Basic representation theory

[draft]

Incomplete notes for a mini-course given at UCAS, Summer 2017

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Version: December 10, 2018

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1 Topological groups

1.1 Definition of topological groups

For any group G, we write $1 = 1_G$ for its identity element and write G^{op} for its opposite group; in other words, G^{op} has the same underlying set as G, but with the new multiplication $(x, y) \mapsto yx$.

Denote by $\mathbb{S}^1$ the group $\{z \in \mathbb{C}^\times : |z| = 1\}$ endowed with its usual topological structure.

Definition 1.1. A topological group is a topological space G endowed with a group structure, such that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. Unless otherwise specified, we always assume that G is Hausdorff.

By a homomorphism $\varphi : G_1 \to G_2$ between topological groups, we mean a continuous homomorphism. This turns the collection of all topological groups into a category TopGrp, and it makes sense to talk about isomorphisms, etc. We write $\mathrm{Hom}(G_1, G_2)$ and $\mathrm{Aut}(G)$ for the sets of homomorphisms and automorphisms in TopGrp.

Let G be a topological group. As is easily checked, (i) G^{op} is also a topological group, (ii) $\mathrm{inv} : x \mapsto x^{-1}$ is an isomorphism of topological groups $G \xrightarrow{\sim} G^{\mathrm{op}}$ with $\mathrm{inv} \circ \mathrm{inv} = \mathrm{id}$, and (iii) for every $g \in G$, the translation map $L_g : x \mapsto gx$ (resp. $R_g : x \mapsto xg$) is a homeomorphism from G to itself: indeed, $L_{x^{-1}} L_x = \mathrm{id} = R_{x^{-1}} R_x$.

By translating, we see that if $\mathscr{N}_1$ is the set of open neighborhoods of 1, then the set of open neighborhoods of $x \in G$ equals $\{xU : U \in \mathscr{N}_1\}$; it also equals $\{Ux : U \in \mathscr{N}_1\}$.

We need a few elementary properties of topological groups. Notation: for subsets $A, B \subset G$, we write $AB = \{ab : a \in A, b \in B\} \subset G$; similarly for ABC and so forth. Also put $A^{-1} = \{a^{-1} : a \in A\}$.

Proposition 1.2. *For any topological group G (not presumed Hausdorff), the following are equivalent:*

(i) G is Hausdorff;

(ii) the intersection of all open neighborhoods containing 1 is $\{1\}$;

(iii) $\{1\}$ is closed in G.

Proof. It is clear that (i) $\implies$ (ii), (iii). Also (ii) $\iff$ (iii) since it is routine to check that $\overline{\{1\}} = \bigcap_{U \ni 1} U$.

Let us prove (ii) $\implies$ (i) as follows. Let $\nu : G \times G \to G$ be the function $(x, y) \mapsto xy^{-1}$. Then G is Hausforff if and only if the diagonal $\Delta_G \subset G \times G$ is closed, whilst $\Delta_G = \nu^{-1}(1)$. Now (ii) implies (iii) which in turn implies Δ_G is closed. □

Proposition 1.3. *Let G be a topological group (not necessarily Hausdorff) and $A, B \subset G$ be subsets.*

1. *If one of A, B is open, then AB is open.*
2. *If both A, B are compact, then so is AB.*
3. *If one of A, B is compact and the other is closed, then AB is closed.*

Proof. Suppose that A is open. Then $AB = \bigcup_{b \in B} Ab$ is open as well.

Suppose that A, B are compact. The image AB of $A \times B$ under multiplication is also compact.

Finally, suppose A is compact and B is closed. If $x \notin AB$, then xB^{-1} is closed and disjoint from A. If we can find an open subset $U \ni 1$ such that $UA \cap xB^{-1} = \varnothing$, then $U^{-1}x \ni x$ will be an open neighborhood disjoint from AB, proving that AB is closed.

To find U, set $V := G \setminus xB^{-1}$. For every $a \in A \subset V$ there exists an open subset $U'_a \ni 1$ with $U'_a a \subset V$. Take an open $U_a \ni 1$ with $U_a U_a \subset U'_a$. Compactness furnishes a finite subset $A_0 \subset A$ such that $A \subset \bigcup_{t \in A_0} U_t t$. We claim that $U := \bigcap_{t \in A_0} U_t$ satisfies the requirements. Indeed, $U \ni 1$ and each $a \in A$ belongs to $U_t t$ for some $t \in A_0$, hence

$$Ua \subset UU_t t \subset U_t U_t t \subset U'_t t \subset V,$$

thus $UA \subset V$. □

Recall that a topological space is *locally compact* if every point has a compact neighborhood.

Proposition 1.4. *Let H be a subgroup of a topological group G (not necessarily Hausdorff). Endow G/H with the quotient topology.*

1. *The quotient map $\pi : G \to G/H$ is open and continuous.*
2. *If G is locally compact, so is G/H.*
3. *G/H is Hausdorff (resp. discrete) if and only if H is closed (resp. open).*

The same holds for $H \backslash G$.

Proof. The quotient map is always continuous. If $U \subset G$ is open, then $\pi^{-1}(\pi(U)) = UH$ is open by 1.3, hence $\pi(U)$ is open.

Suppose G is locally compact. By homogeneity, it suffices to argue that the coset $H = \pi(1)$ has compact neighborhood in G/H. Let $K \ni 1$ be a compact neighborhood in G. Choose a neighborhood $U \ni 1$ such that $U^{-1}U \subset K$. Claim: $\overline{\pi(U)} \subset \pi(K)$. Indeed, if $gH \in \overline{\pi(U)}$, then the neighborhood UgH intersects $\pi(U)$; that is, $ugH = u'H$ for some $u, u' \in U$. Hence $gH = u^{-1}u'H \in \pi(U^{-1}U) \subset \pi(K)$. Therefore $\pi(U) \ni \pi(1)$ is an open neighborhood with $\overline{\pi(U)}$ compact, since $\pi(K)$ is compact.

If G/H is Hausdorff (resp. discrete) then $H = \pi^{-1}(\pi(1))$ is closed (resp. open). Conversely, suppose that H is closed in G. Given cosets $xH \neq yH$, choose an open neighborhood $V \ni 1$ in G such that $Vx \cap yH = \varnothing$; equivalently $VxH \cap yH = \varnothing$. Then choose an open $U \ni 1$ in G such that $U^{-1}U \subset V$. It follows that $UxH \cap UyH = \varnothing$, and these are disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$. All in all, G/H is Hausdorff.

On the other hand, H is open implies $\pi(H)$ is open, thus all singletons in G/H are open, whence the discreteness.

As for $H \backslash G$, we pass to G^{op}. □

Lemma 1.5. *Let G be a locally compact group (not necessarily Hausdorff) and let $H \subset G$ be a subgroup. Every compact subset of $H \backslash G$ is the image of some compact subset of G.*

Proof. Let $U \ni 1$ be an open neighborhood in G with compact closure $\overline{U}$. For every compact subset $K^\flat$ of $H \backslash G$, we have an open covering $K^\flat \subset \bigcup_{g \in G} \pi(gU)$, hence there exists $g_1, \ldots, g_n \in G$ such that

$$K^\flat \subset \bigcup_{i=1}^{n} \pi(g_i U) = \pi\left(\bigcup_{i=1}^{n} g_i U \right) \subset \pi\left(\bigcup_{i=1}^{n} g_i \bar{U} \right)$$

Take the compact subset $K := \bigcup_{i=1}^{n} g_i \bar{U} \cap \pi^{-1}(K^\flat)$ and observe that $\pi(K) = K^\flat$. □

Remark 1.6. The following operations on topological groups are evident.

1. Let $H \subset G$ be a closed normal subgroup, then G/H is a topological group by Proposition 1.4, locally compact if G is. We leave it to the reader to characterize G/H by universal properties in TopGrp.

2. If G_1, G_2 are topological groups, then $G_1 \times G_2$ is naturally a topological group. Again, it has an outright categorical characterization: the product in TopGrp. More generally, one can form fibered products in TopGrp.

3. In a similar vein, $\varprojlim$ exist in TopGrp: their underlying abstract groups are just the usual $\varprojlim$.

Harmonic analysis, in its classical sense, applies mainly to locally compact topological groups. Hereafter we adopt the shorthand *locally compact groups*.

Remark 1.7. The family of locally compact groups is closed under passing to closed subgroups, Hausdorff quotients and finite direct products. However, infinite direct products usually yield non-locally compact groups. This is one of the motivation for introducing *restrict products* into harmonic analysis.

Example 1.8. Discrete groups are locally compact; finite groups are compact.

Example 1.9. The familiar groups $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are locally compact; so are $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{R}^\times, \cdot)$. The identity connected component $\mathbb{R}^\times_{>0}$ of $\mathbb{R}^\times$ is isomorphic to $(\mathbb{R}, +)$ through the logarithm. The quotient group $(\mathbb{R}/\mathbb{Z}, +) \simeq \mathbb{S}^1$ (via $z \mapsto e^{2\pi i z}$) is compact.

1.2 Local fields

Just as the case of groups, a *locally compact field* is a field F with a locally compact Hausdorff topology, such that $(F, +)$ is a topological group, and that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ (on $F^\times$) are both continuous.

Definition 1.10. A *local field* is a locally compact field that is not discrete.

A detailed account of local fields can be found in any textbook on algebraic number theory. The topology on a local field F is always induced by an absolute value $|\cdot|_F : F \to \mathbb{R}_{\geq 0}$ satisfying

$$\begin{gathered} |x|_F = 0 \iff x = 0, \quad |1|_F = 1; \\ |x + y|_F \leq |x|_F + |y|_F; \\ |xy|_F = |x| \cdot |y|_F. \end{gathered}$$

Furthermore F is complete with respect to $|\cdot|$. Up to continuous isomorphisms, local fields are classified as follows.

Archimedean The fields $\mathbb{R}$ and $\mathbb{C}$, equipped with the usual absolute values;

Non-archimedean, characteristic zero The fields $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$ (the p-adic numbers, where p is a prime number) or their finite extensions;

Non-archimedean, characteristic $p > 0$ The fields $\mathbb{F}_q(\!(t)\!) = \mathbb{F}_q[\![t]\!][\frac{1}{t}]$ of Laurent series in the variable t or their finite extensions, where q is a power of p. Here $\mathbb{F}_q$ denotes the finite field of q elements.

Let F be a non-archimedean local field. It turns out the *ultrametric inequality* is satisfied:

$$|x + y|_F \leq \max\{|x|_F, |y|_F\}, \quad \text{with equality when } |x| \neq |y|.$$

Furthermore, $\mathfrak{o}_F = \{x : |x| \leq 1\}$ is a subring, called the *ring of integers*, and $\mathfrak{p}_F = \{x : |x| < 1\}$ is its unique maximal ideal. In fact $\mathfrak{p}_F$ is of the form (ϖ_F); here ϖ_F is called a *uniformizer* of F, and

$$F^\times = \varpi_F^{\mathbb{Z}} \times \mathfrak{o}_F^\times, \quad \mathfrak{o}_F^\times = \{x : |x| = 1\}.$$

The normalized absolute value $|\cdot|_F$ for non-archimedean F is defined by

$$|\varpi_F| = q^{-1}, \quad q := |\mathfrak{o}_F/\mathfrak{p}_F|$$

where ϖ_F is any uniformizer. We will interpret $|\cdot|_F$ in terms of modulus characters in 1.33.

If E is a finite extension of any local field F, then E is also local and $|\cdot|_F$ admits a unique extension to E given by

$$|\cdot|_E = \left|N_{E/F}(\cdot)\right|_F^{1/[E:F]}$$

where $N_{E/F} : E \to F$ is the norm map. It defines the normalized absolute value on E in the non-archimedean case.

Remark 1.11. The example $\mathbb{R}$ (resp. $\mathbb{Q}_p$) above is obtained by completing $\mathbb{Q}$ with respect to the usual absolute value (resp. the p-adic one $|x|_p = p^{-v_p(x)}$, where $v_p(x) = k$ if $x = p^k \frac{u}{v} \neq 0$ with $u, v \in \mathbb{Z}$ coprime to p, and $v_p(0) = +\infty$).

Likewise, $\mathbb{F}_q(\!(t)\!)$ is the completion of the function field $\mathbb{F}_q(t)$ with respect to $|x|_0 = q^{-v_0(x)}$ where $v_0(x)$ is the vanishing order of the rational function x at 0.

In both cases, the local field F arises from completing some *global field* $\mathbb{Q}$ or $\mathbb{F}_q(t)$, or more generally their finite extensions. The adjective "global" comes from geometry, which is manifest in the case $\mathbb{F}_q(t)$: it is the function field of the curve $\mathbb{P}^1_{\mathbb{F}_q}$, and $\mathbb{F}_q(\!(t)\!)$ should be thought as the function field on the "punctured formal disk" at 0.

1.3 Measures and integrals

We shall only consider *Radon measures* on a locally compact Hausdorff space X. These measures are by definition Borel, locally finite and inner regular.

Define the $\mathbb{C}$-vector space

$$\begin{aligned} C_c(X) &:= \left\{ f : X \to \mathbb{C}, \text{ continuous, compactly supported} \right\} \\ &= \bigcup_{\substack{K \subset X \\ \text{compact}}} C_c(X, K), \quad C_c(X, K) := \{ f \in C_c(X) : \mathrm{Supp}(f) \subset K \}. \end{aligned}$$

We write $C_c(X)_+ := \left\{ f \in C_c(X) : f \geq 0 \right\}$.

Remark 1.12. Following Bourbaki, we identify positive Radon measures μ on X and *positive linear functionals* $I = I_\mu : C_c(X) \to \mathbb{C}$. This means that $I(f) \geq 0$ if $f \in C_c(X)_+$. In terms of integrals, $I_\mu(f) = \int_X f \, \mathrm{d}\mu$. This is essentially a consequence of the Riesz representation theorem.

Furthermore, to prescribe a positive linear functional I is the same as giving a function $I : C_c(X)_+ \to \mathbb{R}_{\geq 0}$ that satisfies

- $I(f_1 + f_2) = I(f_1) + I(f_2)$,
- $I(tf) = tI(f)$ when $t \in \mathbb{R}_{\geq 0}$.

To see this, write $f = u + iv \in C_c(X)$ where $u, v : X \to \mathbb{R}$; furthermore, any real-valued $f \in C_c(X)$ can be written as $f = f_+ - f_-$ as usual, where $f_\pm \in C_c(X)_+$. All these decompositions are canonical.

Remark 1.13. The complex Radon measures correspond to linear functionals $I : C_c(C) \to \mathbb{C}$ such that $I|_{C_c(X,K)}$ is continuous with respect to $\|\cdot\|_{\infty,K} := \sup_K |\cdot|$, for all K. In other words, I is continuous for the topology of $\varinjlim_K C_c(X, K) \simeq C_c(X)$.

Now let G be a locally compact group, and suppose that X is endowed with a continuous left G-action. Continuity here means that the action map

$$\begin{aligned} a : G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

is continuous. Similarly for right G-actions. This action transports the functions $f \in C_c(X)$ as well:

$$\begin{aligned} f^g &:= \left[x \mapsto f(gx)\right], \quad \text{left action,} \\ {}^g f &:= \left[x \mapsto f(xg))\right], \quad \text{right action.} \end{aligned}$$

These terminologies are justified as $f^{gg'} = \left(f^g\right)^{g'}$ and ${}^{gg'}f = {}^g\left({}^{g'}f\right)$; they also preserve $C_c(X)_+$. Therefore, for G acting on the left (resp. on the right) of X, it also acts from the same side on the space of positive Radon measures on X by *transport of structure*. In terms of positive linear functionals,

$$\begin{aligned} I &\longmapsto \left[gI : f \mapsto I(f^g)\right], \quad \text{left action,} \\ I &\longmapsto \left[Ig : f \mapsto I({}^g f)\right], \quad \text{right action.} \end{aligned}$$

This pair of definitions is swapped under $G \rightsquigarrow G^{\mathrm{op}}$.

By taking transposes, G also transports complex Radon measures (Remark 1.13). Mnemonic technique:

$$\mathrm{d}(g\mu)(x) = \mathrm{d}\mu(g^{-1}x), \quad \mathrm{d}(\mu g)(x) = \mathrm{d}\mu(xg^{-1}),$$

since a familiar change of variables yields

$$\begin{aligned} \int_X f(x)\,\mathrm{d}(g\mu)(x) &\overset{\text{transpose}}{=} I(f^g) = \int_X f(gx)\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}\mu(g^{-1}x), \\ \int_X f(x)\,\mathrm{d}(\mu g)(x) &\overset{\text{transpose}}{=} I({}^g f) = \int_X f(xg)\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}\mu(xg^{-1}) \end{aligned}$$

and $f \in C_c(X)$ is arbitrary. These formulas extend to all $f \in L^1(X, \mu)$ by approximation.

Definition 1.14. Suppose that G acts continuous on the left (resp. right) of X. Let $\chi : G \to \mathbb{R}^\times_{>0}$ by a continuous homomorphism. We say that a complex Radon measure μ is *quasi-invariant* or an *eigenmeasure* with eigencharacter χ if $g\mu = \chi(g)^{-1}\mu$ (resp. $\mu g = \chi(g)^{-1}\mu$) for all $g \in G$. This can also be expressed as

$$\begin{aligned} \mathrm{d}\mu(gx) &= \chi(g)\,\mathrm{d}\mu(x), \quad \text{left action,} \\ \mathrm{d}\mu(xg) &= \chi(g)\,\mathrm{d}\mu(x), \quad \text{right action.} \end{aligned}$$

When $\chi = 1$, we call μ an *invariant measure*.

Observe that μ is quasi-invariant with eigencharacter χ under left G-action if and only if it is so under the right G^{op}-action.

Remark 1.15. Let χ, η be continuous homomorphisms $G \to \mathbb{C}^\times$. Suppose that $f : X \to \mathbb{C}$ is continuous with $f(gx) = \eta(g)f(x)$ (resp. $f(xg) = \eta(g)f(x)$) for all g and x. Then μ is quasi-invariant with eigencharacter χ if and only if $f\mu$ is quasi-invariant with eigencharacter $\eta\chi$.

In particular, we may let G act on $X = G$ by left (resp. right) translations. Therefore, it makes sense to talk about left and right invariant measures on G.

1.4 Haar measures

In what follows, measures are always nontrivial positive Radon measures.

Definition 1.16. Let G be a locally compact group. A left (resp. right) invariant measure on G is called a left (resp. right) *Haar measure*.

The group $\mathbb{R}^{\times}_{>0}$ acts on the set of left (resp. right) Haar measures by rescaling. For commutative groups we make no distinction of left and right.

Example 1.17. If G is discrete, the counting measure $\mathrm{Count}(E) = |E|$ is a left and right Haar measure. When G is finite, it is customary to take the normalized version $\mu := |G|^{-1}\mathrm{Count}$.

Definition 1.18. Let G be a locally compact. For $f \in C_c(G)$ we write $\check{f} : x \mapsto f(x^{-1})$, $\check{f} \in C_c(G)$. For any Radon measure μ on G, let $\check{\mu}$ be the Radon measure with $\mathrm{d}\check{\mu}(x) = \mathrm{d}\mu(x^{-1})$; in terms of linear functionals, $I_{\check{\mu}}(f) = I_\mu(\check{f})$.

Lemma 1.19. *In the situation above, μ is a left (resp. right) Haar measure if and only if $\check{\mu}$ is a right (resp. left) Haar measure.*

Proof. An instance of transport of structure, because $x \mapsto x^{-1}$ is an isomorphism of locally compact groups $G \xrightarrow{\sim} G^{\mathrm{op}}$. □

Theorem 1.20 (A. Weil)**.** *For every locally compact group G, there exists a left (resp. right) Haar measure on G. They are unique up to $\mathbb{R}^{\times}_{>0}$-action.*

Proof. The following arguments are taken from Bourbaki [1, VII §1.2]. Upon replacing G by G^{op}, it suffices to consider the case of left Haar measures. For the existence part, we seek a left G-invariant positive linear functional on $C_c(G)_+$. Write

$$C_c(G)^*_+ := C_c(G)_+ \smallsetminus \{0\}.$$

For any compact subset $K \subset G$, define $C_c(G,K)_+$ and $C_c(G,K)^*_+$ by intersecting $C_c(G,K)$ with $C_c(G)_+$ and $C_c(G)^*_+$. Observe that for all $f \in C_c(G)_+$ and $g \in C_c(G)^*_+$, there exist $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{R}_{\geq 0}$ and $s_1, \ldots, s_n \in G$ such that

$$f \leq c_1 g^{s_1} + \ldots c_n g^{s_n}, \quad g^{s_i}(x) := g(s_i x).$$

Indeed, there is an open subset $U \subset G$ such that $\inf_U g > 0$; now cover $\mathrm{Supp}(f)$ by finitely many $s_1 U, \ldots, s_n U$.

Given f, g as before, define $(f : g)$ to be the infimum of $c_1 + \ldots + c_n$ among all choices of $(c_1, \ldots, s_1, \ldots)$ satisfying the bound above. We contend that

$$(f^s : g) = (f : g), \quad s \in G, \tag{1.1}$$
$$(tf : g) = t(f : g), \quad t \in \mathbb{R}_{\geq 0}, \tag{1.2}$$
$$(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g), \tag{1.3}$$
$$(f : g) \geq \frac{\sup f}{\sup g}, \tag{1.4}$$
$$(f : h) \leq (f : g)(g : h), \quad h, g \in C_c(G)^*_+, \tag{1.5}$$
$$0 < \frac{1}{(f_0 : f)} \leq \frac{(f : g)}{(f_0 : g)} \leq (f : f_0), \quad f, f_0, g \in C_c(G)^*_+, \tag{1.6}$$

and that for all $f_1, f_2, h \in C_c(G)_+$ with $h|_{\mathrm{Supp}(f_1+f_2)} \geq 1$, and all $\epsilon > 0$, there is a compact neighborhood $K \ni 1$ such that

$$(f_1 : g) + (f_2 : g) \leq (f_1 + f_2 : g) + \epsilon(h : g), \quad g \in C_c(G,K)^*_+. \tag{1.7}$$

The properties (1.1) — (1.4) are straightforward. For (1.5), note that

$$f \leq \sum_i c_i g^{s_i}, \quad g \leq \sum_j d_j h^{t_j} \implies f \leq \sum_{i,j} c_i d_j h^{t_j s_i}$$

hence $(f:h) \le \sum_i c_i \sum_j d_j$. Apply (1.5) to both f_0, f, g and f, f_0, g to obtain (1.6).

To verify (1.7), let $F := f_1 + f_2 + \frac{\epsilon h}{2}$. Use the condition on h to define

$$\varphi_i(x) = \begin{cases} f_i(g)/F(x), & x \in \mathrm{Supp}(f_1+f_2) \\ 0, & \text{otherwise.} \end{cases} \in C_c(G)_+ \quad (i=1,2).$$

Furthermore, given $\eta > 0$ we may choose the compact neighborhood $K \ni 1$ such that $|\varphi_i(x) - \varphi_i(y)| \le \eta$ whenever $x^{-1}y \in K$, for $i = 1,2$. One readily verifies that for all $g \in C_c(G,K)_+$ and $i = 1,2$,

$$\varphi_i g^s \le (\varphi_i(s) + \eta) \cdot g^s, \quad s \in G.$$

Suppose that $F \le c_1 g^{s_1} + \dots c_n g^{s_n}$, then $f_i = \varphi_i F \le \sum_{j=1}^n c_j(\varphi_i(s_j) + \eta) g^{s_j}$ by the previous step. As $\varphi_1 + \varphi_2 \le 1$, we infer that $(f_1 : g) + (f_2 : g) \le (1+2\eta)\sum_{j=1}^n c_j$. By (1.3) and (1.5),

$$\begin{aligned}(f_1:g) + (f_2:g) &\le (1+2\eta)(F:g) \le (1+2\eta)\left((f_1+f_2:g) + \frac{\epsilon}{2}(h:g)\right) \\ &\le (f_1+f_2:g) + \left(\frac{\epsilon}{2} + 2\eta(f_1+f_2:h) + \epsilon\eta\right)(h:g).\end{aligned}$$

Choosing η small enough relative to f_1, f_2, h, ϵ yields (1.7).

Proceed to the construction of Haar measure. We fix $f_0 \in C_c(G)^*_+$ and set

$$I_g(f) := \frac{(f:g)}{(f_0:g)}, \quad f \in C_c(G)_+, \quad g \in C_c(G)^*_+.$$

We want to "take the limit" over $g \in C_c(G,K)^*_+$, where K shrinks to $\{1\}$ and $f, f_0 \in C_c(G)^*_+$ are kept fixed. By (1.6) we see $I_g(f) \in \mathfrak{I} := \left[(f_0:f)^{-1}, (f:f_0)\right]$. For each compact neighborhood $K \ni 1$ in G, let $I_K(f) := \left\{ I_g(f) : g \in C_c(G,K)^*_+ \right\}$. The family of all $I_K(f)$ form a *filter base* in the compact space $\mathfrak{I}$, hence can be refined into an *ultrafilter* which has the required limit $I(f)$.

We refer to [2, I. §6 and §9.1] for the language of filters and its relation with compactness. Alternatively, one can argue using the Moore–Smith theory of *nets* together with Tychonoff's theorem; see [6, (15.25)].

Then $I : C_c(G)_+ \to \mathbb{R}_{\ge 0}$ satisfies G-invariance by (1.1). From (1.3) we infer $I(f_1+f_2) \le I(f_1)+I(f_2)$ which already holds for all I_g; from (1.7) we infer $I(f_1) + I(f_2) \le I(f_1+f_2) + \epsilon I(h)$ whenever $h \ge 1$ on $\mathrm{Supp}(f_1+f_2)$ and $\epsilon > 0$ (true by using g with sufficiently small support), so $I(f_1+f_2) = I(f_1)+I(f_2)$ follows. The behavior under dilation follows from (1.2). All in all, I is the required Haar measure on G.

Now turn to the uniqueness. Let us consider a left (resp. right) Haar measure μ (resp. ν) on G. It suffices by Lemma 1.19 to show that μ and $\check{\nu}$ are proportional. Fix $f \in C_c(G)$ such that $\int_G f \,\mathrm{d}\mu \neq 0$. It is routine to show that the function

$$D_f : x \longmapsto \left(\int_G f \,\mathrm{d}\mu\right)^{-1} \int_G f(y^{-1}x) \,\mathrm{d}\nu(y), \quad x \in G$$

is continuous. Let $g \in C_c(G)$. Since $(x,y) \mapsto f(x)g(yx)$ is in $C_c(G \times G)$, Fubini's theorem implies that

$$\begin{aligned}\int_G f \,\mathrm{d}\mu \cdot \int_G g \,\mathrm{d}\nu = \int_G f(x)\left(\int_G g(y)\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x) &\overset{\mathrm{d}\nu(y)=\mathrm{d}\nu(yx)}{=} \\ \int_G\int_G f(x)g(yx)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) &\overset{\mathrm{d}\mu(x)=\mathrm{d}\mu(yx)}{=} \int_G\left(\int_G f(y^{-1}x)g(x)\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y) \\ &= \int_G g(x)\left(\int_G f(y^{-1}x)\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x) = \int_G f\,\mathrm{d}\mu \cdot \int_G g D_f \,\mathrm{d}\mu(x).\end{aligned}$$

Hence $\int_G g\,\mathrm{d}\nu = \int_G D_f \cdot g\,\mathrm{d}\mu$. Since g is arbitrary and D_f is continuous, $D_f(x)$ is independent of f, hereafter written as $D(x)$. Now

$$D(1)\int_G f\,\mathrm{d}\mu = \int_G f\,\mathrm{d}\check{\nu}$$

for $f \in C_c(G)$ with $\int f\,\mathrm{d}\mu \neq 0$. As both sides are linear functionals on the whole $C_c(G)$, the equality extends and yield the asserted proportionality. □

Proposition 1.21. *Let μ be a left (resp. right) Haar measure on G. Then G is discrete if and only if $\mu(\{1\}) > 0$, and G is compact if and only if $\mu(G) < +\infty$.*

Proof. The "only if" parts are easy: for discrete G we may take μ = Count, and for compact G we integrate the constant function 1.

For the "if" part, first suppose that $\mu(\{1\}) > 0$. Then every singleton has measure $\mu(\{1\})$. Observe that every compact neighborhood K of 1 must satisfy $\mu(K) > 0$, therefore K is finite. As G is Hausdorff, $\{1\}$ is thus open, so G is discrete.

Next, suppose $\mu(G)$ is finite. Fix a compact neighborhood $K \ni 1$ so that $\mu(K) > 0$. If $g_1, \dots, g_n \in G$ are such that $\{g_i K\}_{i=1}^n$ are disjoint, then $n\mu(K) = \mu(\bigcup_i g_i K) \leq \mu(G)$ and this gives $n \leq \mu(G)/\mu(K)$. Choose a maximal collection $g_1, \dots, g_n \in G$ with the disjointness property above. Every $g \in G$ must lie in $g_i K \cdot K^{-1}$ for some $1 \leq i \leq n$, since maximality implies $gK \cap g_i K \neq \emptyset$ for some i. Hence $G = \bigcup_{i=1}^n g_i K \cdot K^{-1}$ is compact by 1.3. □

1.5 The modulus character

Let $\theta : G_1 \to G_2$ be an isomorphism of locally compact groups. A Radon measure μ on G_1 transports to a Radon measure $\theta_*\mu$ on G_2: the corresponding positive linear functional is

$$I_{\theta_*\mu}(f) = I_\mu(f^\theta), \quad f^\theta := f \circ \theta, \quad f \in C_c(G_2).$$

It is justified to express this as $\mathrm{d}(\theta_*\mu)(x) = \mathrm{d}\mu(\theta^{-1}(x))$.

Evidently, θ_* commutes with rescaling by $\mathbb{R}^\times_{>0}$. By transport of structure, $\theta_*\mu$ is a left (resp. right) Haar measure if μ is. For two composable isomorphisms θ, σ we have

$$(\theta\sigma)_*\mu = \theta_*(\sigma_*\mu).$$

Assume hereafter $G_1 = G_2 = G$ and $\theta \in \mathrm{Aut}(G)$. We consider $\theta^{-1}\mu := (\theta^{-1})_*\mu$ where μ is a left Haar measure on G. Theorem 1.20 imlies $\theta^{-1}\mu$ is a positive multiple of μ, and this ratio does not depend on μ.

Definition 1.22. For every $\theta \in \mathrm{Aut}(G)$, define its *modulus* δ_θ as the positive number determined by

$$\theta^{-1}\mu = \delta_\theta \mu, \quad \text{equivalently } \mathrm{d}\mu(\theta(x)) = \delta_\theta\,\mathrm{d}\mu(x),$$

where μ is any left Haar measure on G.

Remark 1.23. One can also use right Haar measures to define the modulus. Indeed, the right Haar measures are of the form form $\check{\mu}$ where μ is a left Haar measure, and

$$\mathrm{d}\check{\mu}(\theta(x)) = \mathrm{d}\mu(\theta(x)^{-1}) = \mathrm{d}\mu(\theta(x^{-1})) = \delta_\theta \cdot \mathrm{d}\mu(x^{-1}) = \delta_\theta \cdot \mathrm{d}\check{\mu}(x).$$

Remark 1.24. The modulus characters are sometimes defined as $\Delta_G(g) = \delta_G(g)^{-1}$ in the literature, such as [1, VII §1.3] or [6]. This ambiguity is responsible for uncountably many headaches. Our convention for δ_G seems to conform with most papers in representation theory.

Proposition 1.25. *The map $\theta \mapsto \delta_\theta$ defines a homomorphism* $\mathrm{Aut}(G) \to \mathbb{R}^\times_{>0}$.

Proof. Use the fact that

$$\left((\theta\sigma)^{-1}\right)_* \mu = \sigma_*^{-1}(\theta_*^{-1}\mu) = \delta_\theta \cdot \sigma_*^{-1}\mu = \delta_\theta\delta_\sigma \cdot \mu$$

for any two automorphisms θ, σ of G and any left Haar measure μ. □

Example 1.26. If G is discrete, then $\delta_\theta = 1$ for all θ since the counting measure is preserved.

Definition 1.27. For $g \in G$, let $\mathrm{Ad}(g)$ be the automorphism $x \mapsto gxg^{-1}$ of G. Define $\delta_G(g) := \delta_{\mathrm{Ad}(g)} \in \mathbb{R}^\times_{>0}$. In other words, $\mathrm{d}\mu(gxg^{-1}) = \delta_G(g)\,\mathrm{d}\mu(x)$.

The following result characterizes δ_G.

Lemma 1.28. *The map $\delta_G : G \to \mathbb{R}^\times_{>0}$ is a continuous homomorphism. For every left (resp. right) Haar measure μ_ℓ (resp. μ_r), we have*

$$\mathrm{d}\mu_\ell(xg) = \delta_G(g)^{-1}\,\mathrm{d}\mu_\ell(x), \quad \mathrm{d}\mu_r(gx) = \delta_G(g)\,\mathrm{d}\mu_r(x).$$

Furthermore, $\check{\mu}_\ell = \delta_G \cdot \mu_\ell$ and $\check{\mu}_r = \delta_G^{-1}\mu_r$.

Proof. Since $\mathrm{Ad}(\cdot) : G \to \mathrm{Aut}(G)$ is a homomorphism, so is δ_G by Proposition 1.25. As for the continuity of δ_G, fix a left Haar measure μ and $f \in C_c(G)$ with $\mu(f) := \int_G f\,\mathrm{d}\mu \neq 0$. Then

$$\delta_G(g)^{-1} = \mu(f)^{-1} \int_G f \circ \mathrm{Ad}(g)\,\mathrm{d}\mu,$$

and it is routine to check that $\int_G f \circ \mathrm{Ad}(g)\,\mathrm{d}\mu$ is continuous in g. The second assertion results from applying $\mathrm{d}\mu_\ell(\cdot)$ and $\mathrm{d}\mu_r(\cdot)$ to

$$xg = g(\mathrm{Ad}(g^{-1})x), \quad gx = (\mathrm{Ad}(g)x)g,$$

respectively.

To deduce the last assertion, note that $\delta_G\mu_\ell$ is a right Haar measure by the previous step and Remark 1.15, thus Theorem 1.20 entails

$$\exists t \in \mathbb{R}^\times_{>0}, \ \check{\mu}_\ell = t\delta_G\mu_\ell.$$

Hence $\mu_\ell^{\vee\vee} = t\delta_G^{-1}\check{\mu}_\ell = t^2\mu_\ell$, and we obtain $t = 1$. A similar reasoning for $\delta_G^{-1}\mu_r$ applies. □

Corollary 1.29. *We have $\delta_G = 1$ if and only if every left Haar measure on G is also a right Haar measure, and vice versa.*

Proposition 1.30. *For every $g \in G$, we have $\delta_G(g)^{-1} = \delta_{G^{\mathrm{op}}}(g)$.*

Proof. Fix a left Haar measure μ for G and denote the multiplication in G^{op} by $\star$. Then $\mathrm{d}\mu(gxg^{-1}) = \delta_G(g)\,\mathrm{d}\mu(x)$ is equivalent to $\mathrm{d}\mu(g^{-1} \star x \star g) = \delta_G(g)\,\mathrm{d}\mu(x)$ for G^{op}. Since μ is a right Haar measure for G^{op}, we infer from Remark 1.23 that $\delta_{G^{\mathrm{op}}}(g^{-1}) = \delta_G(g)$. □

Definition 1.31. A locally compact group G with $\delta_G = 1$ is called *unimodular*.

Example 1.32. The following groups are unimodular.

- Commutative groups.
- Discrete groups: use Example 1.26.

- Compact groups: indeed, the compact subgroup $\delta_G(G)$ of $\mathbb{R}^\times_{>0} \simeq \mathbb{R}$ must be trivial.

The normalized absolute value on a local field F is actually the modulus character for the additive group of F, as shown below.

Theorem 1.33. *For every local field F, define $\|\cdot\|_F$ to be the normalized absolute value $|\cdot|_F$ for non-archimedean F, the usual absolute value for $F = \mathbb{R}$, and $x \mapsto |x\bar{x}|_{\mathbb{R}}$ for $F = \mathbb{C}$. For $t \in F^\times$, let m_t be the automorphism $x \mapsto tx$ for $(F, +)$. Then*

$$\delta_{m_t} = \|t\|_F.$$

Proof. Since $m_{tt'} = m_{t'} \circ m_t$, by Proposition 1.25 we have $\delta_{m_{tt'}} = \delta_{m_{t'}}\delta_{m_t}$. Fix any Haar measure μ on F. By definition

$$\delta_{m_t} = \frac{\mu(tK)}{\mu(K)}$$

for every compact subset $K \subset F$. The assertion $\delta_{m_t} = \|t\|_F$ is then evident for $F = \mathbb{R}$ or $\mathbb{C}$, by taking K to be the unit ball and work with the Lebesgue measure.

Now suppose F is non-archimedean and take $K = \mathfrak{o}_F$. For $t \in \mathfrak{o}_F^\times$ we have $tK = K$, thus $\delta_{m_t} = 1$. The same holds for $|\cdot|_F$ and both are multiplicative. It remains to show $\delta_{m_\varpi} = |\varpi|_F = q^{-1}$ for any uniformizer ϖ of F, where $q := |\mathfrak{o}_F/\mathfrak{p}_F|$. But then $\mu(\varpi\mathfrak{o}_F)/\mu(\mathfrak{o}_F) = (\mathfrak{o}_F : \varpi\mathfrak{o}_F)^{-1} = q^{-1}$. □

1.6 Interlude on analytic manifolds

It is well known that every real Lie group admits an analytic structure. The theory of analytic manifolds generalizes to any local field F. For a detailed exposition, we refer to [9, Part II].

In what follows, it suffices to assume that F is a complete topological field with respect to some absolute value $|\cdot|$. Let $U \subset F^m$ be some open ball. The definition of F-analytic functions $f = (f_1, \ldots, f_n) : U \to F^n$ is standard: we require that each f_i can be expressed as convergent power series on U. For any open subset $U \subset F^m$, we say $f : U \to F^n$ is F-analytic if U can be covered by open balls over which f are F-analytic.

Definition 1.34. An F-analytic manifold of dimension n is a second countable Hausdorff space X, endowed with an atlas $\left\{(U, \phi_U) : U \in \mathscr{U}\right\}$, where

- $\mathscr{U}$ is an open covering of X;
- for each $U \in \mathscr{U}$, ϕ_U is a homeomorphism from U to an open subset $\phi_U(U)$ of F^n.

The condition is that for all $U_1, U_2 \in \mathscr{U}$ with $U_1 \cap U_2 \neq \emptyset$, the transition map

$$\phi_{U_2}\phi_{U_1}^{-1} : \phi_{U_1}(U_1 \cap U_2) \to \phi_{U_2}(U_1 \cap U_2)$$

is F-analytic.

As in the standard theory of manifolds, the F-analytic functions on X are defined in terms of charts, and the same applies to morphisms between F-analytic manifolds; the notion of closed submanifolds is defined in the usual manner. We can also pass to maximal atlas in the definition. The F-analytic manifolds then form a category.

Open subsets of F-analytic manifolds inherit analytic structures. It is also possible to define F-analytic manifolds intrinsically by sheaves.

More importantly, the notion of tangent bundles, cotangent bundles, differential forms and their exterior powers still make sense on an F-analytic manifold X. Let $n = \dim X$. As in the real case, locally a differential form on X can be expressed as $c(x_1, \ldots, x_n)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, where $\left(U, \phi_U = (x_1, \ldots, x_n)\right)$ is any chart for X, and c is an analytic function on $\phi_U(U) \subset F^n$.

Remark 1.35. When $(F, |\cdot|)$ is ultrametric, the theory of F-analytic manifolds is not quite interesting since they turn out to be totally disconnected. Nevertheless, it is a convenient vehicle for talking about measures, exponential maps and so forth.

Example 1.36. Let $\mathbf{X}$ be an F-variety. The set X of its F-points acquires a natural Hausdorff topology (say by reducing to the case to affine varieties, then to affine sets). If $\mathbf{X}$ is smooth, then X becomes an analytic F-manifold. The assignment $\mathbf{X} \mapsto X$ is functorial. The algebraically defined regular functions, differentials, etc. on $\mathbf{X}$ give rise to their F-analytic avatars on X. By convention, objects on $\mathbf{X}$ will often carry the adjective *algebraic*.

Now assume that F is local, and fix a Haar measure on F, denoted abusively by $\mathrm{d}x$ in conformity with the usual practice. This equips F^n with a Radon measure for each $n \geq 0$, and every open subset $U \subset F^n$ carries the induced measure.

Let X be an F-analytic manifold of dimension n. The collection of volume forms will be an $\mathbb{R}^\times_{>0}$-torsor over X. Specifically, a volume form on X is expressed in every chart (U, ϕ_U) as

$$|\omega| = f(x_1, \ldots, x_n) |\,\mathrm{d}x_1| \cdots |\,\mathrm{d}x_n|$$

where $f : V := \phi_U(U) \to \mathbb{R}_{\geq 0}$ is continuous. Define $\|\cdot\|_F : F \to \mathbb{R}_{\geq 0}$ as in Theorem 1.33. A change of local coordinates $y_i = y_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ transports f and has the effect

$$|\,\mathrm{d}y_1| \cdots |\,\mathrm{d}y_n| = \left\| \det\left(\frac{\partial y_i}{\partial x_j}\right)_{i,j} \right\|_F \cdot |\,\mathrm{d}x_1| \cdots |\,\mathrm{d}x_n|. \tag{1.8}$$

In other words, $|\omega|$ is the "absolute value" of a differential form of top degree on X, squared when $F = \mathbb{C}$. Some geometers name these objects as *densities*.

Now comes integration. The idea is akin to the case of C^∞-manifolds, and we refer to [7, Chapter 16] for a complete exposition for the latter. Let $\varphi \in C_c(X)$ and choose an atlas $\{(U, \phi_U) : U \in \mathscr{U}\}$ for X. In fact, it suffices to take a finite subset of $\mathscr{U}$ covering $\mathrm{Supp}(\varphi)$.

1. By taking a partitions of unity $\left(\psi_U : X \to [0, 1]\right)_{U \in \mathscr{U}}$ (with $\sum_U \psi_U = 1$ as usual), and replacing ϕ by $\psi_U \phi$ for each $U \in \mathscr{U}$, the definition of $\int_X \varphi|\omega|$ reduces to the case that $\mathrm{Supp}(\varphi) \subset U$ for some $U \in \mathscr{U}$.

2. In turn, we are reduced to the integration $\int_{V \subset F^n} \varphi f \,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ on some open subset $V \subset F^n$; this we can do by the choice of Haar measure on F. Furthermore, this integral is invariant under F-analytic change of coordinates. Indeed, this is a consequence of (1.8) plus the interpretation of $\|\cdot\|_F$ in Theorem 1.33.

3. As in the case over $\mathbb{R}$, one verifies that $\int \varphi\omega$ does not depend on the choice of charts and partition of unity.

4. It is routine to show that $I : \varphi \mapsto \int_X \varphi|\omega|$ is a positive linear functional, giving rise to a Radon measure on X. The operations of addition, rescaling and group actions on volume forms mirror those on Radon measures.

Remark 1.37. In many applications, X will come from a smooth F-variety $\mathbf{X}$ or its open subsets, and $|\omega|$ will come from a nowhere-vanishing algebraic differential form ω thereon, say by "taking absolute values $\|\cdot\|_F$"; we call such an ω an *algebraic volume form* on $\mathbf{X}$.

Definition 1.38. Let X be an F-analytic manifolds, where F is a local field. Set

$$C^\infty_c(X) := \begin{cases} \left\{ f \in C_c(X) : \text{infinitely differentiable} \right\}, & \text{archimedean } F \\ \left\{ f \in C_c(X) : \text{locally constant} \right\}, & \text{non-archimedean } F \end{cases}$$

where locally constant means that every $x \in X$ admits an open neighborhood U such that $f|_U$ is constant.

1.7 More examples

We begin by pinning down the Haar measures on a local field F, viewed as additive groups.

Archimedean case $F = \mathbb{R}$ Use the Lebesgue measure.

Archimedean case $F = \mathbb{C}$ Use twice of the Lebesgue measure.

Non-archimedean case To prescribe a Haar measure on F, by Theorem 1.20 it suffices to assign a volume to the compact subring $\mathfrak{o}_F$ of integers. A canonical choice is to stipulate that $\mathfrak{o}_F$ has volume 1, but this is not always the best recipe.

Example 1.39. If G is a Lie group over any local field F, i.e. a group object in the category of F-analytic manifolds, one can construct left (resp. right) Haar measures as follows. Choose a basis $\omega_1, \dots, \omega_n$ of the cotangent space of G at 1. The cotangent bundle of G is trivializable by left (resp. right) translation, therefore $\omega_1, \dots, \omega_n$ can be identified as globally defined, left (resp. right) invariant differentials. The nowhere-vanishing top form $\omega := \omega_1 \wedge \cdots \wedge \omega_n$, or more precisely $|\omega|$, gives the required measure on G. The Haar measures constructed in this way are unique up to scaling, as the form ω (thus $|\omega|$) is.

We shall denote the tangent space at $1 \in G$ as $\mathfrak{g}$: we have $\dim_F \mathfrak{g} = n$. The automorphism $\mathrm{Ad}(g) : x \mapsto gxg^{-1}$ induces a linear map on $\mathfrak{g}$ at 1, denoted again by $\mathrm{Ad}(g)$.

Proposition 1.40. *In the circumstance of Example 1.39, one has*

$$\delta_G(g) = \left\| \det(\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}) \right\|_F, \quad g \in G.$$

Proof. This follows from Theorem 1.33. $\square$

Example 1.41. Let F be a local field. For the multiplicative group $F^\times$, the algebraic volume form

$$x^{-1}\,\mathrm{d}x$$

defines a Haar measure: it is evidently invariant under multiplications. More generally, let $G = \mathrm{GL}(n, F)$. We take η to be a translation-invariant nonvanishing algebraic volume form on $M_n(F) \simeq F^{n^2}$. The G-invariant algebraic volume form

$$\omega(g) := (\det g)^{-n}\eta(g), \quad g \in G$$

induces a left and right Haar measure on G.

Example 1.42. Here is a non-unimodular group. Let F be a local field, and $n = n_1 + \cdots + n_2$ where $n_1, n_2 \in \mathbb{Z}_{\geq 1}$. Consider the group of block upper-triangular matrices

$$P := \left\{ g = \begin{pmatrix} g_1 & x \\ & g_2 \end{pmatrix} \in \mathrm{GL}(n, F) : \quad g_i \in \mathrm{GL}(n_i, F), x \in M_{n_1 \times n_2}(F) \right\}$$

the unspecified matrix entries being zero. This is clearly the space of F-points of an algebraic group $\mathbf{P}$. The Lie algebra of $\mathbf{P}$ is simply $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$. A routine calculation of block matrices gives

$$\begin{pmatrix} g_1 & x \\ & g_2 \end{pmatrix} \begin{pmatrix} a & b \\ & d \end{pmatrix} \begin{pmatrix} g_1 & x \\ & g_2 \end{pmatrix}^{-1} = \begin{pmatrix} a & g_1 b g_2^{-1} \\ & d \end{pmatrix}$$

where $a \in M_{n_1}(F)$, $b \in M_{n_2}(F)$ and $b \in M_{n_1 \times n_2}(F)$. Thus Proposition 1.40 and Theorem 1.33 imply

$$\delta_P(g) = \| \det g_1 \|_F^{n_2} \cdot \| \det g_2 \|_F^{-n_1}.$$

1.8 Homogeneous spaces

Let G be a locally compact group. In what follows we consider spaces with right G-actions. One can switch to left G-actions by passing to G^{op}.

Definition 1.43. A G-space in the category of locally compact spaces is a locally compact Hausdorff space X equipped with a continuous right G-action $X \times G \to X$. If the G-action is transitive, X is called a homogeneous G-space.

The G-spaces form a category: the morphisms are continuous G-equivariant maps.

Example 1.44. Let H be a closed subgroup of G. The coset space $H\backslash G$ is locally compact and Hausdorff by Proposition 1.4, and G acts continuously on the right by $(Hx, g) \mapsto Hxg$. This is a homogeneous G-space.

For a general homogeneous G-space X and $x \in X$, let $G_x := \mathrm{Stab}_G(x)$; it is a closed subgroup, and we have $G_{xg} = g^{-1}G_x g$ for $g \in G$. The orbit map $\mathrm{orb}_x : g \mapsto xg$ factors through

$$\begin{aligned} \mathrm{orb}_x : G_x\backslash G &\to X \\ G_x g &\longmapsto xg, \end{aligned}$$

which is a continuous G-equivariant bijection. If orb_x is actually an isomorphism, then $X \simeq G_x\backslash G$ in the category of G-spaces. This indeed holds under mild conditions on G, thereby showing that Example 1.44 is the typical sort of homogeneous spaces.

Proposition 1.45. *Suppose that G is second countable as a topological space. Then for any homogeneous G-space X and any $x \in X$, the map $\mathrm{orb}_x : G_x\backslash G \to X$ is an isomorphism of G-spaces.*

Proof. It suffices to show that orb_x is an open map. Let $U \ni 1$ be any compact neighborhood in G. There is a sequence $g_1, g_2, \ldots \in G$ such that $G = \bigcup_{i\geq 1} Ug_i$, therefore $X = \bigcup_{i\geq 1} xUg_i$. Each xUg_i is compact, hence closed in X.

Baire's theorem implies that some xUg_i has nonempty interior. Since $xUg_i \simeq xU$ by right translation, we obtain an interior point xh of the subset xU of X, with $h \in U$. Then x is an interior point of xUh^{-1}, hence of the bigger subset xUU^{-1}.

Now consider any open subset $V \subset G$ and $g \in V$. Take a compact neighborhood $U \ni 1$ such that $UU^{-1}g \subset V$. Then

$$xg \in xUU^{-1}g \subset xV.$$

As x has been shown to an interior point of xUU^{-1}, so is xg in $xUU^{-1}g \subset xV$. Since g, V are arbitrary, we conclude that $g \mapsto xg$ is an open map. The same holds for orb_x by the definition of quotient topologies. □

Corollary 1.46. *If G is an F-analytic Lie group, where F is a local field, then the condition in Proposition 1.45 holds, thus every homogeneous G-space takes the form $H\backslash G$ for some H.*

Proof. The second countability is built into the definitions. □

Example 1.47. Take X to be the $(n-1)$-sphere $\mathbb{S}^{n-1}$, on which the orthogonal group $\mathrm{O}(n, \mathbb{R})$ acts continuously (in fact analytically, or even algebraically) and transitively. The action is given by $(v, g) \mapsto vg$ where we regard v as a row vector. The subgroup $\mathrm{SO}(n, \mathbb{R})$ also acts transitively on $\mathbb{S}^{n-1}$, since every vector $v \in \mathbb{S}^{n-1}$ is fixed by reflections relative to any hyperplane containing v, which have determinant -1. This gives

$$\mathbb{S}^{n-1} \simeq \mathrm{O}(n-1, \mathbb{R})\backslash \mathrm{O}(n, \mathbb{R}).$$

Indeed, the stabilizer group of $v = (1, 0, \ldots, 0)$ can be identified with $\mathrm{O}(n-1, \mathbb{R})$.

Example 1.48. Another important example of homogeneous space is given by the *Siegel upper half plane*. Here it is customary to work with left actions by setting

$$\mathrm{Sp}(2n,\mathbb{R}) := \left\{ g \in \mathrm{GL}(2n,\mathbb{R}) : g \cdot J \cdot {}^t g = J \right\}, \quad J := \begin{pmatrix} & -1_{n\times n} \\ 1_{n\times n} & \end{pmatrix}.$$

In a coordinate-free language, $\mathrm{Sp}(2n,\mathbb{R})$ is the isometry group of the symplectic form prescribed by J. Take X to be

$$\mathscr{H}_n := \{ Z = R + iS \in M_{n\times n}(\mathbb{C}) : R, S \in M_{n\times n}(\mathbb{R}) \text{ symmetric}, \ S > 0 \}.$$

Let $\mathrm{Sp}(2n,\mathbb{R})$ act on the left of $\mathscr{H}_n$ by

$$gZ := (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathscr{H}_n,$$
$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}), \quad A, B, C, D \in M_{n\times n}(\mathbb{R}).$$

This is manifestly an analytic map in (g, Z). One can show that

- this is indeed a transitive group action, and
- the stabilizer group of $Z = i \cdot 1_{n\times n}$ is the unitary group $U(n)$.

Therefore $\mathscr{H}_n \simeq \mathrm{Sp}(2n,\mathbb{R})/U(n)$. When $n = 1$, it reduces to the usual $\mathscr{H}_1 := \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ with $\mathrm{SL}(2,\mathbb{R})$ acting by linear fractional transformations.

Example 1.49. Let F be a non-archimedean local field, with ring of integers $\mathfrak{o}_F$. By convention, we identify F^n with the space of row vectors, and let $\mathrm{GL}(n,F)$ act from the right.

A *lattice* in an n-dimensional F-vector space V is a free $\mathfrak{o}_F$-submodule $L \subset V$ of rank n such that $L \cdot F = V$; the standard example is $\mathfrak{o}_F^n$ inside F^n. Define the discrete space

$$X := \{\text{lattices } L \subset V\}$$

with right $\mathrm{GL}(V)$-action. Exercise: check that the $\mathrm{GL}(V)$-action is continuous and transitive: in fact $\mathrm{Stab}_{\mathrm{GL}(V)}(L)$ is closed and open in $\mathrm{GL}(V)$.

Now identify V with F^n by choosing a basis. The stabilizer of $L = \mathfrak{o}_F^n$ is $\mathrm{GL}(n,\mathfrak{o}_F)$, hence

$$X \simeq \mathrm{GL}(n,\mathfrak{o}_F)\backslash\mathrm{GL}(n,F).$$

Examples 1.47—1.48 are well-known instances of Riemannian symmetric spaces. Loosely speaking, the space X of lattices is in contrast some non-archimedean counterpart of symmetric spaces of the same type as Example 1.48: it consists of the vertices of the *enlarged Bruhat–Tits building* of $\mathrm{GL}(V)$.

Next, we turn to the measures on the G-spaces $H\backslash G$, where H is a closed subgroup of G.

Lemma 1.50. *Fix a right Haar measure μ_H on H. The map*

$$C_c(G) \longrightarrow C_c(H\backslash G)$$
$$f \longmapsto f^\flat := \left[Hg \mapsto \int_H f(hg)\, \mathrm{d}\mu_H(h) \right]$$

is well-defined, surjective and G-equivariant with respect to the left G-actions on $C_c(G)$ and $C_c(H\backslash G)$ given by $f \mapsto \left[{}^g f : x \mapsto f(xg) \right]$. It maps $C_c(G)_+$ onto $C_c(H\backslash G)_+$.

Proof. The only nontrivial part is the surjectivity. Given $f^\flat \in C_c(H\backslash G)$, by Lemma 1.5 there exists a compact $K \subset G$ such that $g \mapsto f^\flat(Hg)$ is supported on HK. We want to define

$$f(g) := \begin{cases} f^\flat(Hg) \cdot \dfrac{\phi(g)}{\phi^\flat(Hg)}, & g \in HK \\ 0, & g \notin HK \end{cases}$$

for some $\phi \in C_c(G)_+$ depending solely on K. Specifically, take any $\phi \in C_c(G)_+$ such that $\phi|_K > 0$. For any $g \in HK$ we have

$$\phi^\flat(g) \geq \inf_{x \in K} \phi^\flat(Hx) > 0$$

by assumption, hence f is a well-defined element in $C_c(G)$. If $f^\flat \in C_c(H\backslash G)_+$ then $f \in C_c(G)_+$.

It is also readily verified that $\int_H f(hg)\,\mathrm{d}\mu_H(h) = f^\flat(Hg)$ for $g \in HK$, and zero for $g \notin HK$, this affords the required preimage of $f^\flat$. □

Fix a right Haar measure μ_H on H to define $f \mapsto f^\flat$ as in Lemma 1.50. Every positive Radon measure $\mu_{H\backslash G}$ on X induces $\mu^\natural_{H\backslash G}$ on G, characterized by $\int_G f\,\mathrm{d}\mu^\natural_{H\backslash G} = \int_{H\backslash G} f^\flat\,\mathrm{d}\mu_{H\backslash G}$.

Also recall the notion of quasi-invariance from Definition 1.14.

Lemma 1.51. *Fix a right Haar measure μ_H on H and let $\mu_{H\backslash G}$ be a quasi-invariant positive measure on X with eigencharacter $\chi : G \to \mathbb{R}^\times_{>0}$. Then $\mu^\natural_{H\backslash G}$ is of eigencharacter χ (resp. $\delta_H(h)$) under right G-action (resp. left H-action).*

Proof. Since $f \mapsto f^\flat$ respects right G-translation, the eigencharacter of $\mu^\natural_{H\backslash G}$ is the same χ. Now let $f \in C_c(G)$; recall that $f^h(g) = f(hg)$. By Lemma 1.28, $(f^h)^\flat(Hg)$ equals

$$\int_H f^h(kg)\,\mathrm{d}\mu_H(k) = \int_H f(hkg)\,\mathrm{d}\mu_H(k) = \delta_H(h)^{-1} \int_H f(h'g)\,\mathrm{d}\mu_H(h')$$

which is $\delta_H(h)^{-1} f^\flat(Hg)$. This implies that $\int_G f^h\,\mathrm{d}\mu^\natural_{H\backslash G} = \delta_H(h)^{-1} \int_G f\,\mathrm{d}\mu^\natural_{H\backslash G}$ as asserted. □

Theorem 1.52. *Let $\chi : G \to \mathbb{R}^\times_{>0}$ be a continuous homomorphism. Then there exists a quasi-invariant positive measure on $H\backslash G$ of eigencharacter χ if and only if*

$$\chi|_H = \delta_H \left(\delta_G|_H\right)^{-1}.$$

Moreover, such a quasi-invariant measure is unique up to $\mathbb{R}^\times_{>0}$ when $\chi|_H = \delta_H \delta_G|_H$, and every choice of right Haar measures μ_G (resp. μ_H) on G (resp. H) gives rise to such a measure $\mu_{H\backslash G}$, characterized by

$$\int_G f(g)\chi(g)\,\mathrm{d}\mu_G(g) = \int_{H\backslash G} \left(\int_H f(hg)\,\mathrm{d}\mu_H(h) \right) \mathrm{d}\mu_{H\backslash G}(Hg) \tag{1.9}$$

where $f \in C_c(G)$.

Consequently, G-invariant positive measures exists on $H\backslash G$ if and only if $\delta_H = \delta_G|_H$, in which case it is unique up to rescaling.

Proof. First, suppose $\mu_{H\backslash G}$ with eigencharacter χ is given on $H\backslash G$. Let us verify $\chi|_H = \delta_H \left(\delta_G|_H\right)^{-1}$ and (1.9). Fix a right Haar measure μ_H on H to apply Lemma 1.51 to obtain $\mu^\natural_{H\backslash G}$. Lemma 1.51 implies $\mu^\natural_{H\backslash G} = \chi \cdot \mu_G$ for a right Haar measure μ_G on G. Both sides are quasi-invariant under left H-action: the eigencharacter of $\mu^\natural_{H\backslash G}$ is δ_H, whilst that of $\chi \cdot \mu_G$ is $(\chi\delta_G)|_H$ by Remark 1.15 and Lemma 1.28. The formula (1.9) is built into our construction.

Conversely, given χ with $\chi|_H = \delta_H (\delta_G|_H)^{-1}$, fix μ_G and μ_H. We claim that

$$\begin{aligned} I : C_c(H\backslash G) &\longrightarrow \mathbb{C} \\ f^\flat &\longmapsto \int_G f\chi \, \mathrm{d}\mu_G, \quad f \mapsto f^\flat \end{aligned}$$

is well-defined. Granting this, one readily sees that I defines a Radon measure $\mu_{H\backslash G}$ by Lemma 1.50. Furthermore, since $f \mapsto f^\flat$ respects right G-action, $\mu_{H\backslash G}$ will have the same eigencharacter as $\chi\mu_G$, namely χ. What remains to show is that $f^\flat = 0$ implies $\int_G f\chi \, \mathrm{d}\mu_G = 0$. For any $\varphi \in C_c(G)$, Fubini's theorem entails

$$\begin{aligned} 0 &= \int_G \varphi(g) f^\flat(Hg)\chi(g) \, \mathrm{d}\mu_G(g) = \int_G \int_H \varphi(g) f(hg)\chi(g) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_G(g) \\ &= \int_H \int_G \chi(g')\varphi(h^{-1}g') f(g')\chi(h)^{-1}\delta_G(h)^{-1} \, \mathrm{d}\mu_G(g') \, \mathrm{d}\mu_H(h) \end{aligned}$$

by first swapping the integrals and then substituting $g' = hg$, using $\mathrm{d}\mu_G(g) = \delta_G(h)^{-1} \, \mathrm{d}\mu_G(g')$ from Lemma 1.28. Next, substitute $h' = h^{-1}$ into the last integral and use $\mathrm{d}\mu_H(h) = \delta_H(h')^{-1} \, \mathrm{d}\mu_H(h')$ from Lemma 1.28 to arrive at

$$\begin{aligned} 0 = \int_H \int_G \chi(g')\varphi(h'g') f(g') \underbrace{\chi(h')\delta_G(h')\delta_H(h')^{-1}}_{=1} \, \mathrm{d}\mu_H(h') \, \mathrm{d}\mu_G(g') \\ = \int_G \varphi^\flat(Hg') f(g')\chi(g') \, \mathrm{d}\mu_G(g'). \end{aligned}$$

Select φ so that $\varphi^\flat = 1$ on the image of $\mathrm{Supp}(f)$ in $H\backslash G$ to deduce $\int_G f\chi \, \mathrm{d}\mu_G = 0$. □

When G is an F-analytic Lie group and H is a Lie subgroup, the results can also be deduced by considerations of volume forms. We leave this to the curious reader.

Definition 1.53. Denote by μ_G/μ_H the quasi-invariant measure on $H\backslash G$ of eigencharacter χ determined by (1.9). To see the benefits of this notation, note that $(s\mu_G)/(t\mu_H) = (s/t) \cdot \mu_G/\mu_H$ for all $s, t \in \mathbb{R}^\times_{>0}$.

Remark 1.54. For homogeneous spaces under left G-actions, one uses left Haar measures on H to define $f \mapsto f^\flat$, and the condition in Theorem 1.52 becomes

$$\chi|_H = (\delta_H)^{-1} \cdot \delta_G|_H.$$

Indeed, one switches to right G^{op}-actions and applies the previous result; the eigencharacter χ is unaltered, whereas the modulus characters are replaced by inverses by Proposition 1.30.

2 Convolution

Let G be a locally compact group.

2.1 Convolution of functions

Fix a right Haar measure μ on G, and define $L^p(G) := L^p(G, \mu)$ for all $1 \le p \le \infty$. Write $\check{f}(x) := f(x^{-1})$ (sometimes as $f^\vee(x)$) for all functions f on G.

Given two functions f_1, f_2 on G, their *convolution* is defined as the function

$$\begin{aligned} (f_1 \star f_2)(x) &= \int_G f_1(xy^{-1}) f_2(y) \, \mathrm{d}\mu(y) \\ &= \int_G f_1(y^{-1}) f_2(yx) \, \mathrm{d}\mu(y), \quad x \in G \end{aligned} \tag{2.1}$$

whenever it converges. We will also consider (2.1) as an element in various L^p-spaces on G, by estimating $\|f_1 \star f_2\|_{L^p}$ in terms of the norms of f_1 and f_2 (see below).

Remark 2.1. If we use left Haar measures instead, or equivalently work in G^{op}, then (2.1) becomes $\int_G f_1(xy)f_2(y^{-1})\,\mathrm{d}\mu(y) = \int_G f_1(y)f_2(y^{-1}x)\,\mathrm{d}\mu(y)$.

Definition–Proposition 2.2. Let $f_1, f_2 \in L^1(G)$. Then $f_1 \star f_2$ is a well-defined element in $L^1(G)$, satisfying

$$\|f_1 \star f_2\|_{L^1} \leq \|f_1\|_{L^1} \cdot \|f_2\|_{L^1}.$$

The convolution product makes $L^1(G)$ into a Banach algebra, which is in general non-commutative, non-unital.

Proof. To show that $f_1 \star f_2 \in L^1(G)$ is well-defined, evaluate $\iint_{G\times G} |f_1(xy^{-1})| \cdot |f_2(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$ using Fubini's theorem to get $\|f_1\|_{L^2} \cdot \|f_2\|_{L^1}$. It is straightforward to verify that

$$\begin{aligned} f_1 \star (f_2 \star f_3) &= (f_1 \star f_2) \star f_3, \\ f_1 \star (f_2 + f_3) &= f_1 \star f_2 + f_1 \star f_3, \\ (f_1 + f_2) \star f_3 &= f_1 \star f_3 + f_2 \star f_3. \end{aligned}$$

Hence $(L^1(G), \star)$ is a Banach algebra. □

It is important to extend the convolution to more general functions and deduce the corresponding estimates. Following analysts' convention, define a bijection $p \mapsto p'$ from $[1, +\infty]$ to itself by requiring

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 2.3 (Minkowski's inequality)**.** *Suppose that* $1 \leq p \leq \infty$*. Then for* $f_1 \in L^p(G)$ *and* $f_2 \in L^1(G)$*,* (2.1) *is well-defined in* $L^p(G)$ *and we have*

$$\left\|f_1 \star f_2\right\|_{L^p} \leq \|f_1\|_{L^p} \cdot \|f_2\|_{L^1}.$$

Proof. The case $p = 1$ has just been addressed, and the case $p = \infty$ is easy. Assume $1 < p < \infty$. To ease notations, we may and do assume $f_1, f_2 \geq 0$. Apply Hölder's inequality to the measure $f_2(y)\,\mathrm{d}\mu(y)$ to obtain

$$\int_G f_1(xy^{-1})f_2(y)\,\mathrm{d}\mu(y) \leq \left(\int_G f_1(xy^{-1})^p f_2(y)\,\mathrm{d}\mu(y)\right)^{1/p} \left(\int_G f_2(y)\,\mathrm{d}\mu(y)\right)^{1/p'}.$$

Hence

$$\begin{aligned} \|f_1 \star f_2\|_{L^p} &\leq \left(\|f_2\|_{L^1}^{p-1} \cdot \iint_{G\times G} f_1(xy^{-1})^p f_2(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\right)^{1/p} \\ &= \left(\|f_2\|_{L^1}^{p-1} \cdot \iint_{G\times G} f_1(x)^p f_2(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{1/p} \\ &= \left(\|f_2\|_{L^1}^{p-1} \cdot \|f_1\|_{L^p}^p \cdot \|f_2\|_{L_1}\right)^{1/p} = \|f_1\|_{L^p} \cdot \|f_2\|_{L^1} \end{aligned}$$

where the right invariance of μ is used. □

Theorem 2.4 (Young's inequality)**.** *Suppose that* $1 \le p, q, r \le \infty$ *satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then for all f_1, f_2 *such that* $\check{f}_1 \in L^p(G)$ *and* $f_2 \in L^r(G)$*, the convolution* (2.1) *is well-defined in* $L^q(G)$ *and satisfies*

$$\|f_1 \star f_2\|_{L^q} \le \|\check{f}_1\|_{L^p} \cdot \|f_2\|_{L^r}.$$

Proof. The premises imply

$$\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1, \quad \frac{p}{q} + \frac{p}{r'} = 1, \quad \frac{r}{q} + \frac{r}{p'} = 1.$$

First, assume $f_1, f_2 \ge 0$. Apply Hölder's inequality with three exponents (r', q, p') and use the right invariance of μ to deduce that

$$\begin{aligned}
(f_1 \star f_2)(x) &= \int_G f_1(y^{-1}) f_2(yx) \,\mathrm{d}\mu(y) \\
&= \int_G f_1(y^{-1})^{\frac{p}{r'}} \left(f_1(y^{-1})^{\frac{p}{q}} f_2(yx)^{\frac{r}{q}} \right) f_2(yx)^{\frac{r}{p'}} \,\mathrm{d}\mu(y) \\
&\le \left(\int_G (\check{f}_1)^p \,\mathrm{d}\mu \right)^{1/r'} \left(\int_G f_1(y^{-1})^p \cdot f_2(yx)^r \,\mathrm{d}\mu(y) \right)^{1/q} \left(\int_G f_2(yx)^r \,\mathrm{d}\mu(y) \right)^{1/p'} \\
&= \left\|\check{f}_1\right\|_{L^p}^{p/r'} \left(\int_G f_1(y^{-1})^p f_2(yx)^r \,\mathrm{d}\mu(y) \right)^{1/q} \|f_2\|_{L^r}^{r/p'}.
\end{aligned}$$

It follows from Fubini's theorem that

$$\begin{aligned}
\|f_1 \star f_2\|_{L^q} &\le \left\|\check{f}_1\right\|_{L^p}^{p/r'} \|f_2\|_{L^r}^{r/p'} \left(\iint_{G \times G} f_1(y^{-1})^p f_2(yx)^r \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \right)^{1/q} \\
&= \left\|\check{f}_1\right\|_{L^p}^{p/r'} \cdot \|f_2\|_{L^r}^{r/p'} \cdot \|\check{f}_1\|_{L^p}^{p/q} \cdot \|f_2\|_{L^r}^{r/q} = \|\check{f}_1\|_{L^p} \cdot \|f_2\|_{L^r}.
\end{aligned}$$

In general, we can work with $|f_1|, |f_2|$ to reduce to the previous setting. □

Proposition 2.5. *If* $\varphi \in C_c(G)$ *and* $f \in L^r(G)$ *(resp.* $\check{f} \in L^r(G)$*) with* $1 \le r \le \infty$*, then* $\varphi \star f$ *(resp.* $f \star \varphi$*) is defined by a convergent integral and is a continuous function on* G*.*

Proof. Recall $\varphi^g(x) := \varphi(gx)$ and ${}^g\varphi(x) = \varphi(xg)$ for $g, x \in G$. Apply Theorem 2.4 with $q = \infty$ and $\frac{1}{p} + \frac{1}{r} = 1$ to see

$$\left|(\varphi \star f)(gx) - (\varphi \star f)(x)\right| = \left|((\varphi - \varphi^g) \star f)(x)\right| \le \left\|\varphi^\vee - (\varphi^g)^\vee\right\|_{L^p} \left\|f\right\|_{L^r}.$$

We conclude by noting that

$$\left\|\varphi^\vee - (\varphi^g)^\vee\right\|_{L^p} = \left\|\varphi^\vee - {}^{g^{-1}}(\varphi^\vee)\right\|_{L^p} \xrightarrow{g \to 1} 0.$$

The case with $\check{f} \in L^r(G)$ is completely analogous. □

2.2 Convolution of measures

[UNDER CONSTRUCTION]

3 Continuous representations

3.1 General representations

We will work exclusively with topological vector spaces over $\mathbb{C}$. Some words on Hausdorff property: a topological vector space V is also a topological group under $+$, therefore V being Hausdorff is the same as $\overline{\{0\}} = \{0\}$. As easily observed, $\overline{\{0\}} \subset V$ is a closed vector subspace, thus $V/\overline{\{0\}}$ is a Hausdorff topological vector space. The assignment $V \mapsto V/\overline{\{0\}}$ is functorial and is the left adjoint of the inclusion functor

$$\{\text{Hausdorff ones}\} \hookrightarrow \{\text{topological vector spaces}\}.$$

The main topological vector spaces under consideration are *Hausdorff* and *locally convex*; the latter means that there is a basis of neighborhoods consisting of convex subsets. Alternatively, locally convex Hausdorff spaces are exactly the topological vector spaces whose topology is defined by a family of separating semi-norms.

Definition 3.1. Henceforth we denote by $\mathsf{TopVect}$ the category of locally convex Hausdorff topological vector spaces over $\mathbb{C}$. In contrast, Vect will stand for the category of $\mathbb{C}$-vector spaces.

Remark 3.2. The property of being locally convex and Hausdorff is preserved under passing to subspaces and to quotients by a closed subspace. The direct sums of locally convex Hausdorff spaces also carry a locally convex Hausdorff topology, characterized in categorical terms.

More generally, one can form the inductive limits $\varinjlim$ inside the category of locally convex spaces; these limits are not necessarily Hausdorff, but one can take quotient by $\overline{\{0\}}$ to land in $\mathsf{TopVect}$.

Example 3.3. The Hilbert, Banach, Fréchet and LF-spaces are all in $\mathsf{TopVect}$. Every finite-dimensional vector space V admits a unique structure of Hausdorff topological vector space, namely the usual one coming from $V \simeq \mathbb{C}^{\dim V}$, which is also locally convex.

Denote by $\mathrm{Aut}_{\mathsf{TopVect}}(V)$ (resp. $\mathrm{Aut}_{\mathsf{Vect}}(V)$) the group of automorphisms of V taken in $\mathsf{TopVect}$ (resp. in the category of $\mathbb{C}$-vector spaces); $\mathrm{Aut}_{\mathsf{Vect}}(V)$ is much larger than $\mathrm{Aut}_{\mathsf{TopVect}}(V)$ for infinite-dimensional V.

We will consider certain linear left actions of a locally compact group G on a topological vector space V. This is the same as prescribing a homomorphism $\pi : G \to \mathrm{Aut}_{\mathsf{Vect}}(V)$ of groups; it can also be recorded by the corresponding "action map" $a : G \times V \to V$ with $a(g, v) = gv = \pi(g)v$. The issue of continuity is more delicate.

Definition 3.4. Let G be a locally compact group and let V in $\mathsf{TopVect}$. A *continuous representation* of G on the left of V is a homomorphism $\pi : G \to \mathrm{Aut}_{\mathsf{Vect}}(V)$ such that the corresponding action map

$$\begin{aligned} a : G \times V &\longrightarrow V \\ (g, v) &\longmapsto gv := \pi(g)v \end{aligned}$$

is continuous. The case of right actions is defined analogously, with the action written as $(v, g) \mapsto vg$ with the property $v(gg') = (vg)g'$, etc.

We denote a continuous representation of G as (π, V), V or π interchangeably, depending on the context; call $V = V_\pi$ the underlying vector space of π. The shorthand *G-representation* will be used extensively.

Definition 3.5. A morphism of G-representations $\varphi : V_1 \to V_2$ is understood to be continuous homomorphism of topological vector spaces such that $\varphi(gv_1) = g\varphi(v_1)$ for all $g \in G$ and $v_1 \in V_1$. This turns the collection of all G-representations into a category G-Rep, so one can talk about isomorphisms, automorphisms, etc. in the usual manner. The morphisms between G-representations are also called *intertwining operators*.

Definition 3.6. A *subrepresentation* of a G-representation (V, π) is a closed subspace $V_0 \subset V$ that is stable under G-action. A *quotient representation* is a quotient space $\bar{V} = V/V_0$ by a subrepresentation V_0, endowed with the quotient topology. In both cases, V_0 and $\bar{V}$ are naturally G-representations.

The G-action on the direct sum $\bigoplus_{i\in I} V_i$ of G-representations $\{V_i\}_{i\in I}$ makes it into a G-representation. The same holds for more general $\varinjlim$ of G-representations.

Call a G-representation $V \neq \{0\}$ *simple* or *irreducible* if the only subrepresentations (equivalently, quotient representations) are $\{0\}$ and V. Call it *indecomposable* if $V = V_1 \oplus V_2$ with subrepresentations $V_1, V_2 \subset V$ implies that either $V_1 = \{0\}$ or $V_2 = \{0\}$.

One of the basic goals of representation theory is to classify the irreducible G-representations under various constraints.

Remark 3.7. The category G-Rep is additive and $\mathbb{C}$-linear. In particular, the endomorphisms of a G-representation V form a $\mathbb{C}$-algebra $\mathrm{End}_{\mathbb{C}}(V)$. We caution the reader that, unlike the case of finite-dimensional representations, G-Rep is NOT an abelian category: already in the case of $G = \{1\}$ so that G-$\mathsf{Rep} = \mathsf{TopVect}$, every homomorphism $\varphi : V \to W$ of topological vector spaces admits kernel $\varphi^{-1}(0)$ and cokernel $W/\overline{\mathrm{im}(\varphi)}$; by taking φ with dense image, one can show that

$$\ker(\varphi) = 0 = \mathrm{coker}(\varphi), \quad \mathrm{coim}(\varphi) \simeq V \xrightarrow{\varphi} W \simeq \mathrm{im}(\varphi)$$

so that the canonical morphism $\mathrm{coim}(\varphi) \to \mathrm{im}(\varphi)$ is not an isomorphism if φ is not surjective, contradicting the axioms for abelian categories. Such a $\varphi : V \to W$ already exists for Banach spaces.

Let V be in $\mathsf{TopVect}$ with a given homomorphism $\pi : G \to \mathrm{Aut}_{\mathsf{Vect}}(V)$, so that G acts linearly on the left of V. Consider the orbit map for every $v \in V$

$$\begin{aligned} \mathrm{orb}_v : G &\longrightarrow V \\ g &\longmapsto gv. \end{aligned}$$

The following result gives a convenient means to check the continuity of the action map $a : G \times V \to V$ corresponding to π.

Proposition 3.8. *For any V in* $\mathsf{TopVect}$ *and a left linear action of G on V (no continuity condition so far), denote by $\pi(g)$ the map $v \mapsto gv$. The following are equivalent.*

1. *the action map $a : G \times V \to V$ is continuous, i.e. (V, π) is a G-representation;*
2. *the conditions below hold: (a) there is a dense subspace $V_0 \subset V$ such that orb_v is continuous for all $v \in V_0$, and (b) for every compact subset $K \subset G$ the family $\{\pi(g) : g \in K\}$ of operators on V is equicontinuous.*

Proof. Suppose that $a : G \times V \to V$ is continuous, then orb_v is continuous for all $v \in V_0 := V$. Let $K \subset G$ be compact. Given a neighborhood $W \ni 0$ in V, we have to find another neighborhood $U \ni 0$ so that $\pi(g)(U) \subset W$ for each $g \in K$. By the definition of product topology, for each $g \in K$ there is a neighborhood $U'_g \times U_g \ni (g, 0)$ in $G \times V$ that is contained in $a^{-1}(W)$. There is a finite subset $T \subset K$ such that $K = \bigcup_{t\in T} U'_t$; it remains to take $U := \bigcap_{t\in T} U_t$.

Conversely, assume (a) and (b), consider $(g, v) \in G \times V$ and let $W \ni 0$ be a neighborhood in V. We seek a neighborhood $U' \times U$ of (g, v) such that for all $(g', v') \in U' \times U$ we have

$$a(g', v') - a(g, v) = \pi(g')v' - \pi(g)v \in W.$$

For any such neighborhoods U', U, use density to pick $v_0 \in V_0 \cap U \neq \emptyset$; then write $\pi(g)v - \pi(g')v'$ as

$$\left(\pi(g)v - \pi(g)v_0\right) + \left(\pi(g)v_0 - \pi(g')v_0\right) + \left(\pi(g')v_0 - \pi(g')v'\right).$$

Assume U' has compact closure, then $\pi(g)v - \pi(g)v_0$ and $\pi(g')v_0 - \pi(g')v'$ approaches zero when U shrinks, by applying (b) to $K := \bar{U}$. Once $v_0 \in V_0 \cap U$ is chosen, we can further shrink U' to make $\pi(g)v_0 - \pi(g')v_0$ arbitrarily small. This concludes the proof. □

Corollary 3.9. *Let $(V, \|\cdot\|)$ be a Banach space and a left action of G on V (no continuity conditions so far). Then V is a G-representation if and only if $\pi(g)$ is continuous for every g and the orbit map $g \mapsto gv$ is continuous for every $v \in V$.*

Proof. It suffices to prove the "if" part. Take $V_0 = V$ in 3.8 and check the equicontinuity of $\{\pi(g) : g \in K\} \subset \mathrm{Aut}_{\mathsf{TopVect}}(V)$ for every compact $K \subset G$. More precisely, by the uniform boundedness principle of Banach–Steinhaus, it suffices to notice that the continuity of orbit maps implies

$$\sup_{g \in K} \left\| \pi(g)v \right\| < +\infty$$

for every $v \in V$, which in turn implies the asserted equicontinuity. □

Example 3.10. Suppose that X is a locally compact Hausdorff space with a continuous right G-action. Equip $C(X) := \{f : X \to \mathbb{C}, \text{ continuous}\}$ with the topology defined by semi-norms $\|\cdot\|_{\infty,\Omega} := \sup_{\Omega} \|\cdot\|$ where Ω ranges over compact subsets of X. This makes $C(X)$ into a member of $\mathsf{TopVect}$. Let G act on the left of $C(X)$ by

$$gf := {}^{g}f = [x \mapsto f(xg)], \quad f \in C(X),\ g \in G.$$

Let us verify that $C(X)$ is a G-representation. It has been observed that $g(g'f) = (gg')f$. Apply 3.8 with $V = V_0$ as follows.

- For every chosen $f \in C(X)$, we contend that $g \mapsto gf$ is a continuous map from G to $C(X)$. Continuity can be checked just at $g = 1$. This amounts to showing that for every compact $\Omega \subset X$ and every $\epsilon > 0$, there exists a neighborhood $U \ni 1$ in G such that $\sup_{x \in \Omega} |f(xg) - f(x)| < \epsilon$ whenever $g \in U$. Indeed, for every $x \in X$ there exist open $U_x \ni 1$ and $W_x \ni x$ in G and X, such that
  $$(g, y) \in U_x \times W_x \implies |f(yg) - f(y)| < \epsilon.$$
  Now take a finite subcover of the open covering $\{W_x\}_{x \in \Omega}$ of Ω, and take U to be the finite intersection of the corresponding U_x.

- Fix a compact $K \subset G$. For every $g \in K$ and every compact $\Omega \subset X$ we know ΩK is compact, and
  $$\|\pi(g)f_1 - \pi(g)f_2\|_{\Omega,\infty} = \sup_{x \in \Omega} |f_1(xg) - f_2(xg)| \leq \|f_1 - f_2\|_{\Omega K,\infty}.$$
  This shows the equicontinuity of the operators $\{\pi(g) : g \in K\}$.

Remark 3.11. When X is second countable, an increasing countable sequence of Ω exhausts X, and it suffices to treat the corresponding semi-norms. In this case $C(X)$ turns out to be Fréchet. If X is compact, it suffices to take $\Omega = X$ and X is a Banach space.

Example 3.12. The same result in 3.10 holds for G acting on $C_c(X)$, where

$$C_c(X) = \varinjlim_{\Omega} C_c(X, \Omega)$$

is equipped with the inductive limit topology, where

- $\Omega \subset X$ ranges over the compact subsets, and

- $C_c(X, \Omega) := \left\{ f \in C_c(X) : \mathrm{Supp}(f) \subset \Omega \right\}$, topologized by the sup-norm $\|\cdot\|_\infty$.

Then $C_c(X)$ with $gf(x) = f(xg)$ is also a G-representation on an LF-space, and this coincides with $C(X)$ when X is compact.

We check the continuity of $G \times C_c(X) \to C_c(X)$ directly. Since G is locally compact, it suffices to show the continuity of $K \times C_c(X) \to C_c(X)$ a given compact subset $K \subset G$. By the definition of the topology of $\varinjlim$, we further reduce to the composite

$$K \times C_c(X, \Omega) \to C_c(X, \Omega \cdot K) \hookrightarrow C_c(X)$$

where $\Omega \subset X$ is any compact. The continuity of $\hookrightarrow$ is evident, whilst that of $K \times C_c(X, \Omega) \to C_c(X, \Omega \cdot K)$ is straightforward to check, as everything is compact.

Example 3.13. Let X be a locally compact Hausdorff space, endowed with a continuous right G-action. For simplicity, we assume that X admits a G-invariant Radon positive measure μ; this is indeed the case when X is a homogeneous G-space isomorphic to $H\backslash G$ satisfying $\delta_G|_H = \delta_H$, by 1.52.

For every $1 \leq p \leq \infty$, we have G acting on the Banach space $V := L^p(G)$ by $gf = {}^g f : x \mapsto f(xg)$ as usual. This makes sense if $f \in C_c(X)$, and $\|{}^g f\|_p = \|f\|_p$ for all $g \in G$. Therefore the action extends to all $f \in L^p(X)$ by an approximation argument.

Claim: $L^2(X)$ is a G-representation. We apply 3.8 with $V_0 = C_c(X)$ as follows. The condition (b) is satisfied since $\pi(g)$ preserves $\|\cdot\|_p$, hence equicontinuous. It remains to check (a) that $g \mapsto gf$ is continuous for every given $f \in C_c(X)$, i.e. for every $\epsilon > 0$, there exists a neighborhood $U \ni 1$ in G such that $g \in U \implies \|gf - f\|_p < \epsilon$.

Since $\mathrm{Supp}(f)$ is compact, there exists a neighborhood $U \ni 1$ such that

$$|f(xg) - f(x)| < \epsilon\mu(\mathrm{Supp}(f))^{-1/p} \quad \text{for all } x.$$

Then $\|gf - f\|_p \leq \epsilon$ as required; note that the argument also works for $p = \infty$, and in that case we do not need measures.

Representations on function spaces arising from G-actions on X, such as the examples 3.10, 3.12 and 3.13 are called *regular representations*. The most important case is $X = G$. When $X = G$ is finite, we revert to the usual regular representation on the finite-dimensional space $\mathbb{C}[G] = \mathrm{Maps}(G, \mathbb{C})$.

We end these discussions by more terminologies.

Definition 3.14. Let (π, V) be a G-representation. If V is a Hilbert (resp. Banach, Fréchet) space, we say that (π, V) is a Hilbert (resp. Banach, Fréchet) representation. In each case, such representations form a full subcategory of G-Rep.

3.2 Matrix coefficients

Definition 3.15. For (π, V) be in G-Rep. Denote by $V^* := \mathrm{Hom}_{\mathsf{TopVect}}(V, \mathbb{C})$ the topological dual of V. It is customary to write $\langle \lambda, v \rangle := \lambda(v)$ for $\lambda \in V^*$, $v \in V$. Given λ, v, the corresponding *matrix coefficient* is the continuous function

$$\begin{aligned} c_{v\otimes\lambda} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle \lambda, \pi(g)v \rangle . \end{aligned}$$

Matrix coefficients is bilinear in λ and v, which justifies the tensor notation.

Remark 3.16. Suppose $V \neq \{0\}$. The Hahn–Banach theorem implies that for every $v \neq 0$ there exists λ such that $c_{v\otimes\lambda}(1) = \langle \lambda, v \rangle \neq 0$.

We can let G act on the left of V^* by the *contragredient*: $\lambda \mapsto \check{\pi}(g) := \lambda \circ \pi(g^{-1})$. However, the choice of topological structure on V^* is a subtle issue, and in practice it is necessary to pass to some subspace of V^* to obtain interesting continuous representations. At present, we do not put any extra structure on V^*.

A Hilbert representation (π, V) is called *unitary* if $\pi(g)$ is a unitary operator for all $g \in G$. We will say more about this in 4.1.

Remark 3.17. By Riesz's theorem, the topological dual of a Hilbert space $(V, (\cdot|\cdot)_V)$ is its Hermitian conjugate $\bar{V}$: it is the same space with the new scalar multiplication $(z, v) \mapsto \bar{z}v$ for $z \in \mathbb{C}$, and becomes a Hilbert space with $(v|w)_{\bar{V}} = (w|v)_V = \overline{(v|w)_V}$. Every $\bar{w} \in \bar{V}$ corresponds to the linear functional $v \mapsto (v|w)_V$.

The contragredient representation is therefore given by

$$\check{\pi}(g) := {}^*\pi(g)^{-1} : \bar{V} \to \bar{V}$$

where ${}^*(\cdots)$ denotes the Hermitian adjoint. It satisfies $(\check{\pi}(g)w|\pi(g)v) = (w|v)$.

When G is unitary, the contragredient reduces $\check{\pi}(g) = \pi(g)$; to emphasize the complex conjugation, we will also write $(\bar{\pi}, \bar{V}) = (\check{\pi}, \bar{V})$ in the unitary case, and it is again unitary.

Therefore 3.15 specializes as follows.

Definition 3.18. Let (π, V) be a Hilbert representation of G. For $v \in V$ and $w \in \bar{V}$, the corresponding *matrix coefficient* is the continuous map

$$\begin{aligned} c_{v\otimes w} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto (\pi(g)v|w)_V \end{aligned}$$

where $(\cdot|\cdot)$ is the Hermitian form on V.

We record some basic properties of matrix coefficients.

Lemma 3.19. *Let (π, V) be a Hilbert representation of G. The matrix coefficients map $w \otimes v \mapsto c_{v\otimes w}$ is bilinear in $V \times \bar{V}$. Moreover:*

$$c_{w\otimes v}(b^{-1}xa) = c_{\pi(a)v\otimes\check{\pi}(b)w}(x), \quad (a, b) \in G \times G,\ x \in G, \tag{3.1}$$

$$c_{v\otimes w}(g) = \overline{c_{w\otimes v}(g^{-1})}, \quad v, w \in V,\ g \in G, \quad \text{if } \pi \text{ is unitary}; \tag{3.2}$$

Here we write $\bar{\pi}$ for the representation π of G on $\bar{V}$, emphasizing that $c_{v\otimes w}$ is conjugate-linear in w. It is also additive under orthogonal direct sum $V_1 \oplus V_2$ of Hilbert representations, namely

$$c_{(v,v')\otimes(w,w')}(g) = c_{v\otimes w}(g) + c_{v'\otimes w'}(g) \tag{3.3}$$

with $g \in G$, $v, w \in V_1$ and $v', w' \in V_2$.

Proof. When π is unitary, the identity $(\pi(g)w|v) = (w|\pi(g^{-1})v) = \overline{(\pi(g^{-1})v|w)}$ proves (3.2). Next,

$$\left(\pi(h^{-1})\pi(x)\pi(g)v|w\right) = (\pi(x)\pi(g)v|\check{\pi}(h)w)$$

translates into (3.1). □

3.3 Vector-valued integrals

Vector-valued integrals are immensely useful for studying continuous representations. We shall consider a weak version thereof, called *weak integrals* or *Gelfand–Pettis integrals*. The canon is surely Bourbaki [3]; shorter surveys include [8, 3.26 — 3.29] and [4].

Definition 3.20. Let X be a locally compact Hausdorff space and μ a positive Radon measure thereon. Let V be in $\mathsf{TopVect}$, and $f : X \to V$ is a function such that $\lambda \circ f : X \to \mathbb{C}$ is μ-measurable for all λ in the continuous dual V^*. Suppose that $\lambda \circ f$ is μ-integrable for all λ, define the integral $\int_X f \, \mathrm{d}\mu$ as the element

$$\lambda \longmapsto \int_X \langle \lambda, f(x) \rangle \, \mathrm{d}\mu(x), \quad \lambda \in V^*$$

in $\mathrm{Hom}_{\mathsf{Vect}}(V^*, \mathbb{C})$, i.e. the algebraic dual of V^*.

Since we are working with locally convex Hausdorff spaces, V^* separates points on V by the Hahn–Banach theorem, and the natural map $V \to \mathrm{Hom}_{\mathsf{Vect}}(V^*, \mathbb{C})$ is injective. Therefore it makes sense to ask if $\int_X f \, \mathrm{d}\mu$ exists inside V, which will then take a definite value, called the weak integral of $f \, \mathrm{d}\mu$.

When V is a Banach space, the Bochner or the "strong" integrals are instances of weak integrals. Due to its "weak" definition, Gelfand–Pettis integrals satisfy agreeable functorial properties, as illustrated below.

Proposition 3.21. *Let $T : V \to W$ be a morphism in $\mathsf{TopVect}$. If $\int_X f \, \mathrm{d}\mu$ in the situation of 3.20 exists, then so does $\int_X Tf \, \mathrm{d}\mu$ and we have $\int_X Tf \, \mathrm{d}\mu = T \int_X f \, \mathrm{d}\mu$. Here we identify T with the natural linear map $\mathrm{Hom}_{\mathsf{Vect}}(V^*, \mathbb{C}) \to \mathrm{Hom}_{\mathsf{Vect}}(W^*, \mathbb{C})$ it induces.*

Proof. By definition, $T \int_X f \, \mathrm{d}\mu$ is the linear functional on W^* mapping each $\lambda \in W^*$ to $\left\langle \lambda \circ T, \int_X f \, \mathrm{d}\mu \right\rangle$, which is nothing but $\int_X \langle \lambda, Tf \rangle \, \mathrm{d}\mu$. Also, $\langle \lambda, Tf \rangle = \langle \lambda \circ T, f \rangle$ is μ-integrable by assumptions. □

A Radon measure on X is called *bounded* if the integration functional $I : C_c(X) \to \mathbb{C}$ is continuous for the L^∞-norm. A similarly recipe defines the locally bounded measures on X. Denote the support of a measure μ by $\mathrm{Supp}(\mu)$.

Theorem 3.22. *Suppose that μ is bounded in the situation of 3.20. If $f(\mathrm{Supp}(\mu))$ is contained in a convex, balanced, bounded and complete subset $B \subset V$, then $\lambda \circ f$ is μ-integrable for all $\lambda \in V^*$ and*

$$\int_X f \, \mathrm{d}\mu \ \in \mu(X) \cdot B \subset V.$$

Proof. This is in [3, §1.2, Proposition 8]. □

Remark 3.23. Most often we encounter the case of

- $f : X \to V$ is continuous,
- μ has compact support.

In particular, $f(\mathrm{Supp}(\mu))$ is compact. The premises in 3.22 are met whenever compact subsets in V have compact convex hulls. The latter property holds for *quasi-complete* spaces. A topological vector space is called quasi-complete if every bounded closed subset is complete. Completeness implies quasi-completeness as expected. In particular, 3.22 applies to Fréchet spaces.

Another example of quasi-complete spaces are the continuous duals of *barreled spaces*, endowed with the topology of pointwise convergence. For instance, Fréchet spaces and LF spaces are all barreled.

Proposition 3.24. *In the circumstance of 3.22, suppose that* $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ *is a continuous semi-norm on* V. *Then*

$$\left\| \int_X f \, d\mu \right\| \leq \int_X \|f\| \, d\mu.$$

Proof. Take $v = \int_X f \, d\mu \in V$. There exists $\lambda \in V^*$ with $\langle \lambda, v \rangle = \|v\|$ and $|\langle \lambda, w \rangle| \leq \|w\|$ for all $w \in V$ by Hahn–Banach theorem. Insert this λ into the characterization of $\int_X f \, d\mu$. □

3.4 Action of convolution algebra

In what follows, we will often write the integration of a measure f on a locally compact Hausdorff space X as $\int_{x \in X} f(x)$, omitting the useless dx. It is convenient to adopt the notations for functions: given an isomorphism $\Phi : Y \to X$, write $f \circ \Phi$ for the measure transported to Y, so that $\int_Y f \circ \Phi = \int_X f$ holds for tautological reasons.

Let G be a locally compact group. Define the following space of Radon measures on G

$$\begin{aligned}
\mathscr{M}(G) &:= \{\text{bounded Radon measures on } G\}, \\
\mathscr{M}_c(G) &:= \left\{ f \in \mathscr{M}(G) : \operatorname{Supp}(f) \text{ is compact} \right\}, \\
\tilde{\mathscr{M}}(G) &:= \left\{ f \in \mathscr{M}(G) : f \ll \text{any right Haar measure} \right\}, \\
\tilde{\mathscr{M}}_c(G) &:= \tilde{\mathscr{M}}(G) \cap \mathscr{M}_c(G).
\end{aligned}$$

If a right Haar measure μ on G is chosen, then $L^1(G) \, d\mu = \tilde{\mathscr{M}}(G)$.

Recall that a subset of G is called symmetric if it is invariant under $g \mapsto g^{-1}$. Likewise, we say a function or a measure on G is symmetric if it has the same invariance. A topological group always has a symmetric neighborhood basis at 1: replacing each neighborhood $U \ni 1$ by $U \cap U^{-1} \ni 1$ suffices.

Definition 3.25. An *approximate identity* is a symmetric neighborhood basis $\mathfrak{N}$ at $1 \in G$ together with a family $\varphi_U \in \tilde{\mathscr{M}}_c(G)$, where U ranges over $\mathfrak{N}$, such that for all $U \in \mathfrak{N}$,

- φ_U is positive of the form $\phi_U \, d\mu$ where $\phi_U \in C_c(G)$ and μ is a right Haar measure;
- $\int_G \varphi_U = 1$;
- $\operatorname{Supp}(\varphi_U) \subset U$, $\varphi_U(g) = \varphi_U(g^{-1})$.

When $\mathfrak{N}$ is countable, an approximate identity can be conveniently described as a sequence $\varphi_1, \varphi_2, \ldots$ in $\tilde{\mathscr{M}}_c(G)$ with each φ_i symmetric and positive, and $\operatorname{Supp}(\varphi_i)$ shrinks to $\{1\}$ as $\mathfrak{N}$ does.

Proposition 3.26. *For every symmetric neighborhood basis* $\mathfrak{N}$ *at* $1 \in G$, *there exists approximate identities supported on* $\mathfrak{N}$.

Proof. Given $U \in \mathfrak{N}$, use Urysohn's lemma to construct φ_U. We can force $\varphi_U(g) = \varphi_U(g^{-1})$ by averaging. □

Definition 3.27. Let (π, V) be a continuous representation of G on a quasi-complete space. Let $\tilde{\mathscr{M}}_c(G)$ act on the left of V by V-valued integrals

$$\pi(\varphi)v = \int_G \varphi(g)\pi(g)v, \quad v \in V, \ \varphi \in \tilde{\mathscr{M}}_c(G),$$

whose existence is guaranteed by 3.22.

One should imagine that $\pi(g) = \pi(\delta_g)$ where δ_g stands for the Dirac measure at g. In general,

$$\pi(h^{-1})\pi(f)\pi(g^{-1}) = \pi\left({}^h f^g\right), \quad {}^h f^g(x) = f(hxg), \tag{3.4}$$

where the left/right translation should be understood in terms of measures. It preserves the spaces $\mathscr{M}(G)$, etc.

Lemma 3.28. *This makes V into an $\tilde{\mathscr{M}}_c(G)$-algebra, namely*

$$\begin{aligned} \pi(\varphi) &\in \mathrm{End}_{\mathbb{C}}(V), \\ \pi(a\varphi_1 + b\varphi_2)v &= a\pi(\varphi_1)v + b\pi(\varphi_2)v, \\ \pi(\varphi_1 \star \varphi_2) &= \pi(\varphi_1)\pi(\varphi_2)v \end{aligned}$$

for all $\varphi_1, \varphi_2 \in \tilde{\mathscr{M}}_c(G)$ and $a, b \in \mathbb{C}$. Furthermore, if $(\varphi_U)_{U\in\mathfrak{N}}$ is an approximate identity, then

$$\lim_{U\in\mathfrak{N}} \pi(\varphi_U)v = v, \quad v \in V$$

in the sense of filter bases.

Proof. The linearity is clear. As for convolutions, take any $\lambda \in V^*$ and verify that

$$\begin{aligned} \langle \lambda, \pi(\varphi_1 \star \varphi_2)v \rangle &= \int_{g\in G}\int_{h\in H} \varphi_1(h^{-1})\varphi_2(hg)\,\langle \lambda, \pi(g)v\rangle \\ (\because \text{Fubini's theorem}) &= \int_{h\in H} \varphi_1(h^{-1})\left\langle \lambda, \left(\int_{g\in G} \pi(h^{-1})\varphi_2(hg)\pi(hg)v\right)\right\rangle \\ (\because 3.21) &= \int_{h\in H} \varphi_1(h^{-1})\left\langle \lambda, \pi(h^{-1})\left(\pi(\varphi_2)v\right)\right\rangle \\ &= \langle \lambda, \pi(\varphi_1)\pi(\varphi_2)v\rangle. \end{aligned}$$

For the convergence, it suffices to show for every continuous semi-norm $\|\cdot\|$ that

$$\lim_U \|\pi(\varphi_U)v - v\| = 0$$

since V is locally convex. By 3.24, we have

$$\begin{aligned} \|\pi(\varphi_U)v - v\| &= \left\| \int_G \varphi_U(g)(\pi(g)v - v) \right\| \\ &\leq \int_G \varphi_U(g)\|\pi(g)v - v\|. \end{aligned}$$

Given $\epsilon > 0$, for sufficiently small U we have $\|\pi(g)v - v\| < \epsilon$ when $g \in U$. Hence the last expression is bounded by $\epsilon \int_G \varphi_U = \epsilon$. □

Lemma 3.29. *The subspace $\pi\left(\tilde{\mathscr{M}}_c(G)\right)V$ of V is G-stable and dense. In fact, the subspace spanned by $\pi(\varphi_U)V$ is already dense, when φ_U runs over an approximate identity (recall 3.25).*

Proof. Apply 3.28. Note that $\tilde{\mathscr{M}}_c(G)$ is stable under (3.4), thus $\pi\left(\tilde{\mathscr{M}}_c(G)\right)V$ is G-stable as well. □

3.5 Smooth vectors: archimedean case

Let G be a locally compact group and (π, V) be a G-representation. The orbit map $\mathrm{orb}_v : g \mapsto gv$ can also be used to define representation-theoretic properties of vectors $v \in V$.

Definition 3.30. Let **P** stand for a property of continuous functions $G \to V$. For a G-representation V and $v \in G$, we say that v has the property **P** if $\mathrm{orb}_v : G \to V$ has.

Since $v \mapsto \mathrm{orb}_v$ is a $\mathbb{C}$-linear map from V to $\mathrm{Maps}(G, V)$, if the functions with property **P** form a vector space, so are the vectors with property **P**.

In particular, when G is an analytic Lie group over a local field F, we can talk about C^∞ (infinitely differentiable) and C^ω (analytic) vectors in a G-representation, thereby bringing the smooth structure into the picture.

Definition 3.31. The C^∞-vectors in V are called *smooth vectors*. They form a subspace

$$V^\infty := \{v \in V : \text{smooth vector}\}.$$

In the next few sections we will mainly be interested in the case of archimedean F. The same definition pertains to non-archimedean case as well (see 1.38), but it has a quite different flavor. Note that for archimedean F, one can actually talk about C^k-vectors where $k \in \{0, 1, \dots, \infty, \omega\}$.

In what follows we consider only $F = \mathbb{R}$. The case of $\mathbb{C}$ follows by "Weil restriction", namely one can view a $\mathbb{C}$-group as an $\mathbb{R}$-group by dropping its complex structure.

For a Lie group G over $F = \mathbb{R}$, let $\mathfrak{g} := \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexified Lie algebra, and let $U(\mathfrak{g})$ designate the universal enveloping algebra of $\mathfrak{g}$.

Definition 3.32. Suppose that G is a Lie group over $\mathbb{R}$. Let (π, V) be in G-Rep. Define the eponymous action π of the Lie algebra $\mathfrak{g}$ by

$$\pi(X)v := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(g(t))v = \lim_{h\to 0} \frac{\pi(g(h))v - \pi(g(0))v}{h}, \quad X \in \mathfrak{g}$$

for every $v \in V^\infty$, where g is any differentiable curve in G with $g(0) = 1$ and $g'(0) = v$; it is customary to take $g(t) = \exp(tX)$.

Lemma 3.33. *The operators $\pi(X)$ above yield a homomorphism of $\mathbb{C}$-algebras*

$$U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V^\infty).$$

Proof. In other words $(X, v) \mapsto \pi(X)v$ is a representation of the Lie algebra $\mathfrak{g}$. It boils down to check that $\pi(X)(\pi(Y)v) - \pi(Y)(\pi(X)v) = \pi([X, Y])v$ for all $v \in V^\infty$ and $X, Y \in \mathfrak{g}$. This reduces the to basic fact that

$$\exp(tX)\exp(sY) = \exp\left(tX + sY + \frac{st}{2}[X, Y] + \text{higher}\right).$$

□

[TO BE CONTINUED...]

4 Unitary representation theory

Fix a locally compact group G.

4.1 Generalities

Recall that a pre-Hilbert space is a $\mathbb{C}$-vector space V equipped with a positive-definite Hermitian form $(\cdot|\cdot) = (\cdot|\cdot)_V$; if V is complete with respect to the norm $\|v\| := (v|v)^{1/2}$, we say V is a Hilbert space.

Definition 4.1. A *unitary representation* (resp. *pre-unitary representation*) of G is a Hilbert (resp. pre-Hilbert) space $(V, (\cdot|\cdot))$ together with a linear left G-action on V such that

- for every $v \in V$, the map $g \mapsto gv$ from G to V is continuous,
- for each $g \in G$, the operator $\pi(g) : V \to V$ from the G-action is unitary, i.e. $\|\pi(g)(v)\| = \|v\|$ for all $v \in V$.

Remark 4.2. In view of 3.9 with $V_0 = V$, a unitary representation is automatically continuous, and is the same as a Hilbert representation (V, π) of G such that all $\pi(g)$ are unitary operators.

On the other hand, pre-unitary representations can be completed to unitary ones.

Lemma 4.3. *If V is a pre-unitary representation, then the Hilbert space completion $\hat{V}$ of V with respect to $\|\cdot\|$ can be endowed with an induced G-action, and becomes a unitary representation.*

Proof. For every $g \in G$, the operator $\pi(g)$ extends uniquely to a unitary operator on $\hat{V}$. The relation $\pi(g)\pi(g') = \pi(gg')$ also extends to $\hat{V}$. It remains to check the continuity of $g \mapsto \pi(g)v$ where $v \in \hat{V}$ is fixed. To see this, take $v_0 \in V$ and note that

$$\begin{aligned}\|\pi(g)v - \pi(g')v\| &\leq \|\pi(g)v - \pi(g)v_0\| + \|\pi(g)v_0 - \pi(g')v_0\| + \|\pi(g')v_0 - \pi(g')v\| \\ &= \|v - v_0\| + \|\pi(g)v_0 - \pi(g')v_0\| + \|v_0 - v\|.\end{aligned}$$

We conclude from the continuity of $g \mapsto \pi(g)v_0$, by taking $\|v_0 - v\|$ sufficiently small. $\square$

Definition 4.4. A morphism or an *intertwining operator* $\varphi : V_1 \to V_2$ between unitary representations is a $\mathbb{C}$-linear homomorphism φ such that

- φ is unitary, i.e. preserves inner products: $(v|w)_{V_1} = (\varphi(v)|\varphi(w))_{V_2}$ for all $v, w \in V_1$,
- φ respects G-actions: $g\varphi(v) = \varphi(gv)$ for all $v \in V_1$ and $g \in G$.

This turns the collection of all unitary representations into a category, which is a non-full subcategory of Hilbert representations. Isomorphisms between unitary representations are also called *unitary equivalences*.

Example 4.5. Let G acts continuously on a locally compact Hausdorff space; for simplicity assume that X carries a G-invariant positive Radon measure. It is shown in 3.13 with $p = 2$ that $L^2(G)$ is a G-representation. It is actually unitary, since we have pointed out in 3.13 that $f \mapsto gf$ preserves the L^2-norm $\|\cdot\|_2$.

The finite direct sum $\bigoplus_{i=1}^n V_i$ of unitary representations $V_1, \ldots, V_n$ is defined so that the underlying Hilbert space is the orthogonal direct sum of $V_1, \ldots, V_n$. The case of countably infinite sum of $V_1, \ldots$ requires more care: we have to use the completed direct sum $\hat{\bigoplus}_{i\geq 1} V_i$, which is the completion of the pre-Hilbert space $\bigoplus_{i\geq 1} V_i$.

The notion of subrepresentations and irreducibility of unitary G-representations are the same as the case of continuous representations in general. On the other hand, quotients are unnecessary as shown by the following complete reducibility.

Lemma 4.6. *Let W be a subrepresentation of a unitary G-representation V. Then*

$$V = W \oplus W^{\perp}$$

as unitary representations, where $W^{\perp} := \{v \in V : \forall w \in W,\ (v|w) = 0\}$.

Proof. The orthogonal decomposition surely works on the level of Hilbert spaces. It remains to notice that $W^{\perp}$ is stable under G-action. □

In contrast to the case of modules, 4.6 does not imply that V can be written as a direct sum of irreducibles. In fact, in many cases V has no simple submodules, as we shall see in 4.11.

The following is a unitary variant of the familiar Schur's Lemma for irreducible representations of finite groups.

Theorem 4.7. *Let (π, V) be an irreducible unitary representation. Then*

$$\mathrm{End}_{G\text{-}\mathsf{Rep}}(V) = \mathbb{C} \cdot \mathrm{id}_V.$$

Proof. Let $T : V \to V$ be continuous and linear. Suppose that T commutes with G-action, then so is its adjoint ${}^*T$ since

$$(w|g^*Tv) = (Tg^{-1}w|v) = (g^{-1}Tw|v) = (Tw|gv) = (w|^*Tgv)$$

for all $w \in V$ and $g \in G$. Write

$$T = \frac{T + {}^*T}{2} + \sqrt{-1} \cdot \frac{T - {}^*T}{2\sqrt{-1}}.$$

Therefore, to show that T is a scalar, we may suppose ${}^*T = T$.

Now let $\sigma(T) \subset \mathbb{R}$ be the spectrum of T. Apply the spectral decomposition for self-adjoint bounded operators (e.g. [8, 12.23 Theorem]) to express T via its *spectral measure* E:

$$T = \int_{\sigma(T)} \lambda \, \mathrm{d}E(\lambda).$$

Here E can be thought as a projection-valued measure on the Borel subsets of $\sigma(T)$; it is canonically associated to T. The irreducibility of V together with the G-equivariance of T imply that $E(A \cap \sigma(T))$ is either 0 or id_V for all Borel subsets $A \subset \mathbb{R}$; i.e. $\sigma(T)$ is an *atom* in measure-theoretic terms. It follows that there exists a point λ with $E(\{\lambda\}) = \mathrm{id}_V$. Indeed, take any interval I_1 with spectral measure id_V; bisect it and pass to the one with spectral measure id_V, named I_2, and so forth to obtain nested intervals $I_1 \supset I_2 \supset \cdots$ such that $\bigcap_{i \geq 1} I_i = \{\lambda\}$ for some λ. The zero-one dichotomy then implies that E is supported at $\{\lambda\}$. Therefore $T = \lambda \cdot \mathrm{id}_V$. □

Corollary 4.8. *A unitary representation (π, V) is irreducible if and only if $\mathrm{End}_{G\text{-}\mathsf{Rep}}(V) = \mathbb{C} \cdot \mathrm{id}_V$.*

Proof. The "only if" part follows from 4.7. Conversely, if $W \subset V$ is a proper subrepresentation then $V = W \oplus W^{\perp}$ by 4.6, hence

$$\mathrm{End}_{G\text{-}\mathsf{Rep}}(V) \supset \mathrm{End}_{G\text{-}\mathsf{Rep}}(W) \oplus \mathrm{End}_{G\text{-}\mathsf{Rep}}(W^{\perp})$$

has dimension ≥ 2. □

Another important consequence of Schur's Lemma is the existence of central characters in the unitary case.

Theorem 4.9. *Let (π, V) be an irreducible unitary representation. There is a continuous homomorphism $\omega_\pi : Z_G \to \mathbb{S}^1$, where Z_G denotes the center of G, such that $\pi(z) = \omega_\pi(z)\mathrm{id}_V$ for every $z \in Z_G$. We call ω_π the* central character *of π.*

Proof. Central elements act on V by scalar multiplication since they commute with G. This yields the homomorphism ω_π whose continuity and unitarity follow from that of (π, V). □

In practice, it is convenient to allow any closed subgroup Z of Z_G. By abuse of language, we also say that (π, V) has central character ω on Z, if Z acts via the continuous homomorphism $\omega : Z \to \mathbb{S}^1$.

Corollary 4.10. *If G is commutative, the irreducible unitary representations are precisely the one-dimensional unitary representations.*

Proof. For commutative G we have $G = Z_G$ acts via ω_π on irreducible (π, V). Conversely, one-dimensional representations are clearly irreducible. □

Example 4.11. Consider the unitary $\mathbb{R}$-representation $L^2(\mathbb{R})$. It has no irreducible subrepresentations. Indeed, an irreducible subrepresentation must be generated by an L^2-function f with $f(x+t) = \chi(t)f(x)$, where χ is a continuous homomorphism $\mathbb{R} \to \mathbb{S}^1$. The only such L^2-function is 0.

The next enhancement of 4.7 will be needed.

Theorem 4.12. *Let (π, V) and (σ, W) be unitary representations of G and assume that (π, V) is irreducible. Let T be a G-equivariant linear map from a dense G-stable vector subspace $V_0 \subset V$ to W. If the graph of T*

$$\Gamma_T := \{(v, Tv) \in V_0 \times W : v \in V_0\}$$

is closed in $V \oplus W$, then $V_0 = V$ and T is a scalar multiple of a morphism between unitary representations.

Proof. We shall make use of basic facts on *unbounded operators* from Hilbert spaces V to W in general; see for example [8, Chapter 13]. Such an operator T is only defined on some subspace $\mathscr{D}(T)$ of V, and continuity is not presumed. When taking their composites, etc., domains must shrink suitably. In our case $\mathscr{D}(T) = V_0$ is dense.

The first result is the existence of an adjoint ${}^*T$ defined on

$$\mathscr{D}({}^*T) := \left\{w \in W : (T(\cdot)|w)_W \in \mathrm{Hom}_{\mathsf{TopVect}}(V_0, \mathbb{C})\right\},$$

uniquely determined by $(Tv|w)_W = (v|{}^*Tw)_V$. Secondly, assume that Γ_T is closed, then $\mathscr{D}({}^*T)$ is also dense; then consider the unbounded operator $Q := 1 + {}^*TT$ from V to itself. According to [8, Theorem 13.13],

- $\mathscr{D}(Q) = \{v \in \mathscr{D}(T) : Tv \in \mathscr{D}({}^*T)\}$ maps onto V under Q;
- there is a continuous endomorphism B of V such that $\mathrm{im}(B) \subset \mathscr{D}(Q)$ and $QB = \mathrm{id}_V$. In fact $b := Bv$ (for all $v \in V$) is characterized in $W \oplus V$ by

$$(0, v) = (c, {}^*Tc) + (-Tb, b), \quad b \in \mathscr{D}(T),\ c \in \mathscr{D}({}^*T). \tag{4.1}$$

Now apply these results to our setting. The domain $\mathscr{D}({}^*T)$ is a G-stable subspace as $V_0 = \mathscr{D}(T)$ is. One readily checks that ${}^*T : \mathscr{D}({}^*T) \to V$ is also G-equivariant. From the characterization (4.1), we conclude that B is G-equivariant as well, hence $B = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}^\times$ by 4.7, as $QB = \mathrm{id}_V$.

Consequently $\mathscr{D}(Q) = \mathscr{D}(T) = V$ and $Q = \lambda^{-1}\mathrm{id}_{\mathscr{D}(Q)}$. This also implies $\mathscr{D}({}^*T) \supset \mathrm{im}(T)$. Now

$$(Tv|Tv')_V = ({}^*TTv|v')_V = (\lambda^{-1} - 1)(v|v')_V, \quad v, v' \in V_0$$

hence T is a scalar multiple of an isometry $V \to W$. □

4.2 External tensor products

The theory of completed tensor products for general locally convex spaces V_1, V_2 is delicate. For our present purposes, the case of Hilbert spaces suffices.

Definition 4.13. Let $(V_1, (\cdot|\cdot)_1)$ and $(V_2, (\cdot|\cdot)_2)$ be Hilbert spaces. Write $V_1 \otimes V_2 := V_1 \otimes_{\mathbb{C}} V_2$ and equip it with an Hermitian form by requiring

$$(v_1 \otimes v_2 \mid w_1 \otimes w_2) = (v_1|w_1)_1 \cdot (v_2|w_2)_2, \quad v_1, w_1 \in V_1,\ v_2, w_2 \in V_2$$

and extend by sesquilinearity to all $V_1 \otimes V_2$. This is well-defined and positive-definite. Denote by $V_1 \hat{\otimes} V_2$ the corresponding completion of $V_1 \otimes V_2$.

The operation $\hat{\otimes}$ is functorial in V_1 and V_2, turning the collection of Hilbert spaces into a symmetric monoidal category.

Now suppose that V_i is the underlying space of a unitary representations π_i of a locally compact group G_i, for $i = 1, 2$. Let $G_1 \times G_2$ act on $V_1 \otimes V_2$ via $\pi_1 \otimes \pi_2$; this extends to an action on $V_1 \hat{\otimes} V_2$ by unitary operators. In order to stress that it is acted upon by $G_1 \times G_2$, it is customary to designate the resulting representation by $\pi_1 \boxtimes \pi_2$. Summing up, we obtain

$$\left(V_1 \hat{\otimes} V_2, \pi_1 \boxtimes \pi_2\right): \text{ unitary representation of } G_1 \times G_2.$$

Proposition 4.14. *If π_1 and π_2 are both irreducible, then $\pi_1 \boxtimes \pi_2$ is irreducible as a unitary representation of $G_1 \times G_2$.*

Proof. In view of 4.8, it suffices to show that every continuous $G_1 \times G_2$-equivariant endomorphism T of $\pi_1 \boxtimes \pi_2$ is $c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$. Since this property can be tested by taking inner product with all elements from $V_1 \otimes V_2$, which are dense in $V_1 \hat{\otimes} V_2$, it is equivalent to that

$$\left(T(v_1 \otimes v_2) \,\middle|\, w_1 \otimes w_2\right) = c(v_1|w_1)_{V_1} \cdot (v_2|w_2)_{V_2}$$

for all v_1, v_2 and w_1, w_2.

To achieve this, first fix $v_2, w_2 \in V_2$ and define the endomorphism

$$\begin{aligned} \varphi_{w_2,v_2} : V_1 &\longrightarrow V_1 \otimes \mathbb{C} = V_1 \\ v_1 &\longmapsto \left(\mathrm{id}_{V_1} \otimes (\cdot|w_2)_{V_2}\right)(T(v_1 \otimes v_2);) \end{aligned}$$

here we view $(\cdot|w_2)_{V_2}$ as $V_2 \to \mathbb{C}$. It is continuous and G-equivariant, hence 4.7 implies $\varphi_{w_2,v_2} = a_{w_2,v_2} \cdot \mathrm{id}_{V_1}$ for some scalar a_{w_2,v_2}. Now we have a continuous linear map

$$\begin{aligned} A : \overline{V_2} &\longrightarrow (V_2)^* \\ w_2 &\longmapsto \left[v_2 \mapsto a_{w_2,v_2}\right]. \end{aligned}$$

Recall that $(V_2)^* \simeq \overline{V_2}$ via the Hermitian form. A careful verification reveals that A is G_2-equivariant, hence $A = c \cdot \mathrm{id}_{V_2}$ for some $c \in \mathbb{C}$. In other words, $a_{w_2,v_2} = c(v_2|w_2)_{V_2}$ for all $v_2, w_2 \in V_2$. This implies $\left(T(v_1 \otimes v_2) \,\middle|\, w_1 \otimes w_2\right) = c(v_1|w_1)_{V_1} \cdot (v_2|w_2)_{V_2}$. □

Proposition 4.15. *Let (π_i, V_i) and (σ_i, W_i) be irreducible unitary representations of G_i for $i = 1, 2$. If $\pi_1 \boxtimes \sigma_1 \simeq \pi_2 \boxtimes \sigma_2$, then $\pi_i \simeq \sigma_i$ for $i = 1, 2$.*

Proof. Suppose that $T : \pi_1 \boxtimes \sigma_1 \simeq \pi_2 \boxtimes \sigma_2$ is nonzero. Then there must exist $v_2 \in V_2$ and $w_2 \in W_2$ such that

$$\begin{aligned} \varphi_{w_2,v_2} : V_1 &\longrightarrow W_1 \otimes \mathbb{C} = W_1 \\ v_1 &\longmapsto \left(\mathrm{id}_{W_1} \otimes (\cdot|w_2)_{W_2}\right)(T(v_1 \otimes v_2)) \end{aligned}$$

is nonzero. As in the proof of 4.14, it is G_1-equivariant and continuous, hence $\pi_1 \simeq \sigma_1$ by 4.12. Switching the roles of G_1, G_2 yields $\pi_2 \simeq \pi_1$. □

Of particular importance is the case $G_1 = G = G_2$ and $(V_1, V_2) = (V, \bar{V})$, where (π, V) is a unitary representation of G and $(\bar{\pi}, \bar{V})$ is its Hermitian conjugate; see 3.17. Recall that $(x|y)_{\bar{V}} := (y|x)_V$.

By linear algebra, there is a $\mathbb{C}$-linear isomorphism

$$\begin{aligned} V \otimes \bar{V} &\xrightarrow{\sim} \mathrm{End}_{\mathrm{f.r.}}(V) := \left\{ T : V \xrightarrow{/\mathbb{C}} V : \text{finite rank, continuous} \right\} \\ v \otimes w &\longmapsto A_{v\otimes w} := (\cdot|w)v. \end{aligned} \tag{4.2}$$

Equip $\mathrm{End}_{\mathrm{f.r.}}(V)$ with the Hilbert–Schmidt Hermitian inner product, which is positive-definite:

$$(A|B)_{\mathrm{HS}} := \mathrm{Tr}({}^*B \cdot A).$$

It is routine yet amusing to verify that ${}^*A_{w\otimes v} = A_{v\otimes w}$. Hence

$$\left(A_{v\otimes w} \middle| A_{v'\otimes w'}\right)_{\mathrm{HS}} = (v'|v)_V \cdot (w|w')_V = (v|v')_V \cdot (w|w')_{\bar{V}}$$

which equals the Hermitian inner product in $V \otimes \bar{V}$.

Also recall that the space of *Hilbert–Schmidt operators* $V \to V$ is defined as

$$\mathrm{HS}(V) := \text{completion of } \mathrm{End}_{\mathrm{f.r.}}(V) \text{ relative to } (\cdot|\cdot)_{\mathrm{HS}}.$$

The group $G \times G$ acts on $\mathrm{End}_{\mathrm{f.r.}}(V)$ and $\mathrm{HS}(V)$ by

$$(a, b)T = \pi(a) T \pi(b)^{-1}.$$

This is visibly unitary and corresponds to the action on $V \otimes \bar{V}$ under (4.2).

Theorem 4.16. *The identification* (4.2) *extends to an isomorphism*

$$V \boxtimes \bar{V} \simeq \mathrm{HS}(V)$$

between unitary $G \times G$-representations.

Proof. The identification is a $G \times G$-equivariant isometry between $V \otimes \bar{V}$ and $\mathrm{End}_{\mathrm{f.r.}}(V)$. Now pass to completion. □

4.3 Square-integrable representations

Fix a closed subgroup $Z \subset Z_G$. Central characters of irreducible unitary representations will be defined relative to Z. In view of 1.52, up to $\mathbb{R}^\times_{>0}$ there exists a unique right Haar measure on G/Z, since the condition $\delta_G|_Z = \delta_Z$ is trivially satisfied: both are trivial from the very definition of 1.27.

As unitary representations are a special kind of Hilbert representations, it makes sense to consider the matrix coefficients $c_{v\otimes w}(g) = (\pi(g)v|w)_V$ for $v \otimes w \in V \otimes \bar{V}$ as in 3.18. If (π, V) admits a central character ω_π as in 4.9, for example when (π, V) is irreducible, then

$$c_{v\otimes w}(zg) = \omega_\pi(z) c_{v\otimes w}(g), \quad z \in Z,\ g \in G.$$

In this case $|c_{w\otimes v}|$ factors through G/Z.

Definition 4.17. Denote by $L^2(G/Z)$ the L^2-space defined relative to any right Haar measure on G/Z. More precisely, for every continuous homomorphism $\omega : Z \to \mathbb{S}^1\}$, set

$$L^2(G/Z, \omega) := \left\{ f : G \to \mathbb{C}, \begin{array}{l} f(zx) = \omega(z)f(x),\ z \in Z \\ |f| \in L^2(G/Z) \end{array} \right\}.$$

It is a Hilbert space under $\|f\|^2 = \int_{G/Z} |f|^2 \,\mathrm{d}\mu$ once a right Haar measure μ is chosen. Specifically, one starts from measurable f, and take the quotient by functions with $\|\cdot\| = 0$ and so on. Likewise, we define the other function spaces on G with (ω, Z)-equivariance such as

$$C_c(G/Z, \omega) := \left\{ f : G \to \mathbb{C} \text{ continuous}, \begin{array}{l} f(zx) = \omega(z)f(x),\ z \in Z \\ \mathrm{Supp}(f)/Z \text{ is compact} \end{array} \right\}.$$

Notice that $L^2(G/Z, \omega)$ is also a unitary representation of G under $g : f(x) \mapsto f(xg)$ as in 4.5.

Definition 4.18. Suppose G is unimodular. An irreducible unitary representation (π, V) of G is called *square-integrable* or a *discrete series representation* if the matrix coefficient $c_{v\otimes w}$ is nonzero and lies in $L^2(G/Z, \omega_\pi)$ for some w, v.

Such representations are also called *essentially square-integrable* in the literature, while the term square-integrable is reserved to those with $c_{v\otimes w} \in L^2(G)$, i.e. for $Z = \{1\}$. Square-integrable representations in the latter strict sense exists only when Z_G is compact, cf. 1.21 and the integration formula in 1.52; when Z_G is compact, both definitions coincide.

In 4.20 we will extend this definition to non-unimodular groups.

Lemma 4.19. *Let (π, V) be an irreducible unitary representation (π, V) of a unimodular group G. The following are equivalent.*

1. *(π, V) is square-integrable;*
2. *$c_{v\otimes w} \in L^2(G/Z, \omega_\pi)$ for all $v, w \neq 0$, and if $T : v' \mapsto c_{v'\otimes w}$ is not identically zero, T is a G-equivariant linear map from V into $L^2(G/Z, \omega_\pi)$ with closed image, which is a scalar multiple of an isometry.*
3. *there exists an embedding $V \hookrightarrow L^2(G/Z, \omega_\pi)$ as unitary G-representations.*

Proof. We begin with (1) $\implies$ (2). Fix $w, v \in V$ with $c_{v\otimes w} \in L^2(G/Z, \omega_\pi)$ nonzero. We begin by showing that $c_{v'\otimes w} \in L^2(G/Z, \omega_\pi)$ for all v'. Set

$$V_0 := \left\{ v' \in V : c_{v'\otimes w} \in L^2(G/Z, \omega_\pi) \right\}.$$

This is a G-stable vector subspace by (3.1); it contains v, hence we infer from the irreducibility of V that V_0 is dense. Define a linear map

$$T : V_0 \to L^2(G/Z, \omega_\pi), \quad v' \mapsto c_{v'\otimes w}.$$

By (3.1) we see T is G-equivariant. Equip V_0 with the Hermitian pairing

$$(\!(v_1'|v_2')\!) = (v_1'|v_2')_V + (Tv_1'|Tv_2')_{L^2}.$$

We claim that V_0 is a Hilbert space under $(\!(\cdot|\cdot)\!)$, and the graph Γ_T is closed in $V_0 \times L^2(G/Z, \omega_\pi)$.

Indeed, $(\!(\cdot|\cdot)\!)$ is evidently Hermitian and positive definite; we set out to show its completeness. The choice of $(\!(\cdot|\cdot)\!)$ implies that for every Cauchy sequence $(v_i')_{i=1}^\infty$ in V_0, there exists $(v', c) \in V \times L^2(G/Z, \omega_\pi)$ such that $v_i' \to v'$ in V and $Tv_i' \to c$ in $L^2(G/Z, \omega_\pi)$. Observe that $v_i' \to v'$ implies that

$$\left| c_{v_i'\otimes w}(x) - c_{v'\otimes w}(x) \right| = \left| (\pi(x)(v_i' - v')|w) \right| \leq \|v_i' - v'\| \cdot \|w\|$$

so that $Tv'_i \to c_{v'\otimes w}$ uniformly on G. On the other hand, one can pass to a subsequence to ensure $Tv'_i \to c$ pointwise and a.e. Therefore $c_{v'\otimes w} = c$ so that $v' \in V_0$. This proves both the completeness and closedness parts of our claim.

By 4.12, the claim implies that $V_0 = V$ and T is a scalar multiple of an isometry. Such maps between Hilbert spaces always have closed images.

To conclude (2), we must also show that $c_{v\otimes w'} \in L^2(G/Z, \omega_\pi)$ for all $w' \in V$. Via (3.2) and the unimodularity of G/Z, this is subsumed into the previous case.

It is immediate that (2) $\implies$ (3): T furnishes the required embedding.

(3) $\implies$ (1): Assume that (π, V) is a unitary subrepresentation of $L^2(G/Z, \omega_\pi)$ and $v \in V$ is nonzero. There exists $\tilde{w} \in L^2(G/Z, \omega_\pi)$ such that $(v|\tilde{w})_{L^2(G/Z,\omega_\pi)} \neq 0$. We may even assume that $\tilde{w} \in C_c(G/Z, \omega_\pi)$ by density. Then $v' \mapsto (v'|\tilde{w})_{L^2(G/Z,\omega_\pi)}$ restricts to a continuous linear functional on V, of the form $v' \mapsto (v'|w)_V$ for a unique $w \in V$. Thus $c_{v\otimes w}(1) \neq 0$ and it remains to show $c_{v\otimes w} \in L^2(G/Z, \omega_\pi)$.

Denote by μ the chosen right Haar measure on G/Z. Put $u(x) := \overline{\tilde{w}(x^{-1})}$ to express $c_{v\otimes w}(g) = (\pi(g)v|\tilde{w})_{L^2(G/Z,\omega_\pi)}$ as

$$\int_{G/Z} u(x^{-1})v(xg)\,\mathrm{d}\mu(x).$$

Its absolute value is bounded by $(|u| \star |v|)(g)$ (convolution on G/Z). As $|u| \in C_c(G/Z)$ and $|v| \in L^2(G/Z)$, it follows either from a direct analysis or Young's inequality 2.4 with $(p, q, r) = (1, 2, 2)$ that $|u| \star |v| \in L^2(G/Z)$, as required. □

The third condition in 4.19 makes sense for any locally compact G, which leads to the following general definition.

Definition 4.20. Let G be a locally compact group. An irreducible unitary representation (π, V) of G is called *square-integrable* or a *discrete series representation* if it is isomorphic to a unitary subrepresentation of $L^2(G/Z, \omega_\pi)$.

Theorem 4.21 (Schur orthogonality relation)**.** *Let (π, V) and (σ, W) be square-integrable representations of a unimodular group G with $\omega_\pi = \omega_\sigma$, and fix a right Haar measure μ on G/Z.*

1. *If (π, V) and (σ, W) are not isomorphic as unitary representations, then*

$$\int_{G/Z} c_{v\otimes v'}\overline{c_{w\otimes w'}}\,\mathrm{d}\mu = 0$$

for all $v, v' \in V$ and $w, w' \in W$.

2. *In the case $\pi = \sigma$, there exists $d(\pi) \in \mathbb{R}_{>0}$, inverse-proportional to the choice of μ, such that*

$$\int_{G/Z} c_{v\otimes v'}\overline{c_{w\otimes w'}}\,\mathrm{d}\mu = d(\pi)^{-1}(v|w)_V(w'|v')_V$$

for all $v, v', w, w' \in V$.

Proof. Write $T : v \mapsto c_{v\otimes v'}$ and $S : w \mapsto c_{w\otimes w'}$. They are scalar multiples of isometries and are G-equivariant. Now

$$\int_{G/Z} c_{v\otimes v'}\overline{c_{w\otimes w'}}\,\mathrm{d}\mu = (Tv|Sw)_{L^2(G/Z,\omega_\pi)} = ({}^*STv|w)_W.$$

But ${}^*S$ is also continuous and G-equivariant, thus $A := {}^*ST : V \to W$ is a morphism in G-Rep. Then 4.12 implies that A is a scalar multiple of a morphism between unitary representations. If (π, V) and (σ, W) are non-isomorphic, the only possibility is $A = 0$ and the first item is proved.

Next, assume $\pi = \sigma$. Then $A = \lambda_{v',w'} \cdot \mathrm{id}_V$ for some $\lambda_{v',w'} \in \mathbb{C}$. This amounts to

$$\int_{G/Z} c_{v\otimes v'} \overline{c_{w\otimes w'}} \,\mathrm{d}\mu = \lambda_{v',w'} (v|w)_V.$$

Switching $v \leftrightarrow v'$ and $w \leftrightarrow w'$ replaces the matrix coefficients by their complex conjugates. As G/Z is unimodular, we deduce

$$\overline{\lambda_{v',w'}}(w|v)_V = \lambda_{v,w}(v'|w')_V.$$

Fix $v = w \neq 0$ in the equation above to see that $\lambda_{v',w'} = c(w'|v')_V$ for some constant c depending solely on (π, V) and μ; it is proportional to μ. Setting $v = w = w' = v'$ to deduce $\|c_{v\otimes v}\|^2_{L^2(G/Z,\omega_\pi)} = c(v|v)^4_V$. As $c_{v\otimes v}(1) \neq 0$ whenever $v \neq 0$, we see $c > 0$. Now set $d(\pi) = c^{-1}$ to conclude. □

Corollary 4.22. *The convolutions of square-integrable matrix coefficients on G/Z are well-defined. In fact, for irreducible unitary representations (π, V), (σ, W) we have*

$$\begin{aligned} c_{v\otimes v'} \star c_{w\otimes w'}(x) &:= \int_{G/Z} c_{v\otimes v'}(g^{-1}) c_{w\otimes w'}(gx) \,\mathrm{d}\mu(g) \\ &= \begin{cases} d(\pi)^{-1}(v|w')c_{v'\otimes w}(x), & \pi = \sigma \\ 0, & \pi \not\simeq \sigma, \end{cases} \end{aligned}$$

for all $v', v \in V$ and $w, w' \in W$.

Proof. Use 3.2, 3.1 and apply 4.21 to deduce

$$\begin{aligned} \int_{G/Z} c_{v\otimes v'}(g^{-1}) c_{w\otimes w'}(gx) \,\mathrm{d}\mu(g) &= \int_{G/Z} c_{\pi'(x)w\otimes w'}(g) \overline{c_{v'\otimes v}(g)} \,\mathrm{d}\mu(g) \\ &= \begin{cases} d(\pi)^{-1}(\pi'(x)w|v')(v|w'), & \pi = \sigma \\ 0, & \pi \not\simeq \sigma. \end{cases} \end{aligned}$$

In the first case, it equals $d(\pi)^{-1}(v|w')c_{w\otimes v'}(x)$. □

Now enters the bilateral translation on G. Observe that $G \times G$ acts on the right of G by $(a, b) : x \mapsto b^{-1}xa$. It acts on the left of functions $G \to \mathbb{C}$ by

$$(a, b)f : x \mapsto f(b^{-1}xa), \quad x \in G.$$

Proposition 4.23. *The action $((a, b)f)(x) = f(b^{-1}xa)$ makes $L^2(G/Z, \omega)$ into a unitary representation of $G \times G$.*

Proof. Since G/Z is assumed to be unimodular, this action is unitary; in other words G carries a $G \times G$-invariant positive Radon measure. The rest of the verification is akin to 4.5. □

Corollary 4.24. *Let π be a square-integrable unitary representation of G with central character ω on Z. Then $v\otimes w \mapsto d(\pi)^{-1}c_{v\otimes w}$ yields an isomorphism from $\pi \boxtimes \bar{\pi}$ onto the irreducible $G\times G$-subrepresentation of $L^2(G/Z_G, \omega)$ generated by all matrix coefficients of π.*

Proof. The map is $G \times G$-equivariant by (3.1) on $V \otimes \bar{V}$. It is an isometry on $V \otimes \bar{V}$ by 4.21 since

$$(v \otimes v'|w \otimes w')_{V\otimes\bar{V}} = (v|w)_V (v'|w')_{\bar{V}} = (v|w)_V (w'|v')_V.$$

These properties extends to $V\hat{\otimes}\bar{V}$ by density. The image of any isometry is closed. Irreducibility of $\pi \boxtimes \bar{\pi}$ follows from 4.14. □

Definition 4.25. The positive constant $d(\pi)$ is called the *formal degree* of π. It depends only on the isomorphism class of π and G, μ. It is inverse-proportional to the choice of μ and generalizes the notion of dimension, cf. 4.40.

4.4 Spectral decomposition: the discrete part

Fix a closed subgroup $Z \subset Z_G$. We still assume G/Z unimodular, and fix a Haar measure μ on it.

Definition 4.26. Let $\omega : Z \to \mathbb{S}^1$ be a continuous character. The discrete spectrum with central character ω is the subrepresentation

$$L^2_{\mathrm{disc}}(G/Z, \omega) := \overline{\sum \left\{ \pi \subset L^2(G/Z, \omega) : \text{irred. unitary subrep.} \right\}}.$$

By 4.19, every π in the sum is square-integrable and every square-integrable unitary representation π with $\omega_\pi|_Z = \omega$ intertwines into $L^2_{\mathrm{disc}}(G/Z, \omega)$ via matrix coefficients.

Lemma 4.27. *Every irreducible unitary subrepresentation of $L^2(G/Z, \omega)$ lies in the closed subspace generated by matrix coefficients of square-integrable representations of central character ω on Z.*

Proof. Let C stand for the closed subspace generated by all square-integrable matrix coefficients of central character ω on Z. If the assertion does not hold, some irreducible subrepresentation (π, V) would have non-trivial orthogonal projection to $C^\perp$. The projection being G-equivariant, π can be embedded inside $C^\perp$. This is contradictory as explained below.

Assume that $V \subset C^\perp$ and $f \in V$ is nonzero. Express μ as μ_G/μ_Z according to 1.53. There exists a real-valued $\varphi \in C_c(G)$ such that the function

$$\begin{aligned}
\Phi(g) &:= (\check{\varphi} \star f)(g) \quad \text{(continuous by 2.5)} \\
&= \int_G f(hg)\varphi(h)\, \mathrm{d}\mu_G(h) = \int_{G/Z} \int_Z f(hg)\varphi(zh)\omega(z)\, \mathrm{d}\mu_Z(z)\, \mathrm{d}\mu(h) \\
&= \int_{G/Z} f(hg)\overline{\varphi_\omega(h)}\, \mathrm{d}\mu(h)
\end{aligned}$$

is not identically zero, where

$$\varphi_\omega(h) := \overline{\int_Z \varphi(zh)\omega(z)\, \mathrm{d}\mu_Z(h)}, \quad \varphi_\omega \in C_c(G/Z, \omega).$$

This is indeed possible by 3.29 since

$$\Phi(g) = \int_G \check{\varphi}(h) f(h^{-1}g)\, \mathrm{d}\mu_G(h) = L(\check{\varphi}) f(g)$$

where L denotes the continuous G-representation $L(h)f(x) = f(h^{-1}x)$ on $L^2(G/Z, \omega)$.

Note that $\Phi(g) = c_{f\otimes w}(g)$ where $w \in V$ is such that $(\cdot|\varphi_\omega)_{L^2(G/Z,\omega)} = (\cdot|w)_V$. We infer that $(\Phi|c_{f\otimes w}) = \|\Phi\|^2_{L^2(G/Z,\omega)} \neq 0$. On the other hand, since G is unimodular,

$$\begin{aligned}
(\Phi|c_{f\otimes w}) &= \iint_{G/Z\times G/Z} f(hg)\overline{\varphi_\omega(h)} \cdot \overline{c_{f\otimes w}(g)}\, \mathrm{d}\mu(h)\, \mathrm{d}\mu(g) \\
&= \int_{G/Z} \left(\int_{G/Z} f(g)\overline{c_{f\otimes w}(h^{-1}g)}\, \mathrm{d}\mu(g) \right) \overline{\varphi_\omega(h)}\, \mathrm{d}\mu(h) \\
\because (3.1) \quad &= \int_{G/Z} \left(f \,\middle|\, c_{f\otimes\pi(h)w} \right)_{L^2(G/Z,\omega)} \overline{\varphi_\omega(h)}\, \mathrm{d}\mu(h).
\end{aligned}$$

The inner integral vanishes by assumption. Contradiction. □

In general, it may happen that $L^2_{\mathrm{disc}}(G/Z, \omega) = \{0\}$ for all ω. See 4.11.

Theorem 4.28. *For each square-integrable unitary representation π with central character ω on Z, there is an isomorphism of unitary $G \times G$-representations*

$$\mathrm{Coeff} : \hat{\bigoplus_{\pi}} \mathrm{HS}(V_\pi) = \hat{\bigoplus_{\pi}} \pi \boxtimes \bar{\pi} \xrightarrow{\sim} L^2_{\mathrm{disc}}(G/Z, \omega)$$
$$(v_\pi \otimes w_\pi)_\pi \longmapsto \sum_\pi d(\pi) c_{v_\pi \otimes w_\pi}.$$

- *(π, V_π) ranges over the square-integrable representations with central character ω on Z, taken up to isomorphism,*
- *$G \times G$ acts on $L^2(G/Z, \omega)$ by $((a, b)f)(x) = f(b^{-1}xa)$, and*
- *the $\hat{\bigoplus}$ is a completed orthogonal direct sum of unitary $G \times G$-representations.*

Moreover, $\pi \boxtimes \bar{\pi}$ are pairwise non-isomorphic; i.e. the decomposition above of unitary representations is multiplicity-free.

Proof. Combine 4.24 with 4.27. Matrix coefficients from different π are orthogonal, thus the $\overline{\sum \cdots}$ in 4.26 becomes an orthogonal $\hat{\bigoplus}$. The representations $\pi \boxtimes \bar{\pi}$ are pairwise distinct by 4.15. The relation to $\mathrm{HS}(V_\pi)$ is explicated in 4.16. □

Proposition 4.29. *Equip each $V_\pi \hat{\otimes} \overline{V_\pi}$ (resp. $\mathrm{HS}(V_\pi)$) in 4.28 with the $\mathbb{C}$-algebra structure with multiplication*

$$(v \otimes v') \star (w \otimes w') := (v|w')v' \otimes w, \quad (\textit{resp. } A \star B := B \circ A).$$

Then Coeff *restricted to $\bigoplus_\pi \bar{\pi} \boxtimes \pi$ respects the $\star$-multiplications on both sides.*

Proof. This reduces to 4.22 and the rule

$$(\cdot|v')v \star (\cdot|w')w = (v|w')(\cdot|v')w$$

inside each $\mathrm{HS}(V_\pi)$, which mirrors $(v \otimes v') \star (w \otimes w')$. □

The inverse of Coeff can be described on the linear span of matrix coefficients as follows.

Lemma 4.30. *Let (π, V) be a square-integrable unitary representation with central character ω on Z. Consider $\varphi :=$ Coeff$(v \otimes w) \in L^2(G/Z, \omega) \cap C(G/Z, \omega)$, where $v \otimes w \in V \otimes \bar{V}$. Then the vector-valued integrals*

$$\pi(\check{\varphi})v' = \int_{G/Z} \check{\varphi}(g)\pi(g)v \,\mathrm{d}\mu(g), \quad v' \in V$$

defines an operator $V \to V$, which equals the $(\cdot|w)v \in \mathrm{HS}(V)$ determined by $v \otimes w$.

Proof. We determine $\pi(\check{\varphi})v'$ by applying $(\cdot|w') \in V^*$, where $w' \in V$ is arbitrary. By (3.2) and 4.21, this euqlas

$$\int_{G/Z} \check{\varphi}(g)(\pi(g)v'|w') \,\mathrm{d}\mu(g) = d(\pi) \int_{G/Z} c_{v \otimes w}(g^{-1})(\pi(g)v'|w') \,\mathrm{d}\mu(g)$$
$$= \int_{G/Z} c_{v' \otimes w'} \overline{c_{w \otimes v}} \,\mathrm{d}\mu = (v'|w)(v|w').$$

Hence $\pi(\check{\varphi})v' = (v'|w)v$ and defines a linear endomorphism on V. The same description pertains to $(\cdot|w)v$. □

4.5 Usage of compact operators

We make systematic use of the formalism in 3.27.

Lemma 4.31. *Let $\varphi \in \tilde{\mathscr{M}}_c(G)$ and let (π, V) be a unitary representation of G. Define $\check{\varphi}(g) = \varphi(g^{-1})$ in the sense of measures. Then*

$$ {}^*\pi(\varphi) = \pi\left(\overline{\check{\varphi}}\right). $$

In particular, if φ is real-valued and $\varphi(g) = \varphi(g^{-1})$ for all g, then $\pi(\varphi) = {}^*\pi(\varphi)$.

Proof. For all $v, w \in V$,

$$ \begin{aligned} (\pi(\varphi)v|w) &= \int_G \varphi(g)(\pi(g)v|w) = \int_G \varphi(g)(v|\pi(g^{-1})w) \\ &= \int_G \check{\varphi}(g)\overline{(\pi(g)w|v)} = \overline{\int_G \overline{\check{\varphi}(g)}(\pi(g)w|v)} \\ &= \overline{\left(\pi\left(\overline{\check{\varphi}}\right)w\middle|v\right)} = \left(v\middle|\pi\left(\overline{\check{\varphi}}\right)w\right). \end{aligned} $$

□

Example 4.32. Now consider G acting on $L^2(G/Z, \omega)$ where $Z \subset Z_G$ is closed and $\omega : Z \to \mathbb{S}^1$ is a chosen continuous homomorphism; call this representation R. Choose a right Haar measure μ on G/Z, with $\mu = \mu_G/\mu_Z$ as in 1.53. Assume that $\varphi \in C_c(G)\,\mathrm{d}\mu$, then we have

$$ \begin{aligned} \left(R(\varphi)f\right)(x) &= \int_G \varphi(g)f(xg)\,\mathrm{d}\mu_G(g) \\ &= \int_{G/Z} K_\varphi(x,y)f(y)\,\mathrm{d}\mu_{G/Z}(g) \end{aligned} $$

with

$$ K_\varphi(x,y) := \int_Z \varphi(x^{-1}zy)\omega(z)\,\mathrm{d}\mu_Z(z), \quad x, y \in G/Z. \tag{4.3} $$

Clearly $K_\varphi(x,y) : G \times G \to \mathbb{C}$ is continuous and satisfies

$$ K_\varphi(xz', yz'') = \omega(z')\omega(z'')^{-1}K_\varphi(x,y), \quad z', z'' \in Z. $$

It expresses $R(\varphi)$ as an integral transform on $L^2(G/Z, \omega)$ with kernel K_φ. This is slightly different from the usual picture, since a character ω intervenes.

Lemma 4.33 (Gelfand–Graev–Piatetski-Shapiro)**.** *Let (π, V) be a unitary representation of G. Suppose that there exists an approximate identity $(\varphi_U)_{U\in\mathfrak{N}}$ (recall 3.25) such that each $\pi(\varphi_U) : V \to V$ is a compact operator. Then (π, V) is a completed orthogonal sum of its irreducible subrepresentations. Moreover, each irreducible constituent appears in this decomposition with finite multiplicity.*

Proof. The following arguments are due to Langlands. Set

$$ \mathscr{S} := \{\text{sets of mutually orthogonal irreducible subrepresentations } \subset \pi\}. $$

Note that $\mathscr{S}$ is nonempty since $\varnothing \in \Sigma$, and it is partially ordered by set inclusion. Zorn's lemma affords a maximal element $S \in \mathscr{S}$: indeed, every chain in $\mathscr{S}$ has the upper bound furnished by taking union. Consider the subrepresentation $\pi_0 := \hat{\bigoplus}_{\sigma\in S}\sigma$ of π. We claim that $\pi_0 = \pi$.

Let the subrepresentation π_1 of π be the orthogonal complement of π_0. We will derive a contradiction from $\pi_1 \neq \{0\}$ by exhibiting an irreducible subrepresentation inside π_1, which would violate the maximality of S.

Denote by $V_1 \subset V$ the underlying space of π_1. By 3.28, there exists φ_U from the approximate identity such that $T := \pi(\varphi_U)|_{V_1} \neq 0$. Note that T is a self-adjoint compact operator since $\pi(\varphi_U)$ is, by 3.25 and 4.31. The spectral theorem for such operators (eg. [8, Theorems 4.25 + 12.23]) implies that T has an eigenvalue $\lambda \in \mathbb{R} \smallsetminus \{0\}$, and the eigenspace $V_1^{T=\lambda}$ is nonzero and finite-dimensional. Take a subspace $V_1^\flat \subset V_1^{T=\lambda}$ of minimal dimension subject to the condition that $V_1^\flat = W^{T=\lambda} = W \cap V_1^{T=\lambda}$ for some G-stable closed subspace $W \subset V_1$. Among these W, their intersection $W_{\min} := \bigcap\{W : W^{T=\lambda} = V_1^\flat\}$ is the smallest one. We contend that $W_{\min}$ is irreducible. Otherwise there would be an orthogonal decomposition $W_{\min} = A \oplus B$ into nonzero G-stable closed subspaces, and

$$V_1^\flat = W_{\min}^{T=\lambda} = A^{T=\lambda} \oplus B^{T=\lambda}.$$

The minimality assumption on $V_1^\flat$ implies that $V_1^\flat$ equals $A^{T=\lambda}$ or $B^{T=\lambda}$, but this contradicts the minimality of $W_{\min}$.

We conclude that $\pi = \hat{\bigoplus}_{\sigma \in S} \sigma$. Suppose that some irreducible unitary representation σ_0 (considered up to isomorphism) intervenes with multiplicity m. Then for every φ_U in the approximate identity, $\pi(\varphi_U)$ will restrict to m-copies of $\sigma_0(\varphi_U)$. By assumption we may choose φ_U such that $\sigma_0(\varphi_U) \neq 0$, thus has a nonzero eigenvalue λ. As the λ-eigenspace of $\pi(\varphi_U)$ is finite-dimensional, m must be finite as well. □

Proposition 4.34. *In the setting of 4.32, suppose that G/Z is compact, then $L^2(G/Z, \omega)$ decomposes into a completed orthogonal direct sum of irreducible unitary representations, each constituent appearing with finite multiplicity.*

In particular, $L^2(G/Z, \omega) = L^2_{\mathrm{disc}}(G/Z, \omega)$ when G is unimodular.

Proof. In view of 4.33, it suffices to show that each $\varphi \in C_c(G)$ acts on $L^2(G/Z, \omega)$ via compact operators. In fact they act as Hilbert–Schmidt operators: a standard result asserts that integral operators on $L^2(X)$ prescribed by a kernel $K \in L^2(X \times X)$ are Hilbert–Schmidt. If X is compact then every continuous $K : X \times X \to \mathbb{C}$ is L^2. One can take $X = G/Z$ if $\omega = 1$, but the arguments with $\omega : Z \to \mathbb{S}^1$ and K_φ satisfying (4.3) are entirely analogous. □

4.6 The case of compact groups

Suppose that G is a unimodular group and G/Z is compact. Therefore all irreducible unitary representations of G are square-integrable relative to Z. Fix a Haar measure μ on G/Z.

Proposition 4.35. *Every irreducible unitary representation (π, V) of G is finite-dimensional.*

Proof. The argument below is credited to L. Nachbin. Take a nonzero $v \in V$. Define for all $w \in V$ the vector-valued integral

$$Aw := \int_{G/Z} (w|\pi(g)v)_V \cdot \pi(g)v \, \mathrm{d}\mu(g).$$

By 3.24 we know $\|Aw\|_V \leq \|w\|_V \cdot \|v\|_V^2 \cdot \mu(G/Z)$, and A is easily seen to be G-equivariant as G acts unitarily. Hence $A = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$ by 4.7; moreover $(Av|v)_V = \int |(v|\pi(g)v)|^2 \, \mathrm{d}\mu(g)$ implies $\lambda > 0$. Now let $V_0 \subset V$ be any finite-dimensional subspace with the corresponding orthogonal projection $E : V \to V_0 \subset V$. Consider EA restricted to V_0 to obtain

$$\int_{G/Z} \left(\cdot \,|E\pi(g)v \right) \cdot E\pi(g)v \, \mathrm{d}\mu(g) = \lambda \cdot \mathrm{id}_{V_0} : \; V_0 \to V_0.$$

Inside $\mathrm{End}_{\mathbb{C}}(V_0)$, we take traces on both sides and pass it under the integral sign to deduce

$$\mu(G/Z)\|v\|_V^2 \geq \int_{G/Z} \|E\pi(g)v\|^2 \, \mathrm{d}\mu(g) = \lambda \dim V_0.$$

This bound shows that $\dim V$ is finite. □

Theorem 4.36 (Peter–Weyl: the L^2 version)**.** *Let $\omega : Z \to \mathbb{S}^1$ be a continuous homomorphism. There is an isomorphism of unitary $G \times G$-representations*

$$\mathrm{Coeff} : \widehat{\bigoplus_{\pi:irred}} \mathrm{HS}(V_\pi) = \widehat{\bigoplus_{\pi:irred}} \pi \boxtimes \bar{\pi} \xrightarrow{\sim} L^2(G/Z, \omega)$$
$$(v_\pi \otimes w_\pi)_\pi \longmapsto \sum_\pi d(\pi) c_{v_\pi \otimes w_\pi}$$

and the decomposition is multiplicity-free. Consequently, the matrix coefficients of irreducible unitary representations with central character ω span a dense subspace in $L^2(G/Z, \omega)$.

Furthermore, the inverse of Coeff *is determined as follows. Let φ be a matrix coefficient of π, then*

$$\mathrm{Coeff}^{-1}(\varphi) = \pi(\check{\varphi}) := \int_{G/Z} \check{\varphi}(g)\pi(g)\, \mathrm{d}\mu(g) \in \mathrm{End}_{\mathbb{C}}(V_\pi).$$

Proof. To obtain the decomposition, simply combine 4.28 with 4.34. The inverse map follows from 4.30. □

Remark 4.37. In deriving 4.36, we did not use the fact that each irreducible π has finite dimension. In fact, we can deduce 4.35 from 4.36 as follows. Restrict $G \times G$-representations to $G \times \{1\} \simeq G$, so that the spectral decomposition degenerates into an isomorphism of unitary G-representations

$$\widehat{\bigoplus_\pi} \pi \hat{\otimes} V_\pi \simeq L^2(G/Z, \omega)$$

where V_π still carries its Hilbert space structure, but now with trivial G-action. By 4.34 we know π must occur in $L^2(G/Z, \omega)$ with finite multiplicity, which is exactly $\dim V_\pi$.

Corollary 4.38. *The space $L^2(G/Z, \omega)$ is closed under convolution $\star$. Specifically, if each $V_\pi \hat{\otimes} \overline{V_\pi}$ (resp.* $\mathrm{HS}(V_\pi)$*) is endowed with the $\mathbb{C}$-algebra structure from 4.29, then* Coeff *is an isomorphism between $\mathbb{C}$-algebras with multiplication given by $\star$ on both sides.*

Proof. Since $L^2(G/Z, \omega) \subset L^1(G/Z, \omega)$ by compactness, L^2-property is preserved under convolution by 2.3. It suffices to check the compatibility with convolution on a dense subspace of $\widehat{\bigoplus}_\pi \pi \boxtimes \bar{\pi}$, and 4.29 will do the job. □

Finer properties of the spectral decomposition will be studied in the section on direct integrals.

4.7 Basic character theory

We keep the assumptions from §4.6: G is a locally compact group, $Z \subset Z_G$ is closed and G/Z comes with a chosen Haar measure μ. Fix a continuous homomorphism $\omega : Z_G \to \mathbb{S}^1$. Throughout this subsection, we choose the Haar measure normalized by

$$\mu(G/Z) = 1.$$

For endomorphisms of finite-dimensional vector spaces, it is safe to take trace. The following is thus justified.

Definition 4.39. The *character* of a finite-dimensional unitary representation (π, V) of G is the continuous function

$$\Theta_\pi : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \mathrm{Tr}\left(\pi(g) : V \to V\right).$$

We clearly have $\Theta_{\pi\oplus\sigma} = \Theta_\pi + \Theta_\sigma$ and $\Theta_{\bar{\pi}} = \overline{\Theta_\pi}$. Note that Θ_π does not involve choices of measures.

Proposition 4.40. *The formal degree of an irreducible unitary representation* (π, V) *equals* $\dim V$.

Proof. Choose an orthonormal basis $v_1, \ldots, v_d$ of V. Then $\pi(g)v_1, \ldots, \pi(g)v_d$ form an orthonormal basis for every $g \in G$. Then $\left(c_{v_i \otimes v_j}(g)\right)_{1 \le i,j \le d}$ is a transition matrix between orthonormal bases, hence unitary. It follows that $\sum_{i,j} |c_{v_i \otimes v_j}(g)|^2 = d$. Integrate both sides over G/Z and apply 4.21 to see $\sum_{i,j} d(\pi)^{-1} = d$, that is, $d(\pi) = d$. □

Theorem 4.41. *If* $v_1, \ldots, v_d$ *is an orthonormal basis of an irreducible unitary representation* (π, V) *of* G*, then*

$$\Theta_\pi(g) = \sum_{i=1}^{d} c_{v_i \otimes v_i}(g) \in C(G/Z_G, \omega_\pi).$$

Furthermore, for any two irreducible unitary representations π, σ *of* G *with central character* ω *on* Z*,*

$$(\Theta_\pi | \Theta_\sigma)_{L^2(G/Z,\omega)} = \begin{cases} 1, & \pi \simeq \sigma \\ 0, & \pi \not\simeq \sigma. \end{cases}$$

Proof. For any linear endomorphism $A : V \to V$ we have $\mathrm{Tr}(A) = \sum_{i=1}^{d} (Av_i | v_i)$. This proves the first assertion. The second assertion stems from Schur's orthogonality relations 4.21 and 4.40. □

Theorem 4.42. *For the adjoint action* $f(x) \overset{g}{\mapsto} f(g^{-1}xg)$ *of* G *on* $L^2(G/Z, \omega)$*, the invariant-subspace is*

$$L^2(G/Z, \omega)^G = \widehat{\bigoplus_{\substack{\pi:\text{irred} \\ \omega_\pi|_Z = \omega}}} \mathbb{C}\Theta_\pi.$$

Proof. We know that $\pi \boxtimes \bar{\pi} = \mathrm{HS}(V_\pi) = \mathrm{End}_{\mathbb{C}}(V_\pi)$ as $G \times G$-representations by 4.16. The adjoint action $f(x) \mapsto f(g^{-1}xg)$ mirrors the diagonal G-action on $\pi \boxtimes \bar{\pi}$, which in turn corresponds to $A \mapsto \pi(g)A\pi(g)^{-1}$ on $\mathrm{End}_{\mathbb{C}}(V_\pi)$; topology does not matter here since $\dim V_\pi < +\infty$.

Hence the space of G-invariants matches $\mathrm{End}_G(\pi) = \mathbb{C} \cdot \mathrm{id}$ by 4.12. In fact, id corresponds to $\sum_i v_i \otimes v_i \in V_\pi \otimes \overline{V_\pi}$ where $v_1, v_2, \ldots$ is any orthonormal basis; it has norm $d(\pi)$ and maps to $d(\pi)\Theta_\pi \in L^2(G/Z, \omega)$ under Coeff. □

As in the case of finite groups, characters distinguish representations.

Proposition 4.43. *Let* π, σ *be finite-dimensional unitary representations. Then* $\pi \simeq \sigma$ *if and only if* $\Theta_\pi = \Theta_\sigma$.

Proof. Decompose π and σ into orthogonal sums of irreducibles, which is possible by 4.6, and then apply 4.41. □

5 Unitary representation of compact Lie groups (a sketch)

Throughout this section, Lie groups are always taken over $\mathbb{R}$. For a Lie group G we denote by $\mathfrak{g}_0$ its real Lie algebra, and $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Denote the center of $\mathfrak{g}_0$ as $\mathfrak{z}_0$. Below we write $i = \sqrt{-1}$.

5.1 Review of basic notions

Let G be a connected compact Lie group. Below is an incomplete collection of basic results on the structure of such groups. The reader may consult any available text on compact Lie groups for details, such as [5].

- The exponential map $\exp : \mathfrak{g}_0 \to G$ is surjective.
- The Lie algebra is reductive, namely $\mathfrak{g}_0 = \mathfrak{z}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple.
- The Lie subalgebra $\mathfrak{z}_0$ corresponds to the identity connected component $(Z_G)^\circ$ of Z_G, and $[\mathfrak{g}_0, \mathfrak{g}_0]$ corresponds to a closed Lie subgroup G_{ss}. We have $G = (Z_G)^\circ \cdot G_{ss}$. In particular, G_{ss} is a compact semisimple Lie group.
- A *torus* means a connected commutative compact Lie group. Therefore the group Z_G° above is a torus.
- A theorem of Weyl asserts that every compact semisimple Lie group admits a finite covering which is a connected, compact and simply connected Lie group. Applied to G_{ss}, there is a finite covering of G of the form
$$Z \times G_{\mathrm{sc}} \twoheadrightarrow G, \tag{5.1}$$
where Z is a torus and G_{sc} is a connected and simply connected compact Lie group.

Let T be a torus. Since $\exp : \mathfrak{t} \to T$ is an open surjective homomorphism between Lie groups, we see $\mathfrak{t}_0/\mathscr{L} \simeq T$ where $\mathscr{L} := \ker(\exp)$ is a lattice in $\mathfrak{t}_0$. Consequently, $T \simeq (\mathbb{R}/\mathbb{Z})^{\dim T}$ as Lie groups. Define the character lattice as
$$X^*(T) := \mathrm{Hom}(T, \mathbb{S}^1) \hookrightarrow i\mathfrak{t}_0^* \subset \mathfrak{t}^*$$
Here Hom is taken in the category of Lie groups, and the inclusion is given by mapping $\chi : T \to \mathbb{S}^1$ to the $\lambda \in i\mathfrak{t}_0^*$ such that
$$\chi(\exp(X)) = \exp(\langle \lambda, X \rangle), \quad X \in \mathfrak{t}_0;$$
so λ is essentially the derivative of χ at 1. This identifies $X^*(T)$ with a lattice in $i\mathfrak{t}_0^*$, namely
$$\left\{ \lambda \in i\mathfrak{t}_0^* : \lambda(\mathscr{L}) \subset 2\pi i \mathbb{Z} \right\},$$
and $\bar{\chi}$ corresponds to $-\lambda$ whenever χ corresponds to λ. It is sometimes beneficial to read χ as "e^λ".

Now we consider maximal tori in G: they are closed tori $T \subset G$, maximal with respect to inclusion.

- The Lie algebra $\mathfrak{t}_0$ of a maximal torus T is an abelian subalgebra in $\mathfrak{g}_0$, and vice versa.
- The maximal tori in G are all conjugate.
- Given a maximal torus T, every element is conjugate to some element of T. If $t, t' \in T$ are conjugate in G, then they are actually conjugate in the normalizer $N_G(T)$.
- Z_G is contained in the intersection of all maximal tori.

Fix a maximal torus $T \subset G$, there is a decomposition
$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha$$
into T-eigenspaces under the adjoint action $\mathrm{Ad} : T \to \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$. Here $\Phi = \Phi(\mathfrak{g}, \mathfrak{t}) \subset i\mathfrak{t}_0^*$ is the set of roots for $(\mathfrak{g}, \mathfrak{t})$; all the subspaces $\mathfrak{g}_\alpha$ are 1-dimensional. This yields a *reduced root system* on $i\mathfrak{t}_0$. By choosing a system of positive roots, we decompose $\Phi = \Phi^+ \sqcup (-\Phi^+)$.

Given any root α, the associated *coroot* is denoted by $\check{\alpha} \in i\mathfrak{t}_0$.

The positive coroots $\check{\alpha}$ cut out the "acute Weyl chamber"

$$\mathscr{C}_+ := \bigcap_{\alpha \in \Phi^+} \{\check{\alpha} \geq 0\} \subset X^*(T) \otimes \mathbb{R} = i\mathfrak{t}_0^*;$$

the elements of $\mathscr{C}_+$ are called *dominant*. Denote $\mathscr{C}_+^\circ := \bigcap_{\alpha \in \Phi^+} \{\check{\alpha} > 0\}$.

The algebraically defined *Weyl group* $\Omega(\mathfrak{g}, \mathfrak{t})$ is generated by root reflections

$$s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \check{\alpha} \rangle \alpha, \quad \alpha \in \Phi$$

on $i\mathfrak{t}_0^*$. It is a finite group. It also acts on $i\mathfrak{t}_0$ by duality: we require that $\langle w\lambda, X \rangle = \left\langle \lambda, w^{-1}X \right\rangle$ for all $w \in \Omega(\mathfrak{g}, \mathfrak{t})$

It turns out that $\Omega(\mathfrak{g}, \mathfrak{t})$ together with its action on $i\mathfrak{t}^*$ is canonically isomorphic to the analytically defined $\Omega(G, T) = N_G(T)/Z_G(T)$. This is essentially done by reducing to the case $\mathfrak{g}_0 = \mathfrak{su}(2)$. Furthermore, $\mathscr{C}_+$ is a fundamental domain for the $\Omega(G, T)$-action: we have

$$i\mathfrak{t}_0^* = \bigcup_{w \in \Omega(G,T)} w\mathscr{C}_+,$$

$$w \neq w' \implies w\mathscr{C}_+^\circ \cap w'\mathscr{C}_+^\circ = \emptyset,$$

$$(\lambda \in \mathscr{C}_+ \ \wedge \ w\lambda \in \mathscr{C}_+) \implies w\lambda = \lambda.$$

The reflections relative to simple roots in $\Phi(\mathfrak{g}, \mathfrak{t})$ generate $\Omega(G, T)$. We write $\ell : W \to \mathbb{Z}_{\geq 0}$ for the length function for $\Omega(G, T)$ with respect to these generators. Recall that $(-1)^{\ell(\cdot)} : \Omega(G, T) \to \{\pm 1\}$ is a homomorphism of groups.

When G is semisimple, we define

$$P := \{\lambda \in X^*(T) \otimes \mathbb{R} : \forall \alpha \in \Phi,\ \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}\},$$
$$Q := \sum_{\alpha \in \Phi^+} \mathbb{Z}\alpha.$$

It is a general fact about root systems that they are both lattices in $X^*(T) \otimes \mathbb{R}$, and

$$P \supset X^*(T) \supset Q$$

If G is simply connected (resp. $Z_G = \{1\}$, i.e. adjoint), then $P = X^*(T)$ (resp. $Q = X^*(T)$). For semisimple G in general, $P/X^*(T) \simeq \pi_1(G, 1)$. In any case, P contains the half-sum of positive roots

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \ \in P.$$

5.2 Algebraic preparations

Definition 5.1. For any Ω-stable lattice $\Lambda \subset X^*(T) \otimes \mathbb{R}$, and any commutative ring A (usually $A = \mathbb{Z}$), the elements of the group A-algebra $A[\Lambda]$ are expressed uniquely in the exponential notation

$$\sum_{\lambda \in \Lambda} c_\lambda e^\lambda, \quad c_\lambda \in A : \quad \text{finite sum}$$

subject to the relation $e^\lambda e^\mu = e^{\lambda + \mu}$ and the usual laws of algebra. The Weyl group $\Omega(G, T)$ acts A-linearly by mapping e^λ to $e^{w\lambda}$.

Once an isomorphism $\Lambda \simeq \mathbb{Z}^{\oplus r}$ is chosen, $\mathbb{Z}[\Lambda]$ is identifiable with $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$. In particular, $\mathbb{Z}[\Lambda]$ is a unique factorization domain since it is a localization of $\mathbb{Z}[X_1, \ldots, X_r]$. Same for $\mathbb{C}[X^*(T)]$.

Fix a maximal torus $T \subset G$ together with a system of positive roots $\Phi^+ \subset \Phi = \Phi(\mathfrak{g}, \mathfrak{t})$. We will mainly work inside $\mathbb{Z}[X^*(T)]$. Keep in mind that $\lambda \in X^*(T)$ (thus the symbol e^λ) also signifies the character $T \to \mathbb{S}^1$ mapping $\exp(X)$ to $\exp(\langle \lambda, X \rangle)$, for all $X \in \mathfrak{t}_0$. By linearity, we obtain a map

$$\mathbb{C}[X^*(T)] \longrightarrow C(T).$$

As $X^*(T) \subset i\mathfrak{t}_0^*$, it is reasonable to define the complex conjugation in $\mathbb{C}[X^*(T)]$ by

$$\overline{\sum_\lambda c_\lambda e^\lambda} = \sum_\lambda \overline{c_\lambda} e^{-\lambda}.$$

which extends the complex conjugation on $\mathrm{Hom}(T, \mathbb{S}^1) \subset \mathbb{Z}[X^*(T)]$.

Lemma 5.2. *The map* $\mathbb{C}[X^*(T)] \to C(T)$ *is an embedding of* $\mathbb{C}$*-algebras, where* $C(T)$ *is equipped with pointwise addition and multiplication. It is* $\Omega(G,T)$*-equivariant and respects complex conjugation on both sides.*

Proof. For injectivity, apply the linear independence of characters to T. The other assertions are inherent in the construction. □

It is convenient to enlarge $X^*(T)$ to the commensurable lattice $\Lambda := \frac{1}{2}X^*(T)$ to accommodate for elements like $e^{\alpha/2}$, for $\alpha \in \Phi$. Notice that $\Lambda' \subset \Lambda$ implies $\mathbb{Z}[\Lambda'] \subset \mathbb{Z}[\Lambda]$.

Definition 5.3. The *Weyl denominator* is the element

$$\Delta := e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \prod_{\alpha \in \Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2} \right)$$

which lives in $\mathbb{Z}\left[\frac{1}{2}X^*(T)\right]$, or $\mathbb{Z}[P]$ if G is semisimple.

Several algebraic lemmas are in order.

Lemma 5.4. *For every* $w \in \Omega(G,T)$ *we have* $w\Delta = (-1)^{\ell(w)}\Delta$.

Proof. It suffices to check this for $w = s_\beta$ where $\beta \in \Phi^+$ is a simple root. Use $\Delta = \prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$ and the fact that $s_\beta(\alpha) \in \Psi^+$ for all $\alpha \in \Phi^+ \smallsetminus \{\beta\}$, and $s_\beta(\beta) = -\beta$. □

In what follows, it would be convenient to be able to expand $(1 - e^{-\alpha})^{-1}$ in a formal series $\sum_{k \geq 0} e^{-k\alpha}$ for every $\alpha \in \Phi^+$. For every lattice $\Lambda \subset X^*(T) \otimes \mathbb{R}$, we can "complete" $\mathbb{Z}[\Lambda]$ in the direction of the closed cone $_{-}\mathscr{C} \subset X^*(T) \otimes \mathbb{R}$ (the negative "obtuse Weyl chamber") generated by $-(\Phi^+)$ to accommodate such formal series. Specifically, we start with the monoid algebra $\mathbb{Z}[_{-}\mathscr{C} \cap \Lambda]$, take its adic completion with respect to the interior ideal generated by $_{-}\mathscr{C}^\circ \cap \Lambda$; finally, invert $\{e^\lambda : \lambda \in \Lambda\}$ to reach the desired algebra.

Lemma 5.5. *Suppose that* Λ *is a lattice in* $X^*(T) \otimes \mathbb{R}$. *If* $\mu, \nu \in \Lambda \smallsetminus \{0\}$ *are not proportional, then* $1 - e^\mu$, $1 - e^\nu$ *do not generate the same ideal in* $\mathbb{Z}[\Lambda]$.

Proof. By working in some "completion" of $\mathbb{Z}[\Lambda]$ as explained earlier, we expand

$$\frac{1 - e^\mu}{1 - e^\nu} = (1 - e^\mu) \sum_{k \geq 0} e^{k\nu} = \sum_{k \geq 0} e^{k\nu} - \sum_{k \geq 0} e^{\alpha + k\beta}.$$

If α, β are linearly independent in $X^*(T) \otimes \mathbb{R}$, there will be infinitely many nonzero terms, so $1 - e^\nu$ cannot divide $1 - e^\mu$ in $\mathbb{Z}[\Lambda]$. The same reasoning applies to $\frac{1-e^\nu}{1-e^\mu}$. □

Lemma 5.6. *Suppose that Λ is an $\Omega(G,T)$-stable lattice in $X^*(T)\otimes\mathbb{R}$ such that $\langle\Lambda,\check{\alpha}\rangle\subset\mathbb{Z}$ for all $\alpha\in\Phi$. An element $\chi\in\mathbb{Z}[\Lambda]$ satisfies $w\chi=(-1)^{\ell(w)}\chi$ for all $w\in\Omega(G,T)$ if and only if $\chi\in\Delta\cdot\mathbb{Z}[\Lambda]^{\Omega(G,T)}$.*

Proof. Given 5.4, it suffices to prove the "only if" direction. Using

- the unique factorization property in $\mathbb{Z}[\Lambda]$,
- $1-e^{-\alpha}$ and $1-e^{-\beta}$ do not generate the same ideal since Φ is a reduced root system,
- and the fact that $e^{\mu}\in\mathbb{Z}[\Lambda]^{\times}$ for all μ,

it suffices to show that $(1-e^{-\alpha})\mid\chi$ for all $\alpha\in\Phi^+$. Write $\chi=\sum_\lambda c_\lambda e^\lambda$. By assumption we have

$$c_{s_\alpha\lambda}=-c_\lambda,\quad \lambda\in\Lambda$$

and $s_\alpha(\lambda)=\lambda-\langle\lambda,\check{\alpha}\rangle\alpha$. Collecting $\{1,s_\alpha\}$-orbits, we see that χ can be expressed as a $\mathbb{Z}$-linear sum of expressions

$$e^\lambda-e^{\lambda-\langle\lambda,\check{\alpha}\rangle\alpha}=e^\lambda\left(1-e^{-\langle\lambda,\check{\alpha}\rangle\alpha}\right).$$

The last term is divisible by $1-e^{-\alpha}$ as $k:=\langle\lambda,\check{\alpha}\rangle\in\mathbb{Z}$; handle the cases $k\geq 0$ and $k<0$ separately. □

Remark 5.7. There exists a lattice $\Lambda\subset X^*(T)\otimes\mathbb{R}$ such that $\Lambda\supset X^*(T)\cup\{\rho\}$ and $\langle\Lambda,\check{\alpha}\rangle\subset\mathbb{Z}$ for all $\alpha\in\Phi$. For example, in the finite covering $\pi: Z\times G_{\mathrm{sc}}\twoheadrightarrow G$ of (5.1), take a maximal torus in $Z\times G_{\mathrm{sc}}$ of the form $Z\times T_{\mathrm{sc}}$ that surjects onto T, and consider $\Lambda:=X^*(Z\times T_{\mathrm{sc}})$. Pull-back induces $X^*(T)\hookrightarrow\Lambda$ whereas $X^*(T)\otimes\mathbb{Q}=\Lambda\otimes\mathbb{Q}$. On the other hand, the roots remain unaltered under pull-back since π induces an isomorphism on Lie algebras.

Definition 5.8. Take Λ as in 5.7. For $\lambda\in\mathbb{Z}[X^*(T)]\cap\mathscr{C}^+$, define

$$\begin{aligned}\chi_\lambda &:= \Delta^{-1}\sum_{w\in\Omega(G,T)}(-1)^{\ell(w)}e^{w(\lambda+\rho)}\\ &=\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{-1}\sum_{w\in\Omega(G,T)}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}.\end{aligned}$$

It is *a priori* an element in the ring of fractions of $\mathbb{Z}[\Lambda]$.

We will show in 5.10 that χ_λ is independent of the choice of Φ^+.

Lemma 5.9. *For every $\lambda\in\mathbb{Z}[X^*(T)]$, we have $\chi_\lambda\in\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$.*

Proof. Fix λ. Clearly $\sum_{w\in\Omega(G,T)}(-1)^{\ell(w)}e^{w(\lambda+\rho)}$ varies by $(-1)^{\ell(\cdot)}$ under the $\Omega(G,T)$-action on $\mathbb{Z}[\Lambda]$, hence lies in $\Delta\mathbb{Z}[\Lambda]^{\Omega(G,T)}$ by 5.6. On the other hand, the second expression of χ_λ can be expanded into a formal series by completing $\mathbb{Z}[\Lambda]$ (thus $\mathbb{Z}[X^*(T)]$) in the direction of the closed cone generated by Φ^+.

Observe that $w(\lambda+\rho)=w\lambda+(w\rho-\rho)\in X^*(T)$ as

$$w\rho-\rho=-\sum_{\substack{\alpha\in\Phi^+\\ w\alpha\notin\Phi^+}}\alpha\quad\in Q.$$

So the expansion of χ_λ in formal series involves only e^μ with $\mu\in X^*(T)$. A comparison with the previous step shows that $\chi_\lambda\in\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$. □

Remark 5.10. It is a standard fact that the choices of Φ^+ in Φ form a $\Omega(G,T)$-torsor under conjugation. If we pass from Φ^+ to $w\Phi^+$, then χ_λ is also transported by w. The result above shows that χ_λ is actually independent of Φ^+.

Write $\chi_\lambda = \sum_\mu c_\mu e^\mu$ and let $\mathrm{Supp}(\chi_\lambda) := \left\{\mu \in X^*(T) : c_\mu \neq 0\right\}$. As observed above, $\mathrm{Supp}(\chi_\lambda)$ is $\Omega(G,T)$-stable.

For $\lambda, \lambda' \in X^*(T) \otimes \mathbb{R}$, we write

$$\lambda \prec \lambda' \iff \lambda' - \lambda = \sum_{\alpha \in \Phi^+} \underbrace{n_\alpha}_{\geq 0} \alpha. \tag{5.2}$$

Lemma 5.11. *Let $\lambda \in X^*(T)$. We have $\lambda \in \mathrm{Supp}(\chi_\lambda)$ and it appears with coefficient 1. If $\mu \in \mathrm{Supp}(\chi_\lambda)$ then $\mu = \lambda - \sum_{\alpha\in\Phi^+} n_\alpha \alpha$ for some coefficients $n_\alpha \in \mathbb{Z}_{\geq 0}$; in particular $\mu \prec \lambda$.*

Proof. Work in a suitably completed group algebra to write

$$\begin{aligned}\chi_\lambda = e^{-\rho} \prod_{\alpha\in\Phi^+} \left(1 + e^{-\alpha} + e^{-2\alpha} + \cdots\right) \sum_{w \in \Omega(G,T)} (-1)^{\ell(w)} e^{w(\lambda+\rho)} \\ = e^{\lambda} \prod_{\alpha\in\Phi^+} \left(1 + e^{-\alpha} + e^{-2\alpha} + \cdots\right) \sum_{w \in \Omega(G,T)} (-1)^{\ell(w)} e^{w(\lambda+\rho)-(\lambda+\rho)}.\end{aligned}$$

The basic theory of root systems says $w(\lambda+\rho) - (\lambda+\rho)$ is always of the form $-\sum_{\alpha\in\Phi^+} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}_{\geq 0}$, and is trivial only when $w = 1$. □

Lemma 5.12. *Let $\lambda, \mu \in X^*(T) \cap \mathscr{C}_+$. The coefficient of $1 = e^0$ in $\chi_\lambda \Delta \overline{\chi_\mu \Delta}$ equals $|\Omega(G,T)|$ (resp. 0) if $\lambda = \mu$ (resp. $\lambda \neq \mu$).*

Proof. Look at the coefficient of e^0 in

$$\begin{aligned}|\Omega(G,T)|^{-1} \sum_w (-1)^{\ell(w)} e^{w(\lambda+\rho)} \cdot \sum_v (-1)^{\ell(v)} e^{-v(\mu+\rho)} \\ = |\Omega(G,T)|^{-1} \sum_{w,v} (-1)^{\ell(w)+\ell(v)} e^{w(\lambda+\rho)-v(\mu+\rho)}.\end{aligned}$$

Suppose that $w(\rho+\lambda) = v(\rho+\mu)$. Since $\rho+\lambda, \rho+\mu \in \mathscr{C}_+^\circ$, this implies $w = v$ and $\lambda = \mu$. Therefore e^0 appears with multiplicity $|\Omega(G,T)|$ in $\sum_{w,v} (-1)^{\ell(w)+\ell(v)} e^{w(\lambda+\rho)-v(\mu+\rho)}$. □

5.3 Weyl character formula: semisimple case

Still assume G to be a connected compact Lie group with maximal torus T, and choose a system of positive roots $\Phi^+ \subset \Phi = \Phi(\mathfrak{g}, \mathfrak{t})$. Fix the Haar measures μ_G on G and μ_T on T, both of total mass 1.

Notice that since $\Phi = \Phi^+ \sqcup (-\Phi^+)$,

$$\Delta\bar{\Delta} = \prod_{\alpha\in\Phi} (1 - e^\alpha) = \prod_{\alpha\in\Phi} (e^\alpha - 1) \ \in \mathbb{Z}[X^*(T)],$$

thus defines a continuous function on T. We denote this function as $|\Delta|^2$.

Theorem 5.13 (Weyl's integration formula)**.** *If $f : G \to \mathbb{C}$ is measurable, then*

$$\int_G f \,\mathrm{d}\mu_G = \frac{1}{|\Omega(G,T)|} \iint_{(T\backslash G)\times T} f(g^{-1}tg)|\Delta|^2(t) \,\mathrm{d}(\mu \times \mu_T)(Tg, t)$$

where $\mu = \mu_G/\mu_T$ is the measure on $T\backslash G$ prescribed in 1.53.

Proof. Note that $\mu(G/T) = 1$ by plugging $f = 1$ into (1.9). We shall transform the integral by pulling it back through the submersion

$$\begin{aligned} A : (T\backslash G) \times T &\longrightarrow G \\ (Tg, x) &\longmapsto g^{-1}xg. \end{aligned}$$

In doing integration we can work on a dense open subset of G (resp. T). Here we consider *the regular locus* G_{reg} (resp. T_{reg}); $t \in T$ lies in T_{reg} if and only if $Z_G(t) = T$; set $G_{\mathrm{reg}} = \bigcup_{g \in G} g^{-1} T_{\mathrm{reg}} g$. Once restricted to G_{reg}, the map A becomes a $\Omega(G,T)$-torsor with the left $\Omega(G,T)$-action by $(Tg, x) \mapsto (Twg, wxw^{-1})$. Indeed, if $gxg^{-1} = hyh^{-1} \in G_{\mathrm{reg}}$, then x is conjugate to y through a unique $w \in \Omega(G,T)$, with representative $\tilde{w} \in G$. Thus $\tilde{w}^{-1}h^{-1}g$ centralizes x, so that $\tilde{w}^{-1}h^{-1}g \in N_G(T)$ and has trivial image in $\Omega(G,T)$, showing that gxg^{-1} and hyh^{-1} differ by a unique $w \in \Omega(G,T)$.

It remains to pinpoint the Jacobian. Choose the invariant volume forms on $\mathfrak{g}_0$, $\mathfrak{t}_0$ and $\mathfrak{g}_0/\mathfrak{t}_0$ which match the volume forms for μ_G, μ_T and μ at $1 = \exp(0)$ via the exponential. Decompose $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ with $\mathfrak{p} = \bigoplus_\alpha \mathfrak{g}_\alpha$; the latter is actually defined over $\mathbb{R}$, namely $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$, and $\mathfrak{p}_0$ carries the corresponding volume form.

Scrutinize the local behavior of A around a chosen (Tg, x):

$$\mathfrak{p}_0 \times \mathfrak{t}_0 \ni (X, Y) \mapsto g^{-1} \exp(-X) \exp(Y) x \exp(X) g.$$

Using the invariance of Haar measures under bilateral translations, it suffices to consider

$$\begin{aligned} (X, Y) &\mapsto \exp(-X) \exp(Y) x \exp(X) x^{-1} \\ &= \exp(-X) \exp(Y) \exp(\mathrm{Ad}(x^{-1})X); \end{aligned}$$

by writing $\mathrm{Ad}(x^{-1})X = x^{-1}Xx$, its differential at $(0,0)$ is seen to be

$$(X, Y) \mapsto \left(\mathrm{Ad}(x^{-1}) - 1\right) X + Y.$$

Relative to our compatible choice of volume forms, the Jacobian factor at (Tg, x) is precisely the absolute value of $\det\left(\mathrm{Ad}(x^{-1}) - 1 \middle| \mathfrak{p}\right)$. To conclude the computation, note that

$$\det\left(\mathrm{Ad}(x^{-1}) - 1 \middle| \mathfrak{p}\right) = \prod_{\alpha \in \Phi} (\alpha(x)^{-1} - 1) = \prod_{\alpha \in \Phi^+} |\alpha(x) - 1|^2 ;$$

here α is the character corresponding to what we denoted by e^α. □

Remark 5.14. By the basic properties of conjugacy classes on G, the inclusion $T \hookrightarrow G$ induces a bijection $T/W \to G/\mathrm{conj}$. This induces a matching between invariant functions. A conjugation-invariant function $f : G \to \mathbb{C}^\flat$ is continuous if and only if its $\Omega(G,T)$-invariant avatar $f^\flat = f|_T : T \to \mathbb{C}$ is. We supply a low-tech proof of this fact as follows.

The "only if" part is obvious. Conversely, assume $f^\flat$ is continuous and consider any sequence $x_j \to x$ in G; write $x_j = g_j t_j g_j^{-1}$ with $t_j \in T$. Thanks to compactness, we may pass to subsequences to assume that the limits $g_j \to g$ and $t_j \to t$ exist, thus $x = gtg^{-1}$. We deduce that $f(x_j) = f^\flat(t_j) \to f(t) = f(x)$, whence the continuity of f.

Denote the spaces of conjugation-invariant functions on G as $C(G)^{G-\mathrm{inv}}$, $L^2(G)^{G-\mathrm{inv}}$ and so on.

Corollary 5.15. *For $\xi, \eta \in \mathbb{C}[X^*(T)]^{\Omega(G,T)}$, we identify them as $\Omega(G,T)$-invariant continuous functions on T, then viewed as a conjugation-invariant continuous functions on G by 5.14. Let $c_0(\xi, \eta)$ be the coefficient of $e^0 = 1$ in $\xi\Delta\overline{\eta\Delta} = \xi\bar{\eta}|\Delta|^2$, then*

$$(\xi|\eta)_{L^2(G)} = \frac{c_0(\xi, \eta)}{|\Omega(G,T)|}.$$

Proof. Plug these objects into 5.13 to see

$$(\xi|\eta)_{L^2(G)} = \frac{1}{|\Omega(G,T)|} \int_T \xi\overline{\eta}|\Delta|^2 \, \mathrm{d}\mu_T.$$

Recall that the integral over T of a character $\chi : T \to \mathbb{S}^1$ equals zero (resp. $1 = \mu_T(T)$) when $\chi \neq 1$ (resp. $\chi = 1$). The assertion becomes visible after expanding $\xi\Delta\overline{\eta\Delta}$ in $\mathbb{Z}[X^*(T)]$. □

Lemma 5.16. *Assume G is semisimple. The elements χ_λ for $\lambda \in \mathscr{C}_+^\circ \cap X^*(T)$ form a $\mathbb{Z}$-basis for $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$. Moreover, in terms of the embedding $\mathbb{Z}[X^*(T)]^{\Omega(G,T)} \hookrightarrow C(G)^{G-inv}$ of 5.2, they are orthonormal with respect to $(\cdot|\cdot)_{L^2(G)}$.*

Proof. Observe that $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$ has a $\mathbb{Z}$-basis of the form $\xi_\lambda := \sum_{\mu \in W\cdot\lambda} e^\mu$ where $\lambda \in \mathscr{C}_+^\circ \cap X^*(T)$. Write

$$\chi_\lambda = \sum_{\mu \in \mathscr{C}_+^\circ \cap X^*(T)} c_{\lambda,\mu}\xi_\mu, \quad c_{\lambda,\mu} \in \mathbb{Z}.$$

Then 5.11 entails that $c_{\lambda,\lambda} = 1$ and $c_{\lambda,\mu} \neq 0 \implies \mu \prec \lambda$. Note that the $\prec$ defined in (5.2) induces a total order on $X^*(T) \otimes \mathbb{R}$ since $\mathfrak{g}$ is semisimple. This shows that $(c_{\lambda,\mu})_{\lambda,\mu}$ is an upper triangular matrix over $\mathbb{Z}$ when λ, μ are enumerated along $\prec$, with diagonal entries equal to 1. Hence it is invertible, and this implies that $\{\chi_\lambda\}_\lambda$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$.

The orthonormal property follows from 5.12 and 5.15. □

Now apply the results in §4.6 with $Z = \{1\}$. Given a unitary representation π of G with $\dim V_\pi < \infty$, its restriction to T decomposes into irreducibles. By 4.10 this is

$$\pi|_T = \bigoplus_{\mu \in X^*(T)} \mu^{\oplus \mathrm{mult}(\pi:\mu)},$$

where $\mathrm{mult}(\pi : \mu) \in \mathbb{Z}_{\geq 0}$ stands for the *multiplicity* of μ in $\pi|_T$. On the other hand, its character Θ_π restricted to T gives a finite sum

$$\Theta_\pi(t) = \sum_{\mu \in X^*(T)} \mathrm{mult}(\pi : \mu)\mu(t), \quad t \in T.$$

This function is $\Omega(G,T)$-invariant. Therefore $\Theta_\pi|_T$ actually yields an element of $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$, corresponding to $\sum_\mu \mathrm{mult}(\pi : \mu)e^\mu$. We call those μ with nonzero multiplicities as the *weights* of G.

The weight-multiplicities $\{\mathrm{mult}(\pi : \mu)\}_{\mu \in X^*(T) \cap \mathscr{C}_+}$ determine $\Theta_\pi|_T$ and then Θ_π, which in turn determines π up to isomorphism, by 4.43.

Theorem 5.17 (Weyl character formula)**.** *Assume G is semisimple. Then the set of $\lambda \in X^*(T) \cap \mathscr{C}_+$ is in bijection with the set of isomorphism classes of irreducible unitary representations π of G. It is determined by*

$$\lambda \leftrightarrow \pi \iff \Theta_\pi = \chi_\lambda$$

where we identify Θ_π with an element of $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$ as above. In fact, λ is the highest weight *of π relative to Φ^+, namely all the other weights are $\prec \lambda$, see (5.2); it occurs with multiplicity one in π.*

Proof. Given π, we regard Θ_π as an element of $\mathbb{Z}[X^*(T)]^{\Omega(G,T)}$, and expand it into $\sum_\lambda n_\lambda \chi_\lambda$ with $n_\lambda \in \mathbb{Z}$ using 5.16. Since those χ_λ have been shown to be orthonormal, 4.41 yields

$$1 = (\Theta_\pi|\Theta_\pi)_{L^2(G)} = \sum_\lambda n_\lambda^2.$$

Hence there is exact one nonzero n_λ, with $n_\lambda \in \{-1, +1\}$. We must have $n_\lambda = +1$ so that $\Theta_\pi = \chi_\lambda$, otherwise by 5.11, $\mathrm{mult}(\pi : \lambda) = n_\lambda = -1$ would be absurd. That result also implies the assertion about highest weight.

All in all, we have an injection $\pi \mapsto \lambda$ by matching $\Theta_\pi = \chi_\lambda$. Since all the Θ_π form an orthonormal basis of the Hilbert space $L^2(G)^{G-\mathrm{inv}}$, whilst the χ_λ are also orthonormal by 5.16, this map must be surjective as well. □

Proposition 5.18 (Weyl denominator formula)**.** *Assume G is semisimple. Then*

$$\Delta = \sum_{w \in \Omega(G,T)} (-1)^{\ell(w)} e^{w\rho}.$$

Proof. By considerations of the highest weight, the trivial representation of G must correspond to $\lambda = 0$. The assertion then follows from $\chi_0 = 1$. □

Theorem 5.19 (Weyl dimension formula)**.** *Assume G is semisimple. Let π be an irreducible unitary representation with highest weight λ. Then*

$$\dim V_\pi = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \check{\alpha} \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \check{\alpha} \rangle}.$$

Proof. The dimension equals $\Theta_\pi(1)$. Write $\Delta\Theta_\pi = \sum_w (-1)^{w\rho} e^{w(\lambda+\rho)}$. Take appropriate derivatives...... □

[NOT FINISHED YET]

5.4 Extension to non-semisimple groups

[UNDER CONSTRUCTION]

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