarrow$ perverse sheaves $S_{I,W}^{(I_1,\dots,I_k)}$ on the BD-Grassmannian $\mathrm{Gr}_{I,W}^{(I_1,\dots,I_k)}$, normalized relative to X^I + equivariance.
- Local models + $S_{I,W}^{(I_1,\dots,I_k)}$ $\rightsquigarrow$ normalized perverse sheaves $\mathscr{F}_{N,I,W}^{(I_1,\dots,I_k)}$ on the quotient of the moduli of штука $\mathrm{Cht}_{N,I,W}^{(I_1,\dots,I_k)}/\Xi$.
- Take truncation parameter μ and !-push-forward to $(X \smallsetminus N)^I$ to obtain $\mathscr{H}_{N,I,W}^{\leq\mu}$: independent of partition.
- Choose geometric generic point $\bar{\eta} \to \eta$ (resp. $\overline{\eta^I} \to \eta^I$) of X (resp. X^I), and set

$$H_{I,W} := \big(\varinjlim_{\mu} \mathrm{H}^0 \mathscr{H}_{N,I,W}^{\leq\mu}\big|_{\Delta(\bar{\eta})}\big)^{\text{Hecke-finite}}.$$

Everything is functorial in W with various nice properties (eg. "coalescence"), and we recover the earlier $H_{\varnothing,1}$.

- The excursion operator is defined in three stages: **creation**, $\pi_1(\eta,\bar{\eta})^I$-action on the stalk over $\overline{\eta^I}$, then **annihilation**.
- The action of $\pi_1(\eta,\bar{\eta})^I$ (or of $\mathrm{FWeil}(\eta^I,\overline{\eta^I})$) combines the $\pi_1(\eta^I,\overline{\eta^I})$-action and the **partial Frobenius morphisms**, via Drinfeld's Lemma.
- The pairing $\langle h,h'\rangle := \int_{\mathrm{Bun}/\Xi} hh'$ on $H_{\varnothing,1}$ also has a sheaf-theoretic origin: it extends to $\mathscr{H}^{\leq\mu}_{N,I,W}$ and arises ultimately from a functorial isomorphism

$$\mathbf{D}S^{(I_1,\dots,I_k)}_{I,W} \xrightarrow{\sim} S^{(I_1,\dots,I_k)}_{I,W^{\vee,\theta}}, \quad \mathbf{D} : \text{normalized Verdier dual}$$

  where $W^{\vee,\theta} \in \mathrm{Rep}_E({}^LG^I)$ is the contragredient twisted by the Chevalley involution ${}^L\theta$ of ${}^LG^I$.

Remark. The last ingredient is not necessary for newer versions of [Laf]. Nonetheless......

Back to the contragredient conjecture

Let $F \supset \mathbf{F}_p$ be local and $G_{/F}$ reductive.
Given the work of Genestier–Lafforgue, one can try to address "the first layer" of the Adams–Vogan–Prasad conjecture.

Terminology

The semisimple $\phi : \mathrm{W}_F \to {}^L G$ (up to $\hat{G}$-conjugacy) associated to an irrep π is called the **GL-parameter** of π.

Goal: a coarse form of the Adams–Vogan–Prasad conjecture

Show that if ϕ is the GL-parameter of π, then ${}^L\theta \circ \phi$ is the GL-parameter of $\check{\pi}$.

Draw back. We do not look into the internal structures of packets.

The local-global argument

1. Reduce to the case π supercuspidal.
2. Upon twisting by an unramified character, we may even assume π is integral with central character of finite order, defined over some finite $E|\mathbf{Q}_\ell$.
3. Hence (G, π) can be globalized into a cuspidal automorphic representation $\mathring{\pi}$ of $G(\mathbf{A}_F)$ (standard argument: Poincaré series or trace formula), invariant under a suitable lattice Ξ.
4. Take level N sufficiently deep. By [GL], the local GL-parameters of $\mathring{\pi}$ are the semi-simplifications of the global parameter.

Next step. Bring contragredients into the picture.

1. Under the invariant pairing $\langle h, h' \rangle = \int_{\mathrm{Bun}_{G,N}(\mathbf{F}_q)/\Xi} hh'$, we see that $\mathring{\pi}$ must pair non-degenerately with some other cuspidal automorphic representation $\mathring{\pi}'$. As irreducible $G(\mathbf{A}_F)$-representations:

$$(\mathring{\pi}')^{K_N} \neq \{0\}, \quad \mathring{\pi}' \simeq \mathring{\pi}^{\vee}.$$

2. If $\mathring{\pi}^{K_N} \hookrightarrow \mathfrak{H}_\nu$ for some character $\nu : \mathscr{B} \to \overline{\mathbf{Q}_\ell}$, then $(\mathring{\pi}')^{K_N} \hookrightarrow \mathfrak{H}_{\nu^*}$ where

$$\nu^*(S) = \nu(S^*), \quad S \in \mathscr{B},$$

setting S^* := the **transpose** of S with respect to $\langle \cdot, \cdot \rangle$.

3. So we have to describe the transpose of excursion operators. It turns out that

$$S^*_{I,f,\vec{\gamma}} = S_{I,f^\dagger,\vec{\gamma}^{-1}}, \quad f^\dagger(\vec{x}) = f\left({}^L\theta(\vec{x})^{-1}\right).$$

Ideas for computing the transpose

Consider $S_{I,W,x,\xi,\vec{\gamma}}$ and recall its construction.

- The transposes of the creation and annihilation operators have been given in [Laf].
- It remains to show that the duality pairing on $\varinjlim_{\mu} \mathscr{H}_{N,I,W}^{\leq\mu}\big|_{\overline{\eta^I}}$ is invariant under $\pi_1(\eta,\bar{\eta})^I$.
- Furthermore: reduce to the invariance under partial Frobenius morphisms (recall Drinfeld's Lemma).
- Unsurprisingly, it boils down to the invariance of $\mathbf{D}S_{I,W}^{(I)} \simeq S_{I,W^{\vee,\theta}}^{(I)}$ under the Frobenius morphism.

The statement about $S_{I,f,\vec{\gamma}}^{*}$ follows directly.

The contragredient conjecture for GL-parameters is deduced as follows.

- Let σ be the L-parameter of $\mathring{\pi}$ furnished by [Laf].
- Recall: $(\mathring{\pi}')^{K_N}$ lives inside the generalized eigenspace $\mathfrak{H}_{\nu^*}$ of $\nu^* : \mathscr{B} \to \overline{\mathbf{Q}_\ell}$.
- One infers from the theory of pseudo-characters that ν^* gives rise to ${}^L\theta \circ \sigma$: the (global) L-parameter of $\mathring{\pi}'$.
- Local-global compatibility, etc. $\implies$ $\check{\pi}$ has GL-parameter ${}^L\theta \circ \phi$, where $\phi = \sigma^{\mathrm{ss}}$ is the GL-parameter of π.

Duality involutions: Heuristic

Let F be a local field.

Fact (Gelfand–Kazhdan)

For $\mathrm{GL}_n(F)$, the automorphism $g \mapsto {}^t g^{-1}$ takes any irrep π to $\check{\pi}$.

MVW involutions (Mœglin–Vignéras–Waldpsurger)

For any classical group G, there is an involution G taking any smooth irrep π of $G(F)$ to $\check{\pi}$.

- For $G = \mathrm{Sp}_{2n}$, the MVW involution is simply the conjugation by some $g \in \mathrm{GSp}_{2n}(F)$ with similitude factor -1.
- The conjectural generalization below is due to D. Prasad arXiv:1705.03262.

Let G be quasisplit over F. Fix an F-pinning $(B, T, \dots)$ of G and fix $\psi : F \to \overline{\mathbf{Q}_\ell}^\times$.

Definition (D. Prasad)

Set $\iota_G := \iota_{-1} \circ \theta$ where

- θ is the Chevalley involution of G,
- ι_{-1} comes from the $T^{\mathrm{ad}}(F)$-action, as seen earlier; it "flips" the pinning.

It is defined over F and *a priori* depends on the pinning.

Conjecture (D. Prasad)

Let π be a generic irrep of $G(F)$ relative to the pinning and ψ. Then $\pi \circ \iota_G \simeq \check{\pi}$.

This is actually a coarse form of Prasad's original statement. When $F = \mathbf{R}$, it is essentially done by Adams, and ι_G is independent of pinnings.

Motivating the conjecture

Fix $\overline{\mathbf{Q}_\ell} \simeq \mathbf{C}$. Assume LLC for G and let Π_ϕ be a tempered L-packet for G.

The tempered L-packet conjecture (Shahidi)

There exists exactly one generic member $\pi \in \Pi_\phi$, with respect to the given pinning and ψ.

Let $\pi \in \Pi_\phi$ be generic as before. Assume

- the contragredient conjecture for π;
- the tempered L-packet conjecture for Π_ϕ;
- the local **trivial functoriality** with respect to the L-automorphism ${}^L\theta : {}^LG \to {}^LG$. (NOTE: ${}^L\theta$ is "dual" to ι_G).

Then one can show $\pi \circ \iota_G \simeq \check{\pi}$.
Idea: Both sides are generic relative to the pinning and ψ, and belong to $\Pi_{{}^L\theta \circ \phi}$.

Remark

Let $F \supset \mathbf{F}_p$ and state everything in terms of GL-parameters. Assuming the tempered L-packet conjecture for Π_ϕ (uniqueness part only), one can prove Prasad's conjecture for $\pi \in \Pi_\phi$.

NOTE: the (global) "trivial functoriality" is done in [Laf].

Caveat

As the GL-parameters are semi-simplifications of L-parameters, the tempered L-packet conjecture will only hold for a limited class of $\phi : \mathrm{W}_F \to {}^L G$.

EXAMPLE: the regular supercuspidal L-parameters (Kaletha).

Question

For $F \supset \mathbf{Q}_p$, can these techniques be adapted to the setting of Fargues–Scholze?

多謝晒！

Updated on May 9, 2018